# Euler’s formula for intersecting sets
# Newton’s binomial
# Asymptotic combinatorial identities

**Lectures 3-4**

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# Euler's formula for intersecting sets

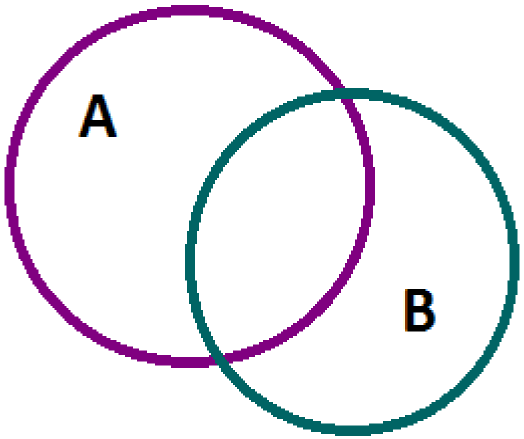
## Formula of inclusions and exclusions for two sets

- Formula of inclusions and exclusions allows us to find capacity of combining different sets if we know the power of their intersections.

- The following formula for the cardinality of the union of two sets is known as the Principle of inclusion and exclusion:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

- The formula is derived from the fact that if the cardinalities of A and B are added, then the elements in $A \cap B$ will be counted twice, so this is corrected by subtracting the cardinality of $A \cap B$ .




Venn diagram of $A \cap B$ ( speckled ) and universal set S

# Formula of inclusions and exclusions for three sets

$$|\boldsymbol{A}\cup\boldsymbol{B}\cup\boldsymbol{C}| = |\boldsymbol{A}| + |\boldsymbol{B}| + |\boldsymbol{C}| - |\boldsymbol{A}\cap\boldsymbol{B}| - |\boldsymbol{B}\cap\boldsymbol{C}| - |\boldsymbol{A}\cap\boldsymbol{C}| + |\boldsymbol{A}\cap\boldsymbol{B}\cap\boldsymbol{C}| \quad (2)$$

- Let $\boldsymbol{x}$ be an element of $A\cup B\cup C$. As such, it's counted once in the left-hand side of equality. $\boldsymbol{x}$ may belong to 1, 2, or 3 sets $A$, $B$, $C$. Assume it belongs only to $A$. Then on the right of (2) it is counted only once in $|A|$.
- Let $\boldsymbol{x}$ belong to $A$ and $B$. On the right, it is counted in $|A|$, $|B|$, and $|A\cap B|$ - twice added, once subtracted.
- Lastly, let $\boldsymbol{x}$ belong to all three sets $A$, $B$, $C$. It is then counted in every term of (2) - 4 times added and 3 times subtracted - again adding up to 1.

# Formula of inclusions and exclusions for n sets

- In the more general case where there are *n* different sets $A_i$, the formula for the Inclusion-Exclusion Principle becomes:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n-1} \left|\bigcap_{i=1}^{n} A_i\right|$$

- On the left there is a number of elements in the union of n sets.
- On the right, we first count elements in each of the sets separately and add the up. If the sets $A_i$ are disjoint, some elements will have be counted more than once.
- Those are the elements that belong to at least two of the sets $A_i$, or the intersections $A_i \cap A_j$. We wish to consider every such intersection, but each only once. Since $A_i \cap A_j = A_j \cap A_i$, to avoid duplications we arbitrarily decide to consider only pairs $(A_i, A_j)$ with $i < j$.

- When we subtract the sum of the number of elements in such pairwise intersections, some elements may have been subtracted more than once. Those are the elements that belong to at least three of the sets $A_i$. We add the sum of the elements of intersections of the sets taken three at a time. (The condition $i < j < k$ assures that every intersection is counted only once.)

- The process goes on with sums being alternately added or subtracted until we come to the last term - the intersection of all sets $A_i$. Whether it's added or subtracted depends on $n$: for **n =2** it was subtracted, for **n = 3** added.

# Examples

- **Example 1:** In a group of 120 students studying computing, 84 can program in C and 66 can program in Java. If 45 can program in both C and Java, how many of the students cannot program in either of these languages?

**Solution** :

Let Ɛ={computing students}, C= {C programming students},and J= {Java programming students}. The Problem is to find $|\overline{C \cup J}|$ . By the Principle of inclusion and exclusion:

$|C \cup J| = |C| + |J| - |C \cap J| = 84 + 66 - 45 = 105$

Therefore $|\overline{C \cup J}|$ = 120-105=15.

There are 15 students who cannot program in either of these languages.

- **Example 2:** How many integers from 1 to 100 are multiples of 2 or 3?

  **Solution**:

  Let A be the set of integers from 1 to 100 that are multiples of 2 ( $|A|=50$).

  Let $B$ be the set of integers from 1 to 100 that are multiples of 3 ($|B| = 33$).

  If $A \cap B$ the set of integers from 1 to 100 that are multiples of both 2 and 3, hence are multiples of 6

  ($|A \cap B| = 16$).

  $|A \cup B|=|A|+|B|-|A \cap B|=50 + 33 - 16 – 67$

- **Example 3**: What is the sum of all integers from 1 to 100 that are multiples of 2 or 3?

**Solution:**

While PIE is often used to count the elements of a set, if we remove the | | symbols, the statement is still true. For example, in two variables, $A \cup B = A \cup B - A \cap B$. The same proof using Venn diagrams works to show that each element is included once.

As such, the sum of elements in $A \cup B$ is equal to the sum of elements in A plus the sum of elements in B minus the sum of elements in $A \cap B$. Letting $\boldsymbol{A}$ be the set of multiples of 2, and $\boldsymbol{B}$ be the set of multiples of 3, then **A∩B** is the set of multiples of 6, hence the sum of A∪B is

$$\frac{(2+100)*50}{2} + \frac{(3+99)*33}{2} - \frac{(6+96)*16}{2} = 3417$$

- **Example 4:** We have 7 balls each of different colors (red, orange, yellow, green, blue, indigo, violet) and 3 boxes each of different shapes (tetrahedron, cube, dodecahedron). How many ways are there to place these 7 balls into the 3 boxes such that each box contains at least 1 ball?

  **Solution:**

  Let X be the total number of ways we can distribute the balls if there are no restrictions. Each ball can be placed into any one of the 3 boxes, so $|X| = 3^7$ .

  Let ***T*** be the set of ways such that the cube boof ways such that the tetrahedron box has no balls, ***C*** be the set of x has no balls and **D** be the set of ways such that the dodecahedron box has no balls. We would like to find **|X|-|T∪C∪D|**

  $$|X| - |T \cup C \cup D| = 3^7 - 3 * 2^7 + 3 * 1^7 - 0 = 1806$$

- We have $|T| = |C| = |D| = 2^7$ , since the balls can be placed into one of the two other boxes, and $|T \cap C| = |C \cap D| = |D \cap T| = 1^7$ ,

  since all the balls must be placed in the remaining box, and $|T \cap C \cap D| = 0$ .

  $|X| - |T \cup C \cup D| = 3^7 - 3 * 2^7 + 3 * 1^7 - 0 = 1806$

- **Example 5 [derangements]:** There are eight guests at a Secret Santa party. Each guest brings a gift and each receives another gift in return. No one is allowed to receive the gift they brought. How many ways are there to distribute the gifts?

  **Solution:**
- Let A denote the set of ways to distribute gifts such that everyone receives a gift, possibly their own. Let  denote the set of ways to distribute gifts such that person ***i*** receives his or her own gift. Then we would like to find

$$|A| - |A_1 \cup A_2 \cup \ldots \cup A_8|$$

- Since  is the set of ways for person i to receive his/her own gift, there are 7 choices of gifts for the next person, 6 choices of gifts for the following person, and so on. By the rule of product,

$$|A_i| = 7 * 6 * \ldots * 2 * 1 = 7!$$

- Since Ai ∩ Aj is the set of ways person i and person j both receive their own gifts, there are 6 choices of gifts for the next person, 5 choices of gifts for the following person, and so on. Again by the rule of product,

$$|A_i \cap A_j| = 6 * 5 * \ldots * 2 * 1 = 6!$$

- By continuing this argument, if k people receive their own gifts, then there are (8 - k)! possible ways. Applying PIE. we obtain

$$|A| - |A_1 \cup A_2 \cup \ldots \cup A_8| = 8! - \binom{8}{1} * 7! + \binom{8}{2} * 6! - \cdots + \binom{8}{8} * 1! = 14833.$$

- **Note**: A **derangement** of n objects is a permutation of the objects such that none of them stay in the same place. The number of ways this can be done is denoted $D_n$, and this calculation shows $\boldsymbol{D_8 = 14833}$.

# Euler's function

- The function φ(m) equal to the number of positive integers not exceeding m and coprime with it, called the **Euler's function**.
- If $m = p_1^{k_1} p_2^{k_2} \dots p_n^{k_n}$
- then $\varphi(\mathrm{m}) = \mathrm{m}\left(1 - \frac{1}{\mathrm{p}_1}\right)\left(1 - \frac{1}{\mathrm{p}_2}\right) \dots \left(1 - \frac{1}{\mathrm{p}_n}\right)$
- **Example**

$$\varphi(12) = \varphi(2^2 * 3) = 2^2 * 3 * \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right) = 4$$

# Newton's binomial
## Permutations

- Take $\boldsymbol{n}$ different elements: $\boldsymbol{a_1, a_2, a_3, \ldots, a_n}$.
- We will rearrange them in all possible ways, keeping their quantity and changing the order of their arrangement.
- Each of the thus obtained combination is called permutation. The total number of permutations of n elements is denoted by $P_n$

$$P_n = n \cdot (n-1) \cdot \ldots \cdot 2 \cdot 1 = n!$$

# Placement

- We will make groups of $m$ distinct elements taken from a set of $n$ elements by placing these m elements in a different order.
- The resulting combinations are called placements of $n$ elements on $k$.

$$A_n^k = n(n-1) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}.$$

# Combinations

- We will make groups of $\boldsymbol{m}$ distinct elements taken from a set of $\boldsymbol{n}$ elements without taking into account the order of these $\boldsymbol{m}$ elements.
- Then we get a combination of $\boldsymbol{n}$ elements on $\boldsymbol{k}$.

$$C_n^k = \binom{n}{k} = \frac{n!}{(n-k)!\,k!} = \frac{n(n-1) * \ldots * (n-k+1)}{k(k-1) * \ldots * 2 * 1}$$

$$C_n^k = C_n^{n-k} \quad , \quad 0 \leq k \leq n \quad , \qquad C_n^k = \frac{A_n^k}{P_k}$$

$$C_{n+1}^k = C_n^k + C_n^{k-1} \quad , \quad 0 < k \leq n$$

$$C_n^n = C_n^0 = 1$$

# Combinations with repetition

- The number of combinations with repetition of ***n*** elements on ***k***

$$H_n^k = \binom{n+k-1}{k} = \frac{(n+k-1)!}{(n-1)!\,k!}$$

# Newton’s binomial

- It is a formula representing the expression $(a + b)^n$ with a positive integer n as a polynomial.

$$(a+b)^n = C_n^0 a^n + C_n^1 a^{n-1} b + C_n^2 a^{n-2} b^2 + \cdots + C_n^{n-1} a b^{n-1} + C_n^n$$

$$(a-b)^n = C_n^0 a^n - C_n^1 a^{n-1} b + \cdots + (-1)^k C_n^k a^{n-k} b^k + \cdots + (-1)^n C_n^n b^n$$

- Note that the sum of the exponents of ***a*** and ***b*** is constant and equal to ***n***.
  Numbers $C_n^1, C_n^2, C_n^3, \ldots, C_n^{n-1}$ are called **binomial coefficients**.

- The other way of writing this is $C_n^k = \binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

# The binomial theorem

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y^n + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}y^n = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^r$$

**Proof**

$(x+y)^n = (x+y)(x+y)\ldots(x+y)$ , (*n* brackets).

- So the coefficient of $x^{n-r}y^r$ in the expansion is the number of ways getting $x^{n-r}y^r$ when the ***n*** brackets are multiplied out.
- Each term in the expansion is the product of one term from each bracket; so $x^{n-r}y^r$ is obtained as many times as we can choose *y* from *r* of the brackets (and *x* from the remaining *n*-*r* brackets).
- But this is just the number of ways of choosing *r* of the *n* brackets, which is $\binom{n}{r}$ .

# Examples

- **Example 1**

$$(a+b)^3 = a^3 + 3a^2b + \frac{3!}{1!\cdot 2!}ab^2 + b^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

- **Example 2**

$$(1+x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$$

## Pascal's triangle

| | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | *1* | | *1* | | | | | | | |
| | | | | | | *1* | | *2* | | *1* | | | | | | |
| | | | | | *1* | | *3* | | *3* | | *1* | | | | | |
| | | | | *1* | | *4* | | *6* | | *4* | | *1* | | | | |
| | | | *1* | | *5* | | *10* | | *10* | | *5* | | *1* | | | |
| | | *1* | | *6* | | *15* | | *20* | | *15* | | *6* | | *1* | | |
| | *1* | | *7* | | *21* | | *35* | | *35* | | *21* | | *7* | | *1* | |
| *1* | | *8* | | *28* | | *56* | | *70* | | *56* | | *28* | | *8* | | *1* |

…………………………………………………………

- In the top line two units are written. All subsequent lines begin and end by one. Intermediate numbers in these lines are obtained by summing adjacent numbers in the previous line.
- The first row in the table contains the binomial coefficients for n = 1; second for n = 2; third for n = 3, etc.

- coefficients:

$$(1+y)^0 = 1$$
$$(1+y)^1 = 1 + y$$
$$(1+y)^2 = 1 + 2y + y^2$$
$$(1+y)^3 = 1 + 3y + 3y^2 + y^3$$
$$(1+y)^4 = 1 + 4y + 6y^2 + 4y^3 + y^4$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

Note the triangular array of coefficients, known as **Pascal's triangle**. Each row consists of the choice numbers: e.g. the bottom row shown consists of:

$$\binom{4}{0} = 1, \binom{4}{1} = 4, \binom{4}{2} = 6, \binom{4}{3} = 4, \binom{4}{4} = 1$$

# Properties of binomial coefficients

- The sum of the coefficients of the expansion of $(a+b)^n$ is $2^n$.

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$$

- Coefficients of terms equidistant from the ends of the expansion are equal.

$$\binom{n}{0}-\binom{n}{1}+\binom{n}{2}-\cdots+(-1)^n\binom{n}{n}=0 \quad (n>0)$$

- The sum of the coefficients of the even terms in the expansion equals to the sum of the coefficients of odd terms of the expansion; each of them is equal to
- For all $m \geq 0 \; and \; n \geq 1$

$$\binom{m}{m}+\binom{m+1}{m}+\cdots+\binom{m+n}{m}=\binom{m+n+1}{m+1}$$