Sets, relations, functions, simplest combinatorial formulas

Lections 1-2

Sets

- It is useful to have a way of describing a collection of "things" and the mathematical name for such a collection is a ***set***
- Any well defined *collection of items*
- The collection of colors {Red, Blue, Green} is a set we might call ***A*** and write as ***A***={Red, Blue, Green}
- The items in a set- the elements of the set
- Order of elements is meaningless

{1,2,3} is the same as {3,2,1}

- It does not matter how often the same element is listed
- $a \in \mathbf{A}$ “a is an element of A”
  “a is a member of A”
- $a \notin \mathbf{A}$ “a is not an element of A”
- $\mathbf{A} = \{a_1, a_2, \ldots, a_n\}$ “A contains…”

Examples

- Empty set- $\boldsymbol{A}=\varnothing$
- $\boldsymbol{N}$={0,1,2,3,…} - the set of integers.
- $\boldsymbol{Z}$={…, -3, -2, -2, 0, 1, 2, 3, …}
- $\boldsymbol{Q}$=the set of fractions
- $\boldsymbol{R}$=the set of real numbers
- The set that contains everything -the universal set written $\boldsymbol{S}$, $\boldsymbol{U}$, or $\boldsymbol{\Omega}$
- We will write $\boldsymbol{\sim A}$ when we mean the set of things which are not in $\boldsymbol{A}$

Subsets

- If the set ***B*** contains all the elements in the set ***A*** together with some others then we write ***A***$\subset$***B***

  A is a subset of ***B***

  {Matthew, Mark, Luke, John} $\subset${Matthew, Mark, Luke, John, Thomas}

- **A** $\subset$ **B** is the same as **B** $\supset$ **A**

-
- if a $\in$ **A** then a $\in$ **B** or

$$a \in \mathbf{A} \Longrightarrow a \in \mathbf{B}$$

- If B is a subset but might possibly be the same as A then we use **A** $\subseteq$ **B**
- **A=B** —**A** contains exactly the same things as **B**
- if **A**$\subseteq$**B** and **B**$\subseteq$**A** then **A=B**

$$(A \subseteq B) \cap (B \subseteq A) \Longrightarrow A=B$$

- $\emptyset$ $\subseteq$**A** for any set **A**

The power set and the cardinality

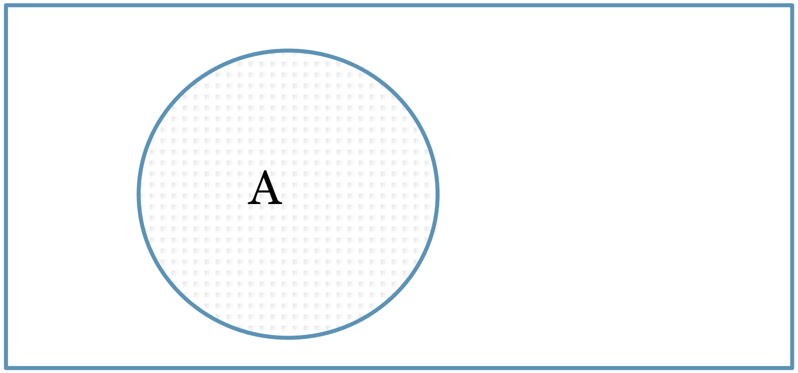
- **P**(**A**) or 2^A is the set of all subset of **A**

  A={Matthew, Mark, Luke} then **P**(**A**) consist of
  {Matthew, Mark, Luke}
  {Matthew, Mark}
  {Matthew, Luke}
  {Mark, Luke}
  {Mathew}
  {Mark}
  {Luke}
  {...}
- The number of element in a set **A** -the cardinality of **A** and written ||**A**||

  if **A**={Matthew, Mark, Luke, John} then ||**A**||=4

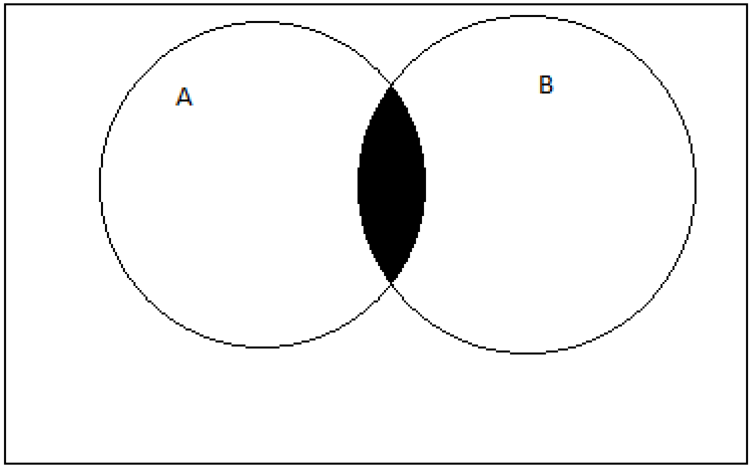
Venn Diagrams and Manipulating Sets

- We can think of the universal set **S** as a rectangle and a set, say **A** as the interior of the circle drawn in **S**



- The speckled area is A, while the remainder of the area of the rectangle is **~A**
- **A** together with **~A** make up **S**

Intersection



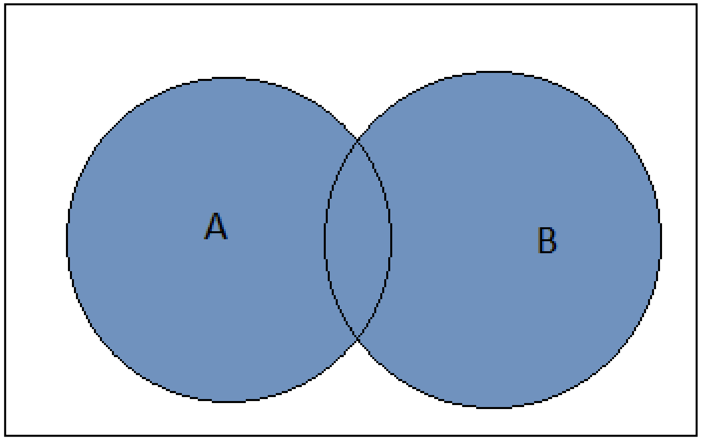
- We can write the set of items that belong to both the set **A** and the set **B** as **A**∩**B**. Formally $(x \in \mathbf{A}) \cap (x \in \mathbf{B}) => (x \in \mathbf{A} \cap \mathbf{B})$
- We call this a *intersection* of **A** and **B** or, less formally, **A** and **B**
- In terms of the Venn diagram in figure the two circles represent **A** and **B** while the overlap (in black) is the intersection

Union

- We can write the set of items that belong to the set **A** or the set **B** or to the both as **A**∪**B**

$$(x \in \mathbf{A}) \cup (x \in \mathbf{B}) \Longrightarrow (x \in \mathbf{A} \cup \mathbf{B})$$

- We call this union of A and B or, less formally, **A** or **B**
- Venn diagrams of set **A**∪**B** (blue) and universal set **S**



Commutative laws	$A\cap B=B\cap A$ $A\cup B=B\cup A$
Associative laws	$A\cup(B\cup C)=(A\cup B)\cup C$
Distributive laws	$A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$ $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$
DeMorgan's laws	$\overline{A\cap B}=\overline{A}\cup\overline{B}$ $\overline{A\cup B}=\overline{A}\cap\overline{B}$
Complement laws	$A\cap\overline{A}=\emptyset$ $A\cup\overline{A}=U$
Double complement law	$\overline{\overline{A}}=A$
Idempotent laws	$A\cup A=A$ $A\cap A=A$
Absorption laws	$A\cup(A\cap B)=A$ $A\cap(A\cup B)=A$
Dominance laws	$A\cap\emptyset=\emptyset$ $A\cup U=U$
Identity laws	$A\cup\emptyset=A$ $A\cap U=A$

Cartesian Product

-
- We define the Cartesian Product **P**=**A**×**B** to be the set of ordered pairs (a,b) where a∈**A** and b∈**B**

$$\mathbf{P}=\{(a,b):(a\in\mathbf{A})\cap(b\in\mathbf{B})\}$$

A={a,b} and **B**={1,2} then
A×**B**={(a,1),(a,2),(b,1),(b,2)}

Relations

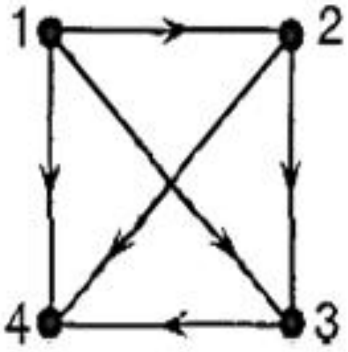
- Given two sets **A** and **B** and the product **A**×**B** we define the relation between **A** and **B** as a subset **R** of **A**×**B**
- a∈**A** and b∈**B** are related if (a,b)∈**R**, more commonly written a**R**b
- Take the simple example of **A**={1,2,3,4,5,6} and **B**={1,2,3,4,5,6} then A×B is the array

(1,1) (1,2) (1,3) (1,4) (1,5) (1,6)
(2,1) (2,2) (2,3) (2,4) (2,5) (2,6)
(3,1) (3,2) (3,3) (3,4) (3,5) (3,6)
(4,1) (4,2) (4,3) (4,4) (4,5) (4,6)
(5,1) (5,2) (5,3) (5,4) (5,5) (5,6)
(6,1) (6,2) (6,3) (6,4) (6,5) (6,6)

- A relation **R** is the subset {(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)} or the set {(i,j): i=j}
- Other example:

R={(i, j): i+j=8} = ((2,6), (3,5), (4,4), (5,3), (6,2)}

Draw the graphical representation of the relation 'less than' on {1, 2, 3, 4}.



Write down the relation matrix for the relation

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} F & T & T & T \\ F & F & T & T \\ F & F & F & T \\ F & F & F & F \end{bmatrix} \end{array}$$

- ***reflexive***: for all $x \in \mathbf{X}$ it follows that $x\mathbf{R}x$

  "greater than or equal to" is a reflexive relation but "greater than" is not
- ***symmetric***: for all x and y in **X** it follows that if $x\mathbf{R}y$ then $y\mathbf{R}x$

  "Is a blood relative of" is a symmetric relation, because x is a blood relative of y if and only if y is a blood relative of x
- ***antisymmetric***: for all x and y in **X** it follows that if $x\mathbf{R}y$ and $y\mathbf{R}x$ then $x = y$

  "Greater than or equal to" is an antisymmetric relation, because if xy and yx, then $x = y$

- ***asymmetric***: for all x and y in **X** it follows that if *x***R***y* then not *y***R***x*
  - ”Greater than” is an asymmetric relation, because if $x > y$ then not $y > x$
- ***transitive***: for all *x*, *y* and *z* in **X** it follows that if *x***R***y* and *y***R***z* then *x***R***z*
  - "Is an ancestor of" is a transitive relation, because if *x* is an ancestor of *y* and *y* is an ancestor of *z*, then x is an ancestor of *z*
- ***Euclidean***: for all *x*, *y* and *z* in **X** it follows that if *x***R***y* and *x***R***z*, then *y***R***z*
- A relation which is reflexive, symmetric and transitive is called an ***equivalence relation***

Partial order relation

- Definition: A relation is a ***partial order relation*** if it is *reflexive, antisymmetric and transitive*

Here are some examples of partial order relations:

- The relation ≤ on the set of real numbers
- The relation ⊆on the power set of a set
- The relation 'is divisible by' on the set of natural numbers
- The relation 'is a subexpression of' on the set of logical expressions (with a given set of variables)

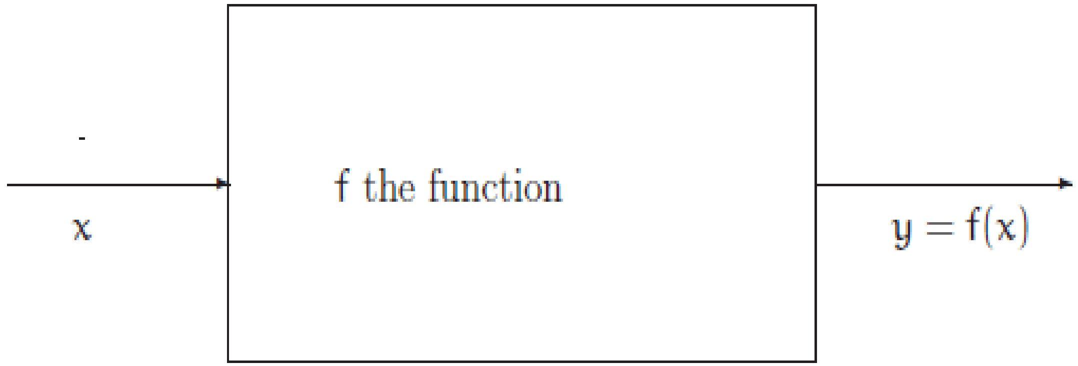
Functions

-
- two sets **X** and **Y** and also a rule which assigns to every $x \in \mathbf{X}$ a UNIQUE value $y \in \mathbf{Y}$
- We will call the rule **f** and say that for each x there is a $y = \mathbf{f}(x)$ in the set **Y**
- the critical point is that

for each x there is a *unique value* y

$$f : X \longrightarrow Y$$

- We can think of the pairs (x, y) or more clearly (x, f(x))

- 

- **X** is called the domain of f, **Y** is the codomain
- Examples

f: X⟶Y

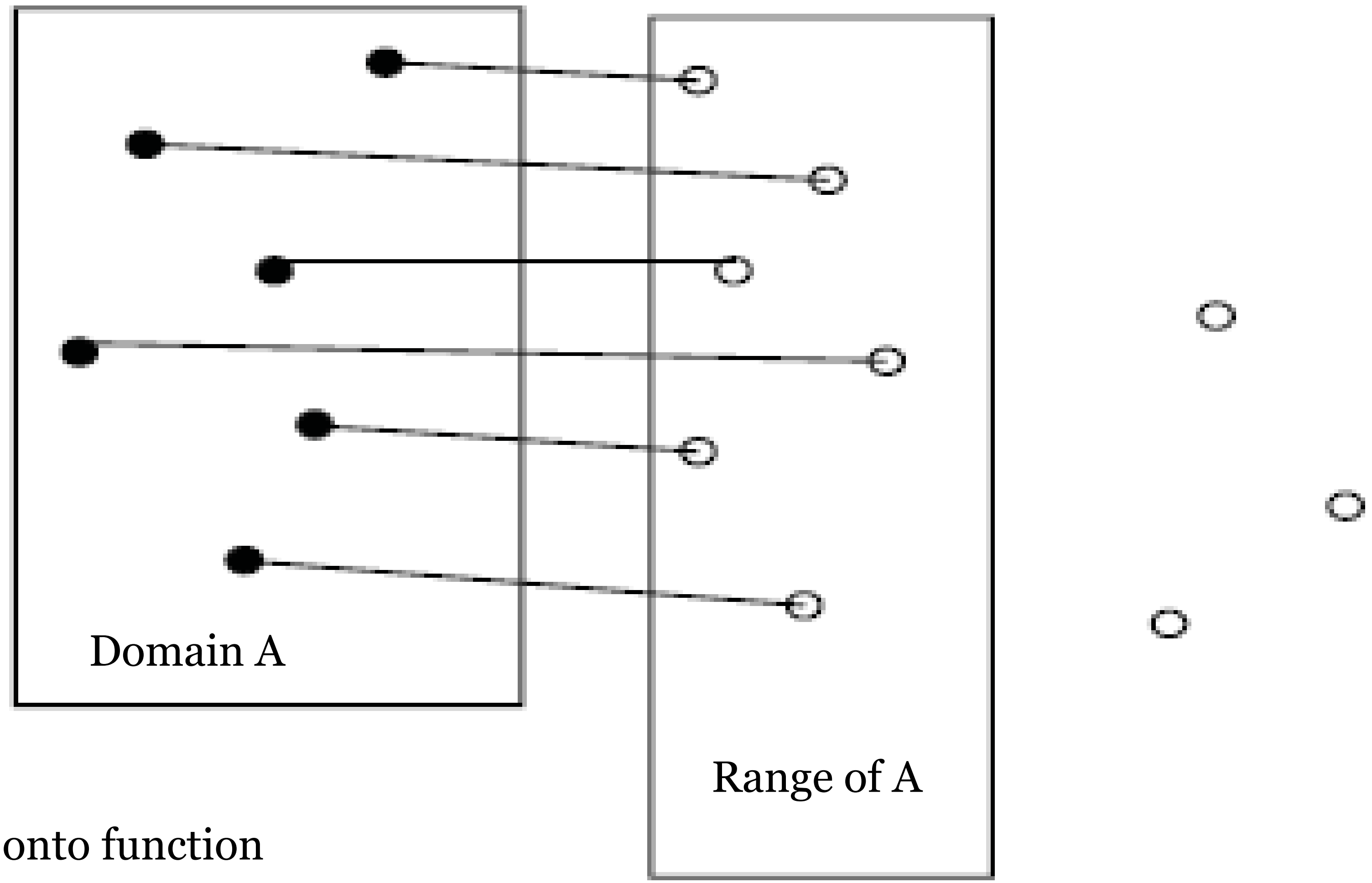
  - $f(x)=2^x$ where X={x:0≤x<∞} and Y={y: 0≤y<∞}
  - $f(x)=\sqrt{x}$ where X={x: 0≤x<∞} and Y={y: 0≤y<∞}
  - $f(x)=\sin^{-1}(x)$ where X={$-\frac{\pi}{2}$ ≤x≤ $\frac{\pi}{2}$} and Y={$-1 \le y \le 1$}

- There may be some points in **Y** (the codomain) which cannot be reached by function f
- If we take all the points in **X** and apply **f** we get a set **R** = {**f**(x) : x∈**X**} which is the ***range*** of the function **f**. Notice **R** is a subset of **Y** i.e. **R**⊂**Y**

Surjections

- Surjections (or onto functions): for every y in the codomain there is an x in the domain such that $\boldsymbol{f}(x) = y$
- codomain is bigger than the range of the function
- If the range and codomain are the same then out function is a *surjection*

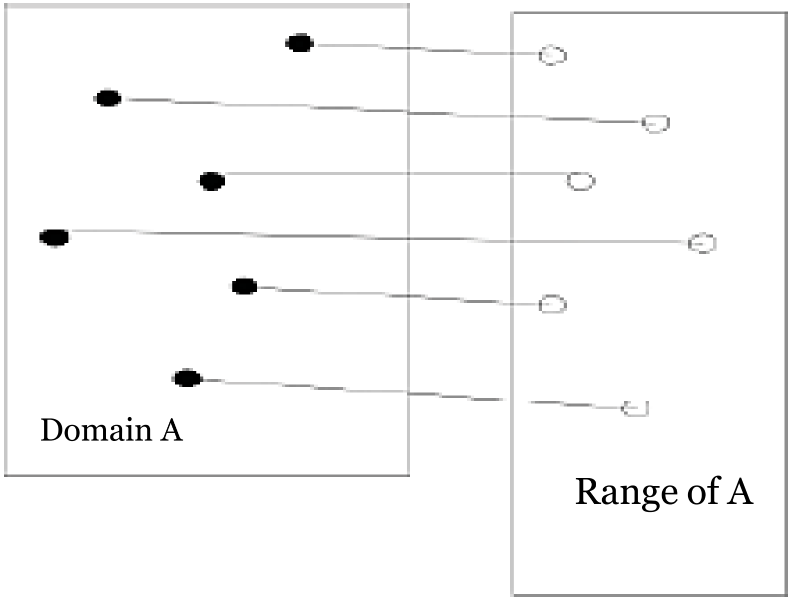
This means every y has a corresponding x for which $y=f(x)$



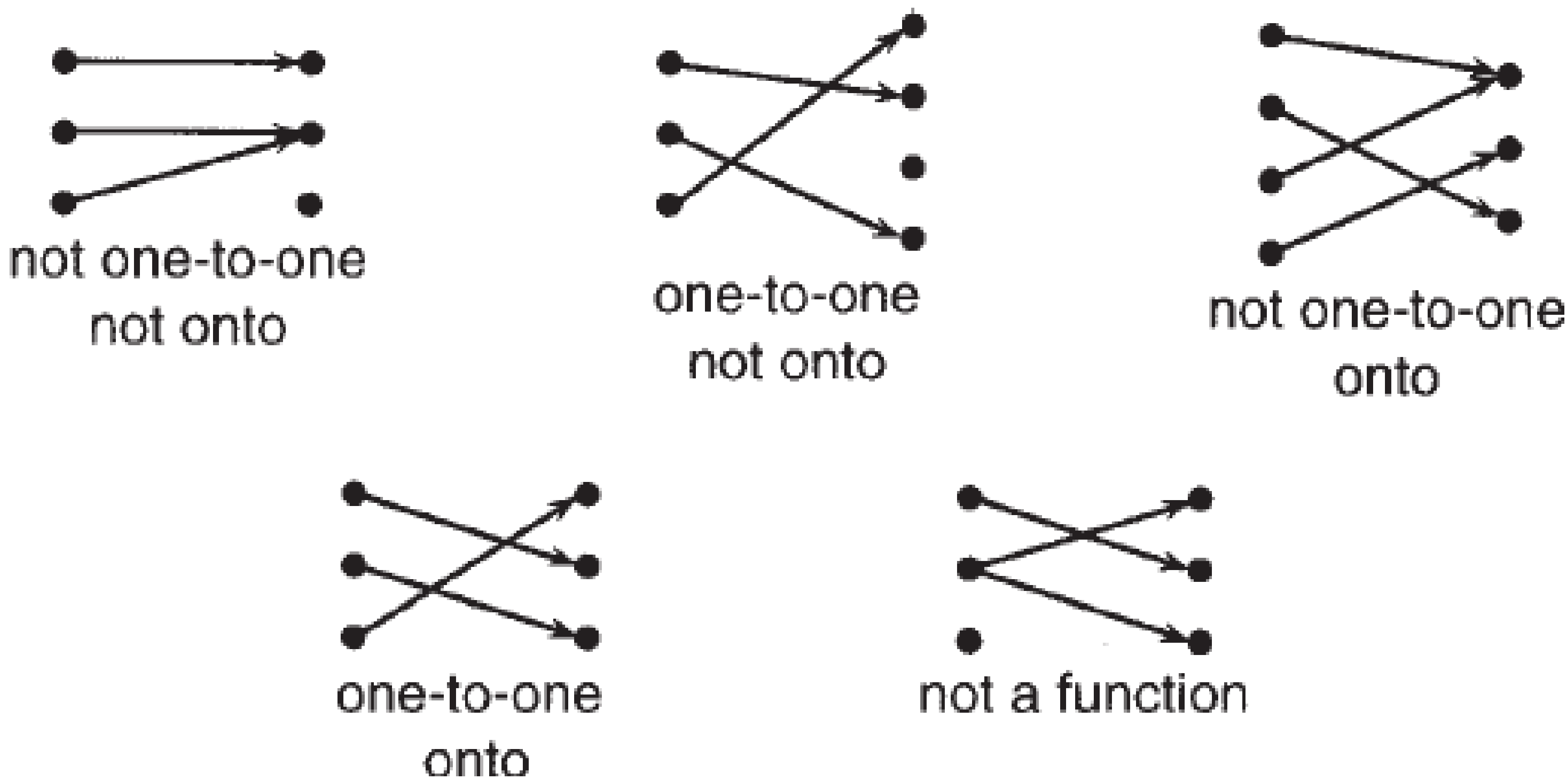
An onto function

Injection

- Another important kind of function is the injection (or one-to-one function), which have the property that x1=x2 then y1 must equal y2



An 1 to 1 function


not one-to-one
not onto
one-to-one
not onto
not one-to-one
onto
one-to-one
onto
not a function

Bijections

- Lastly we call functions ***bijections***, when they are both one-to-one and onto
- Example:

$$\boldsymbol{f}\text{: } \mathbf{X} \longrightarrow \mathbf{Y}$$

f(x)=2^X , $\mathbf{X}$ ={x: $0 \leq x < \infty$} , $\mathbf{Y}$={y: $-\infty < y < \infty$}. The range of the function is R={y: $0 \leq y < \infty$} while the codomain Y has negative values which we cannot reach using our function

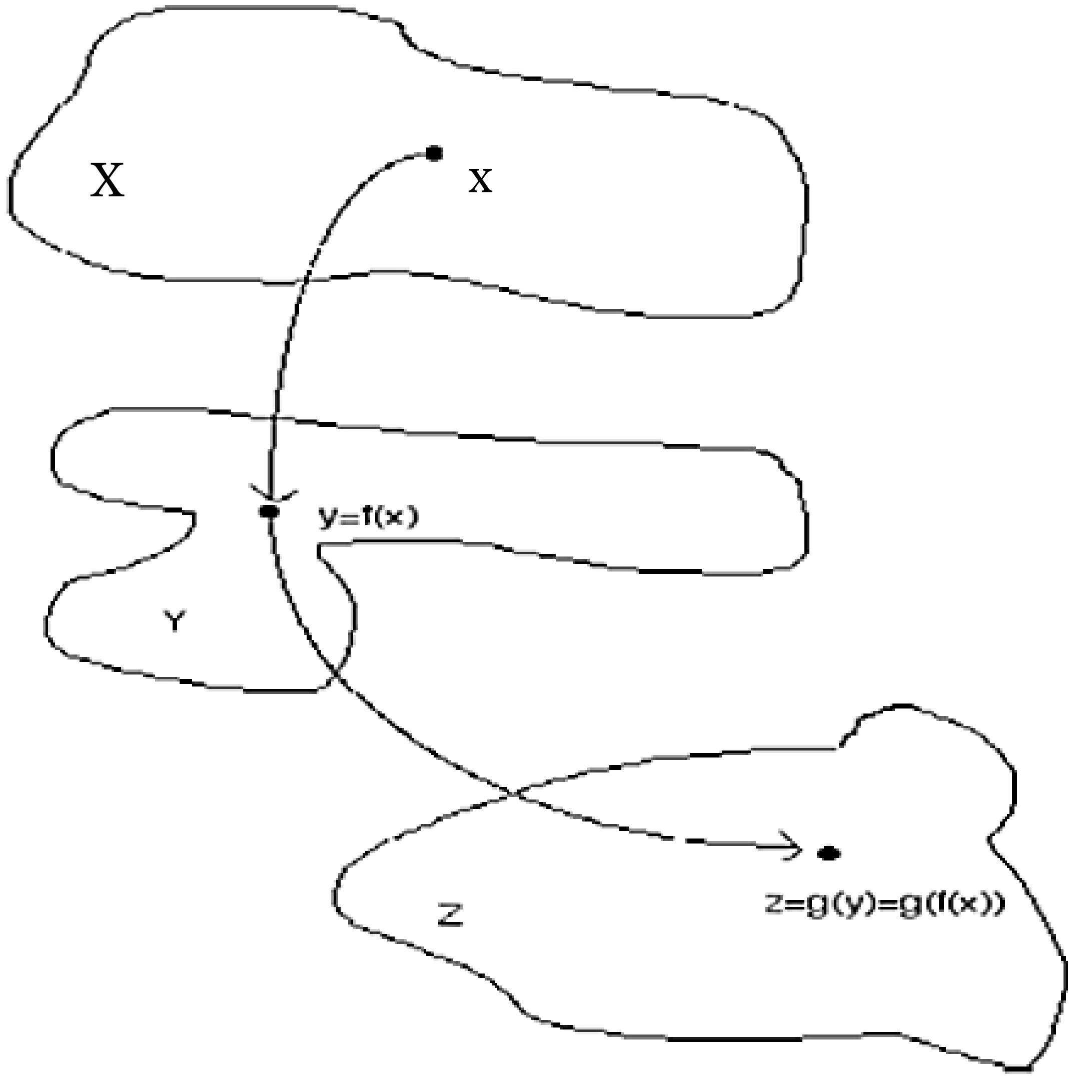
Composition of functions

- The composition of two or more functions uses the output of one function $\boldsymbol{f}$, as the input of another $\boldsymbol{g}$
- The functions $\boldsymbol{f}$: $\mathbf{X}\longrightarrow\mathbf{Y}$ and $\boldsymbol{g}$: $\mathbf{Y}\longrightarrow\mathbf{Z}$ can be composed by applying $\boldsymbol{f}$ to an argument x to obtain $y=\boldsymbol{f}(x)$ and then applying $\boldsymbol{g}$ to y to obtain $z=\boldsymbol{g}(y)$
- The composite function formed in this way from $\boldsymbol{f}$ and $\boldsymbol{g}$ can be written $\boldsymbol{g(f(x))}$ or $\boldsymbol{g \circ f}$

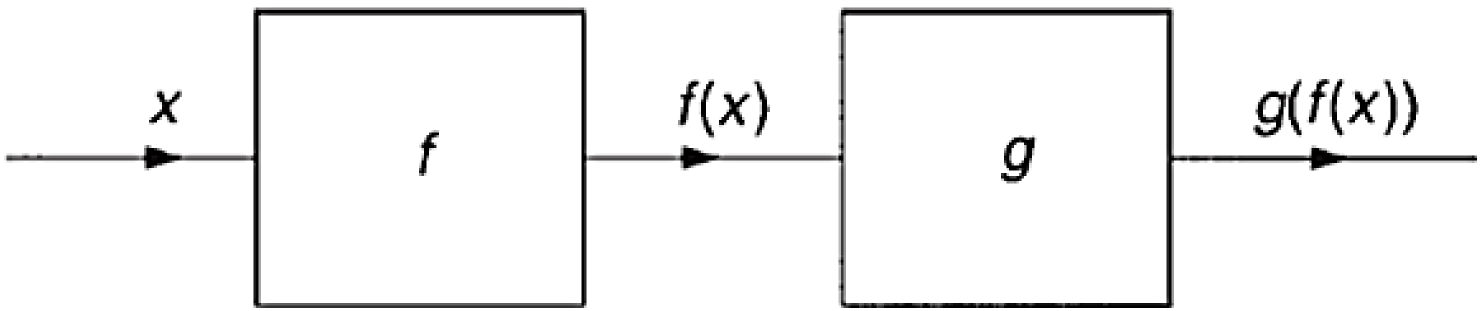
Definition: Let $\boldsymbol{f}$: $\mathbf{A}\longrightarrow\mathbf{B}$ and $\boldsymbol{g}$:$\mathbf{B}\longrightarrow\mathbf{C}$ be functions. The composite function of $\boldsymbol{f}$ and $\boldsymbol{g}$ is the function:

$$\boldsymbol{g} \circ \boldsymbol{f}: \mathbf{A}\longrightarrow\mathbf{C} \ (\boldsymbol{g} \circ \boldsymbol{f})(\mathrm{x})=\boldsymbol{g}(\boldsymbol{f}(x))$$

- Using composition we can construct complex functions from simple ones

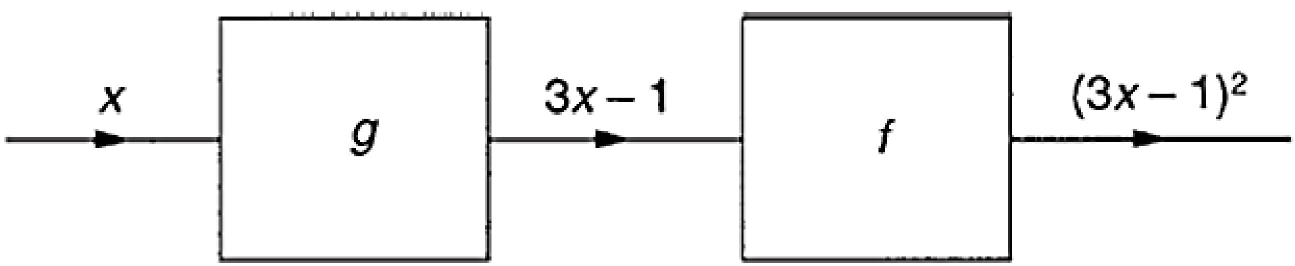


Composition of two functions f and g

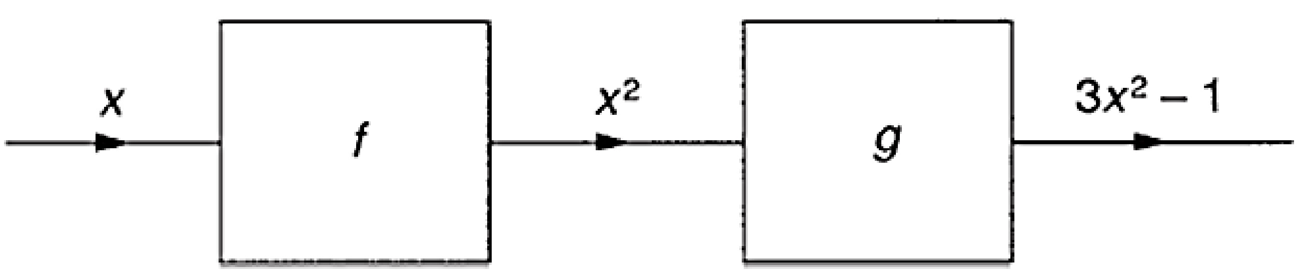


Let $f: R \longrightarrow R, f(x) = x^2$ *and* $g: R \longrightarrow R, g(x) = 3x - 1$. Find $f \circ g$ and $g \circ f$

$f \circ g$:



$g \circ f$:



- Definition: Let **A** be a set. The identity function on **A** is the function:

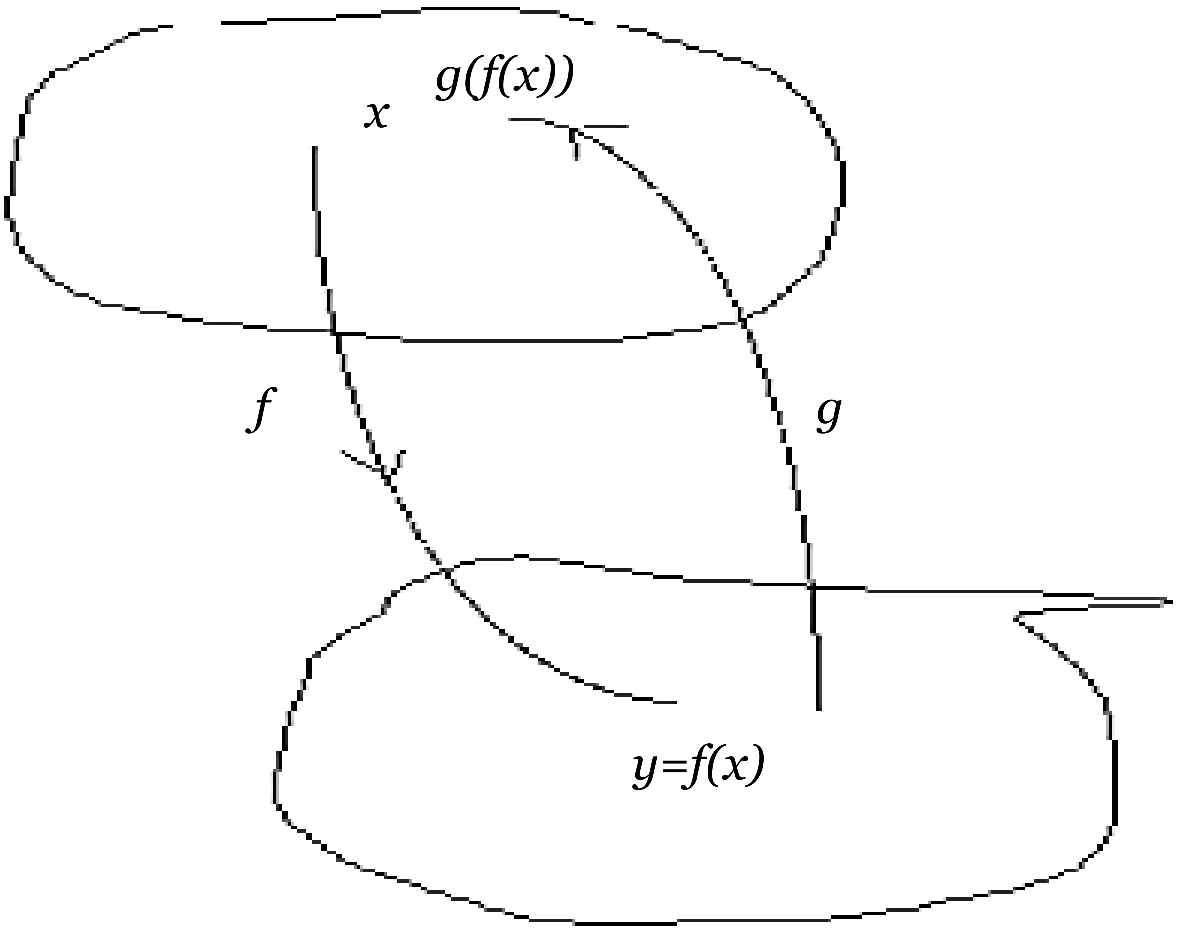
$$\boldsymbol{i}: \mathbf{A} \longrightarrow \mathbf{A},\ \boldsymbol{i}(x)=x$$

- Definition: Let $\boldsymbol{f}$: **A** $\longrightarrow$ **B** and $\boldsymbol{g}$:**B** $\longrightarrow$ **A** be functions. If $\boldsymbol{g} \circ \boldsymbol{f}$: **A** $\longrightarrow$ **A** is the identity function on ***A***, and if $\boldsymbol{f} \circ \boldsymbol{g}$:**B** $\longrightarrow$ **B** is the identity function on **B**, then $\boldsymbol{f}$ is the inverse of $\boldsymbol{g}$ (and $\boldsymbol{g}$ is the inverse of $\boldsymbol{f}$)
- Theorem: A function $\boldsymbol{f}$ has an inverse if and only if $\boldsymbol{f}$ is *onto and one-to-one*

- ***f***, ***g*** for which *x*=***g***(***f***(*x*))- ***g*** *-inverse function*
- Not all functions have inverses, in fact there is an inverse ***g*** written $\boldsymbol{f}^{-1}$ if and only if ***f*** is *bijective*
- $x = \boldsymbol{f}^{-1}(\boldsymbol{f}(x)) = \boldsymbol{f}(\boldsymbol{f}^{-1}(x))$

  $f(x) = x^2$, g(y)=1/y then g(f(x))=$\frac{1}{x^2}$.

  We of course have to take care about the definition if the range and the domain to avoid $x = 0$



The inverse f and $g = f^{-1}$

Simplest combinatorial formulas

- The principle of inclusion and exclusion:

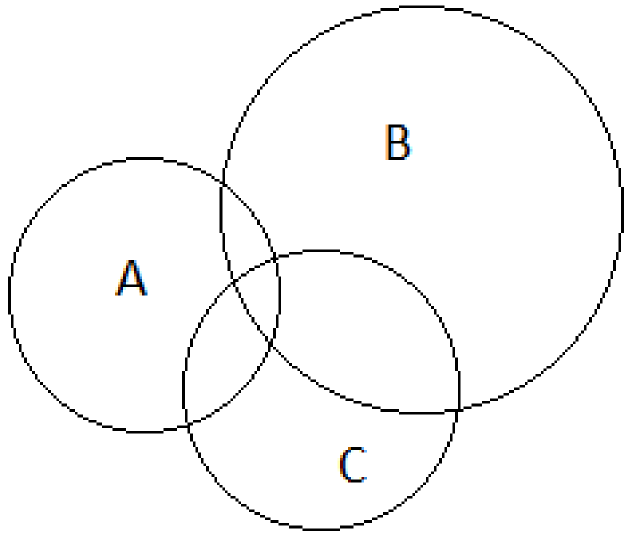
$$|\mathbf{A} \cup \mathbf{B}| = |\mathbf{A}| + |\mathbf{B}| - |\mathbf{A} \cap \mathbf{B}|$$

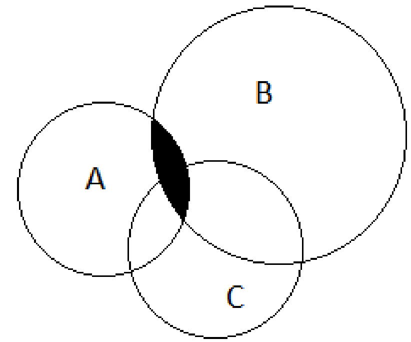
- if the cardinalities of **A** and **B** are added, then the elements in **A∩B** will be counted twice, so this is corrected for by subtracting the cardinality of **A∩B**

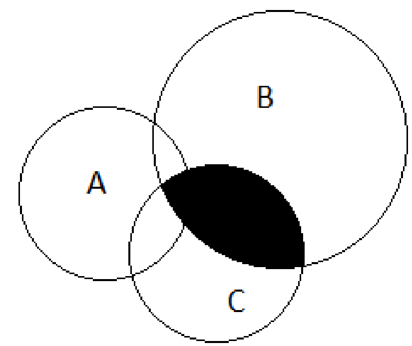
$$\|\mathbf{A} \cup \mathbf{B}\| = \|\mathbf{A}\| + \|\mathbf{B}\| - \|\mathbf{A} \cap \mathbf{B}\|$$

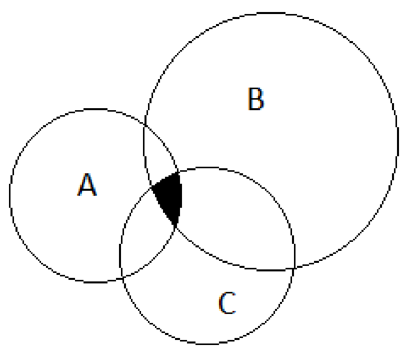
- For 3 sets

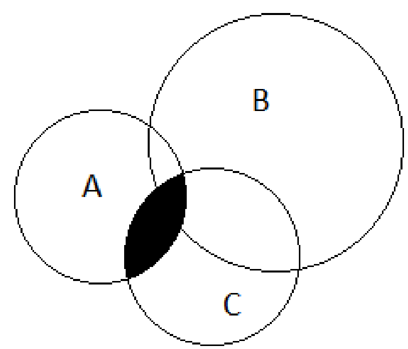
$$||\mathbf{A} \cup \mathbf{B} \cup \mathbf{C}|| = ||\mathbf{A}|| + ||\mathbf{B}|| + ||\mathbf{C}|| - ||\mathbf{A} \cap \mathbf{B}|| - ||\mathbf{B} \cap \mathbf{C}|| - ||\mathbf{A} \cap \mathbf{C}|| + ||\mathbf{A} \cap \mathbf{B} \cap \mathbf{C}||$$


A
B
C


A
B
C


A
B
C


A
B
C


A
B
C

Multiplication principle

- If a selection process consists of n steps, where the selection in the first step can be done in k_1 ways, the selection in the second step can be done in k_2 ways, and so on, then the total number of possible selections is $k_1 k_2 \ldots k_n$

Permutations

- Suppose I have n distinct items and I want to arrange them in a line. I can do this in

$$n \times (n - 1) \times (n-2) \times (n-3) \times \cdots \times 3 \times 2 \times 1 = n!$$

- $1! = 0 \; and \; 0! = 1$

$3! = 3 \times 2 \times 1 = 6$, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

- Suppose we want to select ***r*** elements from a set with ***n*** elements (where ***n* > *r***), and arrange those ***r*** elements in a particular order. In how many ways can this be done?

$$n_{p_r} = \frac{n!}{(n-r)!}$$

- Suppose we have n objects and
  - there are n_1 of type 1
  - n_2 of the type 2
  - … ….
  - n_k of type k

the total number of items in n, so

$n = n_1 + n_2 + \ldots + n_k$ then there are

$$\frac{n!}{n_1!\, n_2!\, n_3! \ldots n_k!}$$

possible arrangements

Examples

- 3 white, 4 red and 4 black balls. They can be arranged in a row in

$$\frac{11!}{3!4!4!} = 11550 \text{ possible ways}$$

- while the letters in WALLY can be arranged in

$$\frac{5!}{2!1!1!1!} = 60 \text{ ways}$$

Combinations

- The number of ways of picking k items from a group of size n is written $\binom{n}{k}$ or (for the traditionalists) C^{n}_{k}

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

- So the number of ways of picking 5 students from a group of 19 is

$$\binom{19}{5} = \frac{19!}{(14)!\,5!} = \frac{19 \times 18 \times 17 \times 16}{4 \times 3 \times 2 \times 2}$$