Linear recurrent sequences and generating functions of sets

Lections 5-6

Contents

Linear recurrent sequences

- Generating function of sequence $\{F\}_{n=0}^{\infty}, F_n \in \mathbb{R}$ is

$$F(x) = \sum_{n=0}^{\infty} x^n F_n \qquad (1)$$

Theorem on solving linear recurrence relations with constant coefficients

If sequence $F_0, F_1, F_2, \dots, F_n, \dots$ satisfy the linear recurrence relation $F_n = a_1 F_{n-1} + a_2 F_{n-2} + \cdots + a_k F_{n-k}$ (2)
with constant coefficients a_i, $n \geq k$, then if $n \geq 0$

$$F_n = \sum_{i=1}^{m} \alpha_i^n P_i(n) \qquad (3)$$

- α_i - is a root of the polynomial
$f(x) = x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_k$ of multiplicity p_i
- P_i - is a polynomial of degree $p_i - 1$,

whose coefficients are determined so that the equation (3) is valid for the first k members of the sequence.

Proof (1/8)

- Sequence $\{F_n\}$ we associate with generating function

$$F(x) = \sum_{n=0}^{\infty} x^n F_n$$

- The left and the right parts of recurrence relation (2) multiple by x^n and sum among all integers n from k to infinity.

$$\sum_{n=k}^{\infty} x^n F_n = \sum_{n=k}^{\infty} x^n (a_1 F_{n-1} + a_2 F_{n-2} + \cdots + a_k F_{n-k})$$

$$\sum_{n=k}^{\infty} x^n F_n = a_1 x \sum_{n=k}^{\infty} x^{n-1} F_{n-1} + \qquad (4)$$

$$+ \; a_2 x^2 \sum_{n=k}^{\infty} x^{n-2} F_{n-2} + \cdots + a_k x^k \sum_{n=k}^{\infty} x^{n-k} F_{n-k}$$

Proof (2/8)

- for the left part (4):

$$\sum_{n=k}^{\infty} x^n F_n = F(x) - (F_0 + xF_1 + \cdots + x^{k-1}F_{k-1})$$

- for the *i* sum of the right part (4):

$$a_i x^i \sum_{n=k}^{\infty} x^{n-i} F_{n-i} = a_i x^i \sum_{m=k-i}^{\infty} x^m F_m =$$

$$= a_i x^i \left(F(x) - \left(F_0 + xF_1 + \cdots + x^{k-i-1}F_{k-i-1}\right)\right)$$

Proof (3/8)

- equation (4) can be written as

$$F(x) - \left(F_0 + xF_1 + \cdots + x^{k-1}F_{k-1}\right) =$$
$$= a_1x^1\left(F(x) - (F_0 + xF_1 + \cdots + x^{k-2}F_{k-2})\right) +$$
$$+a_2x^2\left(F(x) - (F_0 + xF_1 + \cdots + x^{k-3}F_{k-3})\right) + \cdots + a_kx^kF(x)$$

- transfer all terms with *F(x)* to the left side:

$$F(x)\left(1 - a_1x - a_2x^2 - \cdots - a_kx^k\right) =$$
$$= F_0\left(1 - a_1x - a_2x^2 - \cdots - a_{k-1}x^{k-1}\right) +$$
$$+F_1\left(1 - a_1x - a_2x^2 - \cdots - a_{k-2}x^{k-2}\right) + \cdots \; +x^{k-1}F_{k-1}$$

- we denote the polynomial on the right side as $H_{k-1}(x)$

Proof (4/8)

$$F(x) = \frac{H_{k-1}(x)}{1 - a_1 x - a_2 x^2 - \cdots - a_k x^k} \tag{5}$$

- Suppose that polynomial from the denominator of the right part has m different roots $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \dots, \frac{1}{\alpha_m}$. Multiplicities are $p_1, p_2, \dots, p_m$ respectively.
- Decompose denominator into multipliers:

$$F(x) = \frac{H_{k-1}(x)}{(1 - \alpha_1 x)^{p_1} (1 - \alpha_2 x)^{p_2} \dots (1 - \alpha_k x)^{p_m}} \tag{6}$$

$$p_1 + p_2 + \cdots + p_m = k$$

Proof (5/8)

- Represent the right side of equation as the sum of fractions

$$F(x) = \frac{\beta_{p_1}^1}{(1-\alpha_1 x)^{p_1}} + \cdots + \frac{\beta_{p_1-j}^1}{(1-\alpha_1 x)^{p_j-1}} + \cdots + \frac{\beta_1^1}{(1-\alpha_1 x)} +$$
$$+ \frac{\beta_{p_2}^2}{(1-\alpha_2 x)^{p_2}} + \cdots + \frac{\beta_{p_2-j}^2}{(1-\alpha_2 x)^{p_2-j}} + \cdots + \frac{\beta_1^2}{(1-\alpha_2 x)} + \cdots$$
$$+ \frac{\beta_{p_m}^m}{(1-\alpha_m x)^{p_m}} + \cdots + \frac{\beta_{p_m-j}^m}{(1-\alpha_m x)^{p_m-j}} + \cdots + \frac{\beta_1^m}{(1-\alpha_m x)}$$

- β_i^j - constants (possible to be complex)

$$\frac{1}{(1-\alpha x)^k} = \sum_{n=0}^{\infty} (\alpha x)^n \binom{n+k-1}{n}$$

Proof (6/8)

- $$\frac{\beta_p}{(1-\alpha x)^p} + \cdots + \frac{\beta_{p-j}}{(1-\alpha x)^{p-j}} + \cdots + \frac{\beta_1}{1-\alpha x}$$
- The coefficient of x^n is $\alpha^n \sum_{j=0}^{p} \beta_{p-j} \binom{n+p-1-j}{n}$
- So, $F_n = \sum_{i=1}^{m} \alpha_i^n \sum_{j=0}^{p} \beta_{p_i-j}^{i} \binom{n+p_i-1-j}{n}$ (7)
- Note that for constants $\beta_{p-1}, \dots, \beta_0$ there are constants $\gamma_{p-1}, \dots, \gamma_0$ so that :

$$\sum_{j=0}^{p} \beta_{p-j} \binom{n+p-1-j}{n} = \sum_{j=0}^{p} \gamma_{p-j}\, n^{n+p-1-j}$$

Proof (7/8)

- (7) can be rewritten as

$$F_n = \sum_{i=1}^{m} \alpha_i^n \sum_{j=0}^{p} \gamma_{p_i-j}^i \, n^{n+p_i-1-j} \qquad (8)$$

constants γ_i^j can be found by substitution n by 0,1, ... ,k-1 in (8) and further solving system of linear equations.

- Recurrence equation (2) associates with its characteristic polynomial

$$f(x) = x^k - a_1 x^{k-1} - a_2 x^{k-2} - \cdots - a_k \qquad (9)$$

- Substitute x by $1/y$ in this polynomial and then multiply it by y^k : $f^*(y) = 1 - a_1 y - a_2 y^2 - \cdots - a_k y^k$
It is equal to the polynomial in the denominator of the right part of (5).

Proof (8/8)

$$f^*(y) = y^k f\left(\frac{1}{y}\right)$$

- If α is the root of $f^*(y) = 0$, then $\frac{1}{\alpha}$ is the root of $f(y) = 0$
- Hence from (8) n term of the recurrence sequence (2) can be written as

$$F_n = \sum_{i=1}^{m} (\alpha_i)^n P_i(n)$$

- α_i is the root of the characteristic polynomial (9) of multiplicities p_i, P_i is a polynomial of degree $p_i - 1$, whose coefficients are determined from k first terms of that sequence.

QED

Homogeneous and inhomogeneous recurrence relation

- In the recurrence relation (2) transfer all terms (which are not equal to zero) to the left.

$$F_n - a_1 F_{n-1} - a_2 F_{n-2} - \cdots - a_k F_{n-k} = 0$$

Such relation is called homogeneous and is a special case of recurrence relation.

$$F_n - a_1 F_{n-1} - a_2 F_{n-2} - \cdots - a_k F_{n-k} = f(n) \qquad (10)$$

- If $f(n) \neq 0$ then such relation is called inhomogeneous.

To solve inhomogeneous recurrent relation we can use the method of generating functions.

The example of solving the inhomogeneous recurrent relation

- Let in sequence $\{F_n\}_0^\infty$ first two terms are equal to 1 and other terms satisfy to inhomogeneous recurrent relation

$$F_n = 3F_{n-1} - 2F_{n-2} + n$$

Assume $$F(x) = \sum_{n=0}^{\infty} F_n x^n$$

then $$\sum_{n=2}^{\infty} F_n x^n = 3\sum_{n=2}^{\infty} F_{n-1} x^n - 2\sum_{n=2}^{\infty} F_{n-2} x^n + \sum_{n=2}^{\infty} n \cdot x^n$$

or $$F(x) - F_0 - xF_1 = 3xF(x) - 3xF_0 - 2x^2F(x) + \frac{x}{(1-x)^2}$$

Because $F_0 = 1$ and $F_1 = 1$, so we will get

The example of solving the inhomogeneous recurrent relation

- $$F(x) = \frac{1-3x}{1-3x+2x^2} + \frac{x}{(1-x)^2(1-3x+2x^2)}$$

As $1-3x+2x^2 = (1-x)(1-2x)$, then

$$F(x) = \frac{H_3(x)}{(1-x)^3(1-2x)}$$

$H_3(x)$ - polynomial of the third degree

Hence, $F_n = an^2 + bn + c + d2^n$ (12)

From (11) when *n=2* and *n=3* we can find $F_2 = 3, F_3 = 10$

Substitute in (12) instead of *n* numbers *0, 1, 2* and *3,* we will get a system of linear equations.

The example of solving the inhomogeneous recurrent relation

- $$\begin{cases} 1 = c + d \\ 1 = a + b + c + 2d \\ 3 = 4a + 2b + c + 4d \\ 10 = 9a + 3b + c + 8d \end{cases}$$
- From this system $a = -\frac{1}{2}, b = -\frac{5}{2}, c = -2, d = 3$
- Hence, $$F_n = -\frac{1}{2}n^2 - \frac{5}{2}n - 2 + 3 \cdot 2^n$$
- The possibility of solution (10) depends on the function *f(n)*.
- If *f(n)* is quasi polynomial, i.e. $f(n) = \alpha^n P(n)$,α - constant, $P(n)$ - polynomial, then (10) can be solved almost the same way as the inhomogeneous one.