Euler formula for intersecting sets Newton binomial, asymptotic combinatorial identities

Theorem 1

If $n \geq 6 \quad \left(\frac{n}{3}\right)^n < n! < \left(\frac{n}{2}\right)^n$

Proof

Prove by induction on n the upper estimate.

Assume that $n! \leq \left(\frac{n}{2}\right)^n$ then

$$\left(\frac{n+1}{2}\right)^{n+1} = \frac{\left(\frac{n+1}{2}\right)^n (n+1)}{2} = \frac{\left(\frac{n}{2}\right)^n \left(1+\frac{1}{n}\right)^n (n+1)}{2}$$
$$\geq n!\left(1+1+\frac{\binom{n}{2}}{n^2}+\cdots+\frac{\binom{n}{n}}{n^n}\right)\frac{n+1}{2} > n! \cdot 2 \cdot \frac{n+1}{2} = (n+1)!$$

Similarly prove the lower estimate

Let us assume that $n! \geq \left(\frac{n}{3}\right)^n$ then considering $n! \geq 2^{n-1}$ if $n \geq 2$

$$\left(\frac{n+1}{3}\right)^{n+1} = \frac{\left(\frac{n+1}{3}\right)^n (n+1)}{3} = \frac{\left(\frac{n}{3}\right)^n \left(1+\frac{1}{n}\right)^n (n+1)}{3}$$
$$\leq \frac{n!\left(1+1+\frac{\binom{n}{2}}{n^2}+\frac{\binom{n}{3}}{n^3}+\cdots+\frac{n^n}{n!n^n}\right)(n+1)}{3}$$
$$< \frac{n!\left(1+1+\frac{n^2}{2!n^2}+\frac{n^3}{3!n^3}+\cdots+\frac{n^n}{n!n^n}\right)(n+1)}{3}$$
$$< \frac{n!\left(1+1+\frac{1}{2^1}+\frac{1}{2^2}+\cdots+\frac{1}{2^{n-1}}\right)(n+1)}{3} < (n+1)!$$

QED

Theorem 2

If $n \geq 1 \qquad e \cdot \left(\frac{n}{e}\right)^n \leq n! \leq ne \cdot \left(\frac{n}{e}\right)^n$

Proof

When n=1, 2 we can verify the inequality by substitution of these values of n. Further it is easy to see if k≥2 for ln k

$$\int_{k-1}^{k} \ln x \, dx < \ln k < \int_{k}^{k+1} \ln x \, dx \quad (2.1)$$

therefore

$$\int_{1}^{n} \ln x \, edx < \ln n! < \int_{2}^{n+1} \ln x \, dx \quad (2.2)$$

Transform the right-hand inequality (2.2) given that $n \geq 3$

$$\ln n! < \int_{2}^{n+1} \ln x \, dx = (xln\, x - x)|_{2}^{n+1} = (n+1)\ln(n+1) - (n+1) - 2ln2 + 2$$
$$= (n+1)ln\frac{n+1}{e} - 2ln2 + 2$$
$$= (n+1)\ln\frac{n}{e} + (n+1)\ln\left(1+\frac{1}{n}\right) - 2ln2 < (n+1)\ln\frac{n}{e} + 2$$

Hence

$$n! < ne < \left(\frac{n}{e}\right)^{n}$$

QED

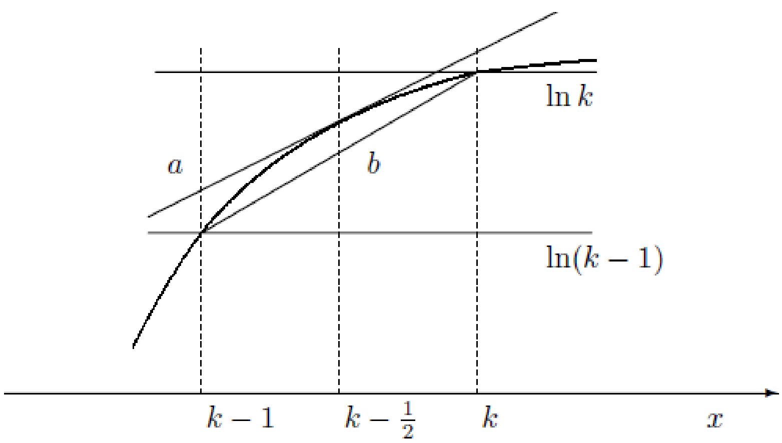
Lemma 1

If $k \geq 2$

$$\int_{k-1}^{k} \ln x \, dx + (\ln 2k - \ln(2k-1)) \leq \ln k \leq \int_{k-1}^{k} \ln x \, dx + \frac{1}{2}(\ln k - \ln(k-1))$$

Proof

The image below shows the part of the graph of the function ln x between points x=k-1 and x=k



It can be seen that $\int_{k-1}^{k} \ln x \, dx$ exceeds the difference between the values ln k and the area of a triangle, which is bounded by a segment b and straight lines x = k-1 and y = ln k. Thus

$$\int_{k-1}^{k} \ln x \, dx \geq \ln k - \frac{1}{2}(\ln k - \ln(k-1))$$

It is also can be seen, that $\int_{k-1}^{k} \ln x \, dx$ does not exceed the area of the trapezoid, which is bounded by a line a and lines x=k-1, x=k and y=0. As the area of the trapezoid is equal to the multiplication of the mean line, which is equal to $\ln(k-\frac{1}{2})$ and the height, which is equal to one, then

$$\int_{k-1}^{k} \ln x \, dx \leq \ln\left(k-\frac{1}{2}\right) = \ln k + \ln\left(1-\frac{1}{2k}\right) = \ln k - \ln(2k) + \ln(2k-1).$$

Using these inequalities we estimate ln k

$$\int_{k-1}^{k} \ln x \, dx + \ln(2k) - \ln(2k-1) \leq \ln k \leq \int_{k-1}^{k} \ln x \, dx + \frac{1}{2}(\ln k - \ln(k-1)).$$

QED

Theorem 3

$$0.8 \cdot e\sqrt{n}\left(\frac{n}{e}\right)^{n} \leq n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^{n}$$

Proof

It is sufficient to sum the inequalities from the previous lemma with $k \in [2, n]$

Sum the right inequalities

$$\sum_{k=2}^{n} \ln k \leq \int_{1}^{n} \ln x \, dx + \frac{1}{2}(\ln n - \ln 1) = n \ln n - n + 1 + \frac{1}{2}\ln n\,.$$

Hence $\boldsymbol{n}! < \boldsymbol{e}\sqrt{\boldsymbol{n}}\left(\frac{\boldsymbol{n}}{\boldsymbol{e}}\right)^{\boldsymbol{n}}$

To estimate the sum of the left inequalities assume that

$$a_1 = \sum_{k=2}^{n} (\ln(2k) - \ln(2k-1)), \;\; a_2 = \sum_{k=2}^{n} (\ln(2k+1) - \ln(2k)).$$

It can be seen that $a_1 + a_2 = \ln(2n+1) - \ln 3$

Since $a_1 > a_2$

$$a_1 > \frac{1}{2}\ln(2n+1) - \frac{1}{2}\ln 3 > \frac{1}{2}\ln n - \frac{1}{2}\ln\frac{3}{2}.$$

$$\sum_{k=2}^{n} \ln k > n\ln n - n + 1 + \frac{1}{2}\ln n - \frac{1}{2}\ln\frac{3}{2}.$$

Note that $\sqrt{\frac{2}{3}} > 0.8$

Hence $n! > 0.8 \cdot e \cdot \sqrt{n}\left(\frac{n}{e}\right)^n$

QED

Lemma 2

$$\int_0^{\frac{\pi}{2}} \sin^{2n} dx = \frac{(2n-1)(2n-3)\cdot \ldots \cdot 3 \cdot 1}{2n(2n-2.\ldots 4\cdot 2} \cdot \frac{\pi}{2} = \frac{(2n-1)!!}{2n!!}\cdot\frac{\pi}{2}$$

$$\int_o^{\frac{\pi}{2}} \sin^{2n+1} dx = \frac{2n(2n-2)\cdot \ldots 4 \cdot 2}{(2n+1)(2n-1)\cdot \ldots \cdot 3 \cdot 1} = \frac{2n!!}{(2n+1)!!}$$

Proof

Denote $\int_0^{\frac{\pi}{2}} \sin^n dx = I_n$ then

$$I_n = -\int_o^{\frac{\pi}{2}} \sin^{n-1} d\cos x = -\sin^{n-1} x \cos x \,\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \, d\sin^{n-1} x = (n-1)\int_0^{\frac{\pi}{2}} \cos^2 x \sin^{n-2} x \, dx = (n-1)\int_0^{\frac{\pi}{2}} (1-\sin^2 x)\sin^{n-2} x \, dx = (n-1)(I_{n-2} - I_n)$$

Hence

$I_n = \frac{n-1}{n} \cdot I_{n-2}$

Consistently applying it to the integrals $I_{2n} and\ I_{2n+1}$

$$I_{2n} = \frac{(2n-1)(2n-3)\cdot\ldots\cdot 3\cdot 1}{2n(2n-2)\cdot\ldots 4\cdot 2}\cdot I_0$$

$$I_{2n+1} = \frac{2n(2n-2)\cdot\ldots\cdot 4\cdot 2}{(2n+1)(2n-1)\cdot\ldots\cdot 3\cdot 1}\cdot I_1$$

As $I_0 = \frac{\pi}{2}\, and\ I_1 = 1$

Substituting these values into the above expression obtain the required equality

QED

Lemma 3

$$\frac{2^{2n}}{\sqrt{\pi n}} e^{-\frac{1}{4n}} \le \binom{2n}{n} \le \frac{2^{2n}}{\sqrt{\pi n}}$$

Proof

As sin x varies from 0 to 1 between 0 and π/2, then

$$\sin^{2n+1} x \le \sin^{2n} x \le \sin^{2n-1} x\, , x \in [0, \frac{\pi}{2}]$$

Hence

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\, dx \le \int_0^{\frac{\pi}{2}} \sin^{2n} x\, dx \le \int_0^{\frac{\pi}{2}} \sin^{2n-1} x\, dx$$

Using lemma 2

$$\frac{2n!!}{(2n+1)!!} \le \frac{(2n-1)!!}{2n!!}\cdot\frac{\pi}{2} \le \left(\frac{(2n-2)!!}{(2n-1)!!}\right)$$

$$\frac{2n!!\cdot 2n!!}{(2n+1)!!\cdot(2n-1)!!} \le \frac{\pi}{2} \le \frac{2n!!\cdot(2n-2)!!}{(2n-1)!!\cdot(2n-1)!!}$$

$$\frac{1}{\sqrt{2n+1}}\cdot\frac{2n!!}{(2n-1)!!} \le \sqrt{\frac{\pi}{2}} \le \frac{1}{\sqrt{2n}}\cdot\frac{2n!!}{(2n-1)!!}$$

Divide all members of the resulting inequalities on 2n!! and $\sqrt{\frac{\pi}{2}}$ and multiply it on (2n-1)!! and 2^{2n}

$$\frac{1}{\sqrt{1+\frac{1}{2n}}}\cdot\frac{2^{2n}}{\sqrt{\pi n}} \le 2^{2n}\cdot\frac{(2n-1)!!}{2n!!} \le \frac{2^{2n}}{\sqrt{\pi n}}\ (*)$$

Note that

$$\frac{(2n-1)!!}{2n!!} = \frac{(2n-1)!!\ \cdot 2n!!}{2n!!\ \cdot 2n!!} = \frac{(2n)!}{2^{2n}\ \cdot n!\ \cdot n!} = \binom{2n}{n} \cdot 2^{-2n}$$

Given that if $0 < x < 1\ e^{-x}\ \le 1/(1+x)$ substitute the last equality in (*)

and obtain the required estimates for $\binom{2n}{n}$

QED

Theorem 4

$$\sqrt{2\pi n}\ \cdot \left(\frac{n}{e}\right)^n e^{-1/4n}\ \le n!\ \le \sqrt{2\pi n}\ \cdot \left(\frac{n}{e}\right)^n e^{1/4n}$$

Proof

As $\binom{2n}{n} = \frac{2n!}{n!\cdot n!} = \frac{2n(2n-1)\cdot\ldots\cdot(n+1)}{n!}$, it can be seen that $n! = 2n(2n-1)\cdot\ldots\cdot(n+1)/\binom{2n}{n}$

Estimate the logarithm of $2n(2n-1)\cdot\ldots\cdot(n+1)$

We will use

$$\ln k \ge \int_{k-1}^{k} \ln x\, dx + \ln(2k) - \ln(2k-1) \qquad (1)$$
$$\ln k \le \int_{k-1}^{k} \ln x\, dx + \frac{1}{2}\left(\ln(k) - \ln(k-1)\right) \qquad (2)$$

from lemma 1

Summing the inequalities (2) for $k \in [n+1, 2n]$

$$\sum_{k=n+1}^{2n} \ln k \le \int_{n}^{2n} \ln x\ dx + \frac{1}{2}\big((\ln(2n) - \ln n)\big)$$
$$= \int_{n}^{2n} \ln x\, dx + \frac{1}{2}\ln 2 = n\ln n + 2n\ln 2 - n + \frac{1}{2}\ln 2$$

To estimate the sum of inequalities (1) assume

$$a_1 = \sum_{k=n+1}^{2n} (\ln(2k) - \ln(2k-1)), \qquad a_2 = \sum_{k=n+1}^{2n} (\ln(2k+1) - \ln(2k))$$
$$a_1 + a_2 = \ln(4n+1) - \ln(2n+1)$$

As $a_1 > a_2$ then

$$a_1 > \frac{1}{2}\ln(4n+1) - \frac{1}{2}\ln(2n+1) = \frac{1}{2}\ln\left(2\cdot\frac{2n+\frac{1}{2}}{2n+1}\right) = \frac{1}{2}\ln 2 + \frac{1}{2}\ln\left(1 - \frac{1}{4n+2}\right) \ge \frac{1}{2}\ln 2 - \frac{1}{4n}.$$

Thus

$$\sum_{k=n+1}^{2n} \ln k > n \ln n + 2n \ln 2 - n + \frac{1}{2}\ln 2 - \frac{1}{4n}$$

Hence

$$\sqrt{2}\left(\frac{n}{e}\right)^n 2^{2n} e^{-\frac{1}{4n}} \leq 2n\,(2n-1)\cdot \ldots \cdot (n+1) \leq \sqrt{2}\left(\frac{n}{e}\right)^n 2^{2n}$$

From lemma 3

$$\frac{\sqrt{\pi n}}{2^{2n}} \leq \frac{1}{\binom{2n}{n}} \leq \frac{\sqrt{\pi n}}{2^{2n}}\, e^{\frac{1}{4n}}$$

Term by term, we multiply the last inequalities

$$\sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n e^{-\frac{1}{4n}} \leq n! \leq \sqrt{2\pi n}\cdot\left(\frac{n}{e}\right)^n e^{\frac{1}{4n}}$$

QED

Theorem 5

If $\min(k, n-k) \to \infty$

$$\binom{n}{k} = \frac{2^n H\left(\frac{k}{n}\right)}{\frac{\sqrt{2\pi k\,(n-k)}}{n}}(1 + o(1))$$

Proof

Using Stirling's formula

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \sim \frac{\sqrt{2\pi n}}{\sqrt{2\pi k \cdot 2\pi(n-k)}} \cdot \frac{\left(\frac{n}{e}\right)^n}{\left(\frac{k}{e}\right)^k \left(\frac{n-k}{e}\right)^{n-k}}$$

$$= \frac{1}{\sqrt{\frac{2\pi k\,(n-k)}{n}}} \cdot \frac{n^n}{k^k (n-k)^{n-k}} = \frac{1}{\sqrt{\frac{2\pi k\,(n-k)}{n}}} \cdot \left(\frac{k}{n}\right)^{-k} \cdot \left(1 - \frac{k}{n}\right)^{-(n-k)}$$

$$= \frac{1}{\sqrt{\frac{2\pi k\,(n-k)}{n}}} \cdot 2^{nH\left(\frac{k}{n}\right)}$$

QED

Theorem 6

If $n \to \infty$ and $t = o(n^{\frac{2}{3}})$ then

$$\binom{n}{\frac{n}{2-t}} = \frac{2^n e^{-\frac{2t^2}{n}}}{\sqrt{\frac{\pi n}{2}}}\left(1 + o(1)\right) = \binom{n}{\left\lfloor\frac{n}{2}\right\rfloor} e^{-\frac{2t^2}{n}}\left(1 + o(1)\right).$$

Proof

With the use of the previous theorem we transform the exponent on the right part of its equality

$$H\left(\frac{\frac{n}{2} - t}{n}\right) = H\left(\frac{1}{2}\left(1 - \frac{2t}{n}\right)\right)$$
$$= -\frac{1}{2}\left(1 - \frac{2t}{n}\right)\log_2 \frac{1}{2}\left(1 - \frac{2t}{n}\right)$$
$$-\frac{1}{2}\left(1 + \frac{2t}{n}\right)\log_2 \frac{1}{2}\left(1 + \frac{2t}{n}\right)$$
$$= 1 - \frac{1}{2}\left(\left(1 - \frac{2t}{n}\right)\log_2(1 - \frac{2t}{n}) + \left(1 + \frac{2t}{n}\right)\log_2\left(1 + \frac{2t}{n}\right)\right)$$

Using (3)

$$-\left(1 - \frac{2t}{n}\right)\log_2\left(1 - \frac{2t}{n}\right) - \left(1 + \frac{2t}{n}\right)\log_2\left(1 + \frac{2t}{n}\right) = -\left(\frac{4t^2}{n^2} + \mathcal{O}\left(\frac{t^3}{n^3}\right)\right)\log_2 e.$$

so

$$nH\left(\frac{\frac{n}{2} - t}{n}\right) = n\left(1 - \frac{1}{2\left(\frac{4t^2}{n^2} + \mathcal{O}\left(\frac{t^3}{n^3}\right)\right)\log_2 e}\right) = n - \left(\frac{2t^2}{n} + \mathcal{O}\left(\frac{t^3}{n^2}\right)\right)\log_2 e.$$

Substituting this equality in equality from theorem 5, taking into account $t = o(n^{\frac{2}{3}})$ we will have

$$\binom{n}{\frac{n}{2} - t} = \frac{2^n e^{-\frac{2t^2}{n} + \mathcal{O}\left(\frac{t^3}{n^2}\right)}}{\sqrt{\frac{2\pi\left(\frac{n}{2} - t\right)\left(\frac{n}{2} + t\right)}{n}}}\left(1 + o(1)\right) = \frac{2^n e^{-\frac{2t^2}{n}}}{\sqrt{\frac{\pi n}{2}}}\left(1 + o(1)\right).$$

QED

Theorem 7

If $n \to \infty$ and $k = o(n^{\frac{2}{3}})$ then

$$\binom{n}{k} = \frac{n^k e^{-\frac{k^2}{2n}}}{k!}(1 + o(1))$$

Proof

Using Stirling's formula, equality $(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$ and $k = o(n^{\frac{2}{3}})$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k!)} \sim \frac{\sqrt{2\pi n}}{\sqrt{2\pi(n-k)}} \cdot \frac{\left(\frac{n}{e}\right)^n}{k!\left(\frac{n-k}{e}\right)^{n-k}}$$
$$= \frac{n^k}{k!} \cdot \frac{e^{-k}}{\sqrt{1-\frac{k}{n}}} \cdot \left(1 - \frac{k}{n}\right)^{k-n} \sim \frac{n^k e^{-k}}{k!} \cdot e^{(k-n)\ln\left(1-\frac{k}{n}\right)}$$
$$= \frac{n^k e^{-k}}{k!} \cdot e^{(k-n)\left(-\frac{k}{n}-\frac{k^2}{2n^2}-O\left(\frac{k^3}{n^3}\right)\right)} = \frac{n^k e^{-k}}{k!} \cdot e^{k-\frac{k^2}{2n}-O\left(\frac{k^3}{n^2}\right)}$$
$$= \frac{n^k e^{-\frac{k^2}{2n}}}{k!}\left(1 + o(1)\right).$$

QED

Theorem 8

If $1 \le \varphi(n) \le \frac{\sqrt{n}}{2}$ then

$$\sum_{k=0}^{\frac{n}{2}-\sqrt{n\varphi(n)}} \binom{n}{k} \le \frac{2^{n-3}}{\varphi(n)}$$

Proof

We estimate the sum of the binomial coefficients, the lower index of which differs from $\frac{n}{2}$ more than on t units.

$$\sum_{k:\left|\frac{n}{2}-k\right|>t} \binom{n}{k}$$
$$= \sum_{k:\left|\frac{n}{2}-k\right|>t} \frac{\left(\frac{n}{2}-k\right)^2}{\left(\frac{n}{2}-k\right)^2}\binom{n}{k}$$
$$\le \frac{1}{t^2} \sum_{k:\left|\frac{n}{2}-k\right|>t} \left(\frac{n}{2}-k\right)^2 \binom{n}{k} \le \frac{1}{t^2} \sum_{k=0}^{n} \left(\frac{n}{2}-k\right)^2 \binom{n}{k}$$

(4)

Find the sum of the right side of the inequality

$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{n}{2}-k\right)^2 = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{n^2}{4}-nk+k^2\right) = \frac{n^2}{4}\sum_{k=0}^{n}\binom{n}{k} - \sum_{k=0}^{n}\binom{n}{k}(n-k)k.$$

The first sum of the right side is equal to $n^2 2^{n-2}$, now find the second sum

$$\sum_{k-0}^{n}(n-k)k\binom{n}{k}$$

$$= \sum_{k=1}^{n-1}(n-k)k\binom{n}{k}$$

$$= \sum_{k=1}^{n-1}\frac{(n-k)kn!}{(n-k)!\,k!}$$

$$= \sum_{k=1}^{n-1}\frac{(n(n-1)(n-2)!)}{(n-k-1)!\,(k-1)!} = n(n-1)\sum_{k=0}^{n-2}\binom{n-2}{k} = n(n-1)2^{n-2}$$

From the two previous inequalities

$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{n}{2}-k\right)^2 = n^2 2^{n-2} - n(n-1)2^{n-2} = n2^{n-2}$$

Substitute this equality into the right side of (4) and assume $t = \sqrt{n\varphi(n)}$

$$\sum_{k:\left|\frac{n}{2}-k\right|>\sqrt{n\varphi(n)}}\binom{n}{k} \le \frac{n2^{n-2}}{n\varphi(n)} = \frac{2^{n-2}}{\varphi(n)}$$

QED

Theorem 9

If $1 \le t \le \frac{n}{2}$ then

$$\sum_{k=0}^{t}\binom{n}{k} \le 2^{nH\left(\frac{t}{n}\right)}$$

Proof

Assume $0 < x < 1$

$$\sum_{k=0}^{t}\binom{n}{k} \le \sum_{k=0}^{t}x^{k-t}\binom{n}{k} = \frac{1}{x^t}\sum_{k=0}^{t}x^k\binom{n}{k} \le \frac{1}{x^t}\sum_{k=0}^{n}x^k\binom{n}{k} = \frac{(1+x)^n}{x^t}$$

Differentiate the function $f(x) = \frac{(1+x)^n}{x^t}$ on x

$$\left(\frac{(1+x)^n}{x^t}\right)' = \frac{n(1+x)^{n-1}}{x^t} - \frac{t(1+x)^n}{x^{t+1}} = \frac{(1+x)^{n-1}}{x^{t+1}}\left(nx - t(1+x)\right)$$

Its derivative between 0 and 1 has the only root $x_0 = \frac{t}{n-t}$

As f(x) increases without limit as x tends to 0 on the right and $f(1) = 2^n$ then on the interval (0, 1) f(x) reaches its minimum value at x_0

$$\sum_{k=0}^{t}\binom{n}{k} \le \left(\frac{t}{n-t}\right)^{-t}\left(1+\frac{t}{n-t}\right)^{n} = \left(\frac{t}{n-t}\right)^{-t}\left(\frac{n}{n-t}\right)^{n} = \left(\frac{t}{n}\right)^{-t}\left(\frac{n}{n-t}\right)^{n-t}$$

$$= \left(\frac{t}{n}\right)^{-t}\left(1-\frac{t}{n}\right)^{-(n-t)} = \left(\left(\frac{t}{n}\right)^{-\frac{t}{n}}\left(1-\frac{t}{n}\right)^{-\left(1-\frac{t}{n}\right)}\right)^{n} = 2^{nH\left(\frac{t}{n}\right)}$$

QED

Theorem 10

If $\boldsymbol{0 \le t \le \frac{n}{2}}$ $\quad \sum_{k=0}^{\frac{n}{2}-t}\binom{n}{k} \le 2^n e^{-\frac{2t^2}{n}}$

Proof

From theorem 9 and

$$\sum_{k=0}^{\frac{n}{2}-t}\binom{n}{k} M \le 2^{nH\left(\frac{\frac{n}{2}-t}{n}\right)} \le 2^{nH\left(\frac{1}{2\left(1-\frac{2t}{n}\right)}\right)} \le 2^{(n(1-1/2((1-\frac{2t}{n})\log_2(1-\frac{2t}{n})+(1+\frac{2t}{n})\log_2(1+\frac{2t}{n})))}$$

$$H\left(\frac{\frac{n}{2}-t}{n}\right) = 1 - 1/2(\left(1-\frac{2t}{n}\right)\log_2\left(1-\frac{2t}{n}\right) + \left(1+\frac{2t}{n}\right)\log_2\left(1+\frac{2t}{n}\right)). \quad (5)$$

To estimate exponent on the right side of the inequality show that

$$f(x) = (1-x)\ln(1-x) + (1+x)\ln(1+x) - x^2 \ge 0$$

$$x \in (-1,1)$$

f(x) is an even function, so we can prove it only for $[0,1)$ and as $f(0) = 0$ it is enough to prove that on this interval derivative of a function f(x) is non-negative.

$$f'(x) = -\frac{1-x}{1-x} - \ln(1-x) + \frac{1+x}{1+x} + \ln(1+x) - 2x = \ln(1+x) - \ln(1-x) - 2x$$

$$f'(0) = 0$$

$$f''(x) = \frac{1}{1+x} + \frac{1}{1-x} - 2 = \frac{2}{1-x^2} - 2$$

These derivatives are non-negative on the interval $[0,1)$ hence

$$(1-x)\ln(1-x)+(1+x)\ln(1+x) \geq x^2 \text{for all } x \in (-1,1)$$

Hence

$$-\left(1-\frac{2t}{n}\right)\log_2\left(1-\frac{2t}{n}\right)-\left(1+\frac{2t}{n}\right)\log_2\left(1+\frac{2t}{n}\right) \leq -\frac{4t^2}{n^2\log_2 e}$$

Substitute this inequality into (5)

$$\sum_{0}^{\frac{n}{2}-t}\binom{n}{k} \leq 2^{n\left(1-\frac{1}{2}\cdot\frac{4t^2}{n^2}\cdot\log_2 e\right)} = 2^n\, e^{-\frac{2t^2}{n}}$$

QED

Литература:

А.В. Чашкин “Лекции по дискетной математике”

A First Course in Discrete Mathematics 2nd ed

Discrete Mathematics for Computing