Asymptotic combinatorial identities

Estimates of n!

- This estimation is a component part in many combinatorial identities.
- There are three bilateral inequalities for this function, wherein the accuracy of inequalities of every next pair is higher than accuracy of the previous inequalities.

Theorem 1

If $n \geq 6$, then $\left(\frac{n}{3}\right)^n < n! < \left(\frac{n}{2}\right)^n$

Theorem 2

If $n \geq 1$, then $e \cdot \left(\frac{n}{e}\right)^n \leq n! \leq ne \cdot \left(\frac{n}{e}\right)^n$

Lemma 1

If $k \geq 2$, then

$$\int_{k-1}^{k} \ln x\, dx + (\ln 2k - \ln(2k-1)) \leq \ln k \leq \int_{k-1}^{k} \ln x\, dx + \frac{1}{2}(\ln k - \ln(k-1))$$

Theorem 3

$$0.8 \cdot e\sqrt{n}\left(\frac{n}{e}\right)^{n} \leq n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^{n}$$

Stirling's formula

Lemma 2

$$\int_0^{\frac{\pi}{2}} \sin^{2n} dx = \frac{(2n-1)(2n-3)\cdot \ldots \cdot 3\cdot 1}{2n(2n-2. \ldots 4\cdot 2}\cdot\frac{\pi}{2} = \frac{(2n-1)!!}{2n!!}\cdot\frac{\pi}{2}$$

$$\int_o^{\frac{\pi}{2}} \sin^{2n+1} dx = \frac{2n(2n-2)\cdot \ldots 4\cdot 2}{(2n+1)(2n-1)\cdot \ldots \cdot 3\cdot 1} = \frac{2n!!}{(2n+1)!!}$$

Lemma 3

$$\frac{2^{2n}}{\sqrt{\pi n}} e^{-\frac{1}{4n}} \leq \binom{2n}{n} \leq \frac{2^{2n}}{\sqrt{\pi n}}$$

Theorem 4

$$\sqrt{2\pi n} \cdot \binom{n}{e}^n e^{-1/4n} \leq n! \leq \sqrt{2\pi n} \cdot \binom{n}{e}^n e^{1/4n}$$

Stirling's formula

The ratio of upper and lower inequalities from theorem 4 does not exceed $e^{\frac{1}{2n}}$ and if $n \to \infty$ converges to 1. Thus from the theorem 4 Stirling's formula follows

$$n! = \sqrt{2\pi n} \cdot \binom{n}{e}^n \left(1 + o(1)\right)$$

Binomial coefficients

If $k \geq 6$, $\left(\frac{2(n-k)}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{3n}{k}\right)^k$

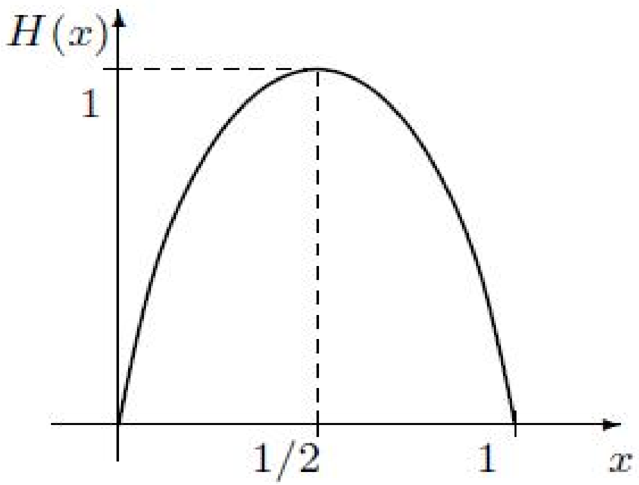
- Using Stirling's formula we establish three asymptotically precise formulas for $\binom{n}{k}$.
- They are valid under constraints on k.

Entropy function

For each $0 < x < 1$

$$H(x) = -x\log_2 x - (1-x)\log_2(1-x)$$

$$H(0) = H(1) = 0$$



Theorem 5

If $\min(k, n-k) \to \infty$,

then $\binom{n}{k} = \dfrac{2^n H\left(\frac{k}{n}\right)}{\sqrt{\frac{2\pi k(n-k)}{n}}}(1+o(1))$

Theorem 6

If $n \to \infty$ and $t = o(n^{\frac{2}{3}})$,

then $\binom{n}{\frac{n}{2-t}} = \dfrac{2^n e^{-\frac{2t^2}{n}}}{\sqrt{\frac{\pi n}{2}}}(1+o(1))$

Theorem 7

If $n \to \infty\ and\ k = o(n^{\frac{2}{3}})$,

then $\binom{n}{k} = \dfrac{n^k e^{-\frac{k^2}{2n}}}{k!}(1+o(1))$

The sum of the binomial coefficients

Theorem 8

If $1 \le \varphi(n) \le \frac{\sqrt{n}}{2}$, then $\sum_{k=0}^{\frac{n}{2}-\sqrt{n\varphi(n)}} \binom{n}{k} \le \frac{2^{n-3}}{\varphi(n)}$

Theorem 9

If $1 \le t \le \frac{n}{2}$, then $\sum_{k=0}^{t} \binom{n}{k} \le 2^{nH\left(\frac{t}{n}\right)}$

Consequence

If $0 \le k \le n$, then $\binom{n}{k} \le 2^{nH\left(\frac{k}{n}\right)}$

Theorem 10

If $0 \le t \le \frac{n}{2}$, then $\sum_{k=0}^{\frac{n}{2}-t} \binom{n}{k} \le 2^{n} e^{-\frac{2t^2}{n}}$

Literature

- А.В. Чашкин “Лекции по дискетной математике”
- A First Course in Discrete Mathematics 2nd edition
- Discrete Mathematics for Computing