Generating functions of sets

- Consider set A and function $w: A \to \{0, 1, \dots\}$, which is positive on this set of elements.
- Function w is a **weight function** on the set A.
- **Generating function** of the set A by weight function w is

$$F_A^w(x) = \sum_{\alpha \epsilon A} x^{w(\alpha)}$$

- Assume, for example, that set A consists of all binary sequences of finite length and value $w(\alpha)$ of weight function on sequence α is equal to the length of this sequence. Then

$$F_A^w(x) = \sum_{n=1}^{\infty} 2^n x^n = \frac{2x}{1-2x}$$

Theorem 3

- If w - weight function on the set *N*, *A* and *B* – disjoint subsets of set *N*, then

$$F_{A\cup B}^{w}(x) = F_{A}^{w}(x) + F_{B}^{w}(x)$$

- **Proof**

  Next equation follows from the definition of generating functions of sets

$$F_{A\cup B}^{w}(x) = \sum_{\sigma\epsilon A\cup B} x^{w(\sigma)} = \sum_{\sigma\epsilon A} x^{w(\sigma)} + \sum_{\sigma\epsilon B} x^{w(\sigma)} =$$

$$= F_{A}^{w}(x) + F_{B}^{w}(x)$$

QED

Theorem 4

- If w_a, w_b, w - weight functions on sets A and B and in direct product $A \times B$. If for all (α, β) from $A \times B$ and

$$w\big((\alpha,\beta)\big) = w_a(\alpha) + w_b(\beta)$$

then $$F^w_{A\times B}(x) = F^w_A(x) \cdot F^w_B(x)$$

- **Proof**

Next equation follows from the definition of generating functions of sets

$$F^w_{A\times B}(x) = \sum_{(\alpha,\beta)\epsilon A\times B} x^{w((\alpha,\beta))} = \sum_{\alpha\epsilon A} x^{w_a(\alpha)} \cdot \sum_{\beta\epsilon B} x^{w_b(\beta)} =$$

$$= F^w_A(x) \cdot F^w_B(x)$$

QED

Note

- Further we will omit the sign of the direct product.
- Iteration of the set A is a set $A^* = \bigcup_{n=0}^{\infty} A^n$ where A^0 is an empty set.
- Later we will denote empty set as λ and assume its weight equals to zero.

Theorem of the iterations generating function

- **Theorem 5**

  If w – weight function on the set A, then

$$F_{A^*}^{w}(x) = \frac{1}{1 - F_A^w(x)}$$

  if function $\frac{1}{1 - F_A^w(x)}$ decomposes into expansion in zero.

- **Proof**

  As $A^* = \bigcup_{n=0}^{\infty} A^n$ then from two previous theorems follows

$$F_{A^*}^{w}(x) = \sum_{n=0}^{\infty} F_{A^n}^{w}(x) = \sum_{n=0}^{\infty} (F_A^w(x))^n = \frac{1}{1 - F_A^w(x)}$$

- **QED**

Operation of sets subtraction

- If on sets A and B the same weight function w is defined and $A \subseteq B$, then $F^{w}_{B \setminus A}(x) = F^{w}_{B}(x) - F^{w}_{A}(x)$.

Examples of the theorems usage

- **Example 1**

  Find generating functions for set $A = \bigcup_{n=1}^{\infty} \{0, 1\}^n$

  of all sequences (0, 1) of finite length with weight function equal to the length of sequence.
- Set A can be $A = (0 \cup 1)^* - \lambda$ or $A = (0 \cup 1)^*(0 \cup 1)$.

$$F^{w}_{(0\cup 1)^* - \lambda}(x) = \frac{1}{1-2x} - 1 = \frac{2x}{1-2x}$$

$$F^{w}_{(0\cup 1)^*(0\cup 1)}(x) = \frac{1}{1-2x} \cdot 2x = \frac{2x}{1-2x}$$

- different formulas, which define the same set, leads to the same generating function.

Example 2 (1/3)

- **Example 2**

  Find generating function for the set C which consists of all (0, 1) sequences of finite length without two adjacent zeros with weight function of sequence equal to its length.
- Every sequence in C is divided into three parts.
- The first part consists only of ones and if sequence begins with zero assume that the length of first part equals to zero.
- The third part consists of zero if sequence ends with zero and if sequence ends with one assume that the length of third part equals to zero.
- The second part can be divided into blocks each of which begins with zero and after there is a nonzero number of ones.

Example 2 (2/3)

- For example, in sequence 1110110101 the first part is 111, the second part is 0110101 and the length of the third part is zero.
- Sequence 10 consists only of two parts – first part 1 and third part 0, the length of the second part is zero.
- Set C can be represented by formula as $1^*(011^*)^*(0 \cup \lambda)$. Each element of this set is generated by this formula uniquely.

$$F_C^w(x) = F_{1^*}^w(x) F_{(011^*)^*}^w(x) F_{0 \cup \lambda}^w(x) =$$

$$= \frac{1}{1-x}\left(1 - \frac{x^2}{1-x}\right)^{-1}(1+x) = \frac{1+x}{1-x-x^2}$$

Example 2 (3/3)

- Now obtain an explicit formula for the number of sequences from *C* with the length n.
- This can be done by expanding the function in the series as this was done in the proof of Theorem 1.

Composition of sets, s-fragments

- Let B is a set.

  Assume that each element of this set can be divided into fragments.

  Among fragments select a subset. Elements of that subset we will call **s-fragments**.
- For example, if B consists of all (0, 1) sequences of finite length, then as fragments can be considered zeros and ones and as s-fragments - zeros.

Composition of sets

- Consider three sets *A, B, C*.

  The set *C* is a **composition of sets** *A* and *B* ($C = B \circ A$) if such set of s-fragments of elements from *B* exists that each element of set *C* can be uniquely derived from some element of the set *B* by substitution in this element all of its s-fragments by elements of the set *A*.
- **For example**, let *A*=0(0*), *B* consists of (0, 1) sequences of finite length, which doesn't have two adjacent zeros, the set of s-fragments elements from *B* consists of unique fragment "0".
- As any (0, 1) sequence can be uniquely derived from appropriate sequence of the set *B*, in which each zero is substituted by some sequence of zeros then set *C* of all finite sequences of (0, 1) is a composition of sets *A* and *B*

$$C = B \circ A$$

Weight function

- Consider $C = B \circ A$

- w_s is a **weight function** on the set B, that for element β from B, $w_s(\beta)$ is equal to the number of s-fragments in element β.

- For the previous example $w_s(\beta)$ is equal to the number of zeros in the set β.

Theorem 6

- **Theorem 6**

Assume w and w'– weight functions on the sets A and $B \circ A$. If for each element $\sigma \in B \circ A$ derived from element $\beta \in B$ with weight $w_s(\beta) = m$ and elements $\alpha_1, \dots, \alpha_m \in A$ equation exists

$$w'(\sigma) = w(\alpha_1) + \cdots + w(\alpha_m) \text{ , then}$$

$$F_{B \circ A}^{w'}(x) = F_B^{w_\beta}(F_A^w(x)) \text{ ,}$$

if the given substitution is acceptable.

Proof (1/2)

- Assume $\sigma \in B \circ A, \alpha^m = (\alpha_1, \ldots, \alpha_m) \in A^m$ and $\beta \in B, w_s(\beta) = m$

  We use equation $\sigma = (\beta, \alpha^m)$ to show that element σ is derived by substitution of elements $\alpha_1, \ldots, \alpha_m$ into element β . Then

Proof (2/2)

$$F_{B\circ A}^{w'}(x) = \sum_{\sigma \in B\circ A} x^{w'(\sigma)} = \sum_{m\geq 0} \sum_{\substack{\sigma=(\beta,\alpha^m)\\ w_s(\beta)=m}} x^{w'(\sigma)} =$$

$$= \sum_{m\geq 0} \sum_{\substack{\sigma=(\beta,\alpha^m)\\ w_s(\beta)=m}} x^{w(\alpha_1)+\cdots+w(\alpha_m)} = \sum_{m\geq 0} \sum_{w_s(\beta)=m} \left(\sum_{\alpha\epsilon A} x^{w(\alpha)}\right)^m =$$

$$= \sum_{m\geq 0} \sum_{w_s(\beta)=m} \left(\sum_{\alpha\epsilon A} x^{w(\alpha)}\right)^{w_s(\beta)} = \sum_{\beta\epsilon B} \left(\sum_{\alpha\epsilon A} x^{w(\alpha)}\right)^{w_s(\beta)} =$$

$$= F_B^{w_s}(F_A^w(x))$$

QED

Problem of the number of formulas

- The sequence of left and right brackets is called formula only if it satisfies
  1. Sequence ***( )*** is a formula
  2. If $\boldsymbol{f}$ is a formula then $\boldsymbol{(f)}$ is a formula
  3. If $\boldsymbol{f_1}$ and $\boldsymbol{f_2}$ are formulas then $\boldsymbol{f_1 f_2}$ is a formula

- Each formula contains the same number of left and right brackets.
- The number of left (or right) brackets in this formula is called the **length** of formula.
- Find the number of formulas with the length $k \geq 1$.

Problem of the number of formulas

- On the set *A*, which consists of all formulas including empty formula which doesn't contain any bracket, consider weight function w equal to the length of the formula.

- Let, $F_A^w(x)$ - generating function of the set *A*.
- Consider set *B* which consists of all formulas, each of which can be represented as *(f)*, where *f* – formula.

- For example, formula (() ()) belongs to *B* and formula () (() ()) doesn't belongs to *B*.
- Function $xF_A^w(x)$ is a generating function of the set *B*.

Problem of the number of formulas

- Consider the set C, which consists of empty formula and all of the formulas ()…() in which after each left bracket there is a right bracket.
- C is an iteration of the formula (), so $F_C^w(x) = \dfrac{1}{1-x}$
- Note that each formula from A can be uniquely derived from a formula of the set C, if each pair of brackets in this formula is substituted by appropriate formula from B.
- For example, formula $(\,)\big((\,)(\,)\big) \in A$ is derived from formula $(\,)(\,) \in C$, if in this formula first pair of brackets is substituted by formula $(\,) \in B$ and the second pair by $\big((\,)(\,)\big) \in B$.

Problem of the number of formulas

- Hence $A = C \circ B$, where in formulas from C s-fragments are the pair of brackets ( ).
- In this case, generating functions $F_C^w(x), F_C^{w_s}(x)$ are equal.

  Equation $w(f) = w(f_1) + \cdots + w(f_m)$ exists for every formula $f \in A$, which is derived from some formula of the set C by substitution in this formula the pair of brackets ( ) to formulas $f_1, \ldots, f_m \in B$.
- With the use of Theorem 6

$$F_A^w(x) = F_C^{w_s}\left(F_B^w(x)\right) = \frac{1}{1 - xF_A^w(x)}$$

Problem of the number of formulas

- $$x\left(F_A^w(x)\right)^2 - F_A^w(x) + 1 = 0$$

Function $F_A^w(x)$ is expansible into a series in the neighborhood of zero, so we take only one root of the equation, then

$$F_A^w(x) = \frac{1 - \sqrt{1 - 4x}}{2x} \qquad (20)$$

With the use of binomial formula we obtain

Problem of the number of formulas

- $\sqrt{1-4x} = 1 + \sum_{k=1}^{\infty} \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\ldots\left(\frac{1}{2}-k+1\right)}{k!}(-4)^k x^k =$

$$= 1 + \sum_{k=1}^{\infty} \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(-\frac{2k-3}{2}\right)}{k!}(-4)^k x^k =$$

$$= 1 - \sum_{k=1}^{\infty} \frac{2^k(2k-3)!!}{k!} x^k = 1 - \sum_{k=1}^{\infty} \frac{2^k(2k-3)!!\,(k-1)!}{k!\,(k-1)!} x^k =$$

$$= 1 - \sum_{k=1}^{\infty} \frac{2(2k-3)!!\,(2k-2)!!}{k!\,(k-1)!} x^k = 1 - \sum_{k=1}^{\infty} \frac{2(2k-2)!}{k(k-1)!\,(k-1)!} x^k =$$

$$= 1 - \sum_{k=1}^{\infty} \frac{2}{k}\binom{2k-2}{k-1} x^k$$

Problem of the number of formulas

- Substituting it into formula (20) we obtain

$$F_A^w(x) = \sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k} x^k \quad .$$

- Thus, there are $\frac{1}{k+1}\binom{2k}{k}$ formulas with the length k.

Generating functions of several variables

- Theorems 3 - 6 are also correct for generating functions with larger number of variables if its conditions are satisfied with respect to some of the variables.
- As example of generating functions of several variables consider new solution of the task about the number of (0, 1)-sequences, in which there are no two adjacent zeros.
- Assume $A = 1(1^*)$, ***B*** is the set of (0, 1)-sequences, in which there is no two adjacent identical symbols
- ***C*** is the set of (0, 1)-sequences in which there are no two adjacent zeros.
- Assume the set of s-fragments of *B* contain the only fragment "1". So, $C = B \circ A$

Generating functions of several variables

- On the set A weight function w_a is the length of sequence.
- On sets B and C weight function w such that $w(\beta) = (i, j)$, where i equals to the number of zeros and j - to the number of ones in β.
- Functions w and w_a such as the second component w_2 of function w and function w_a satisfied Theorem 6,
- for any σ derived by substitution of s-fragments in sequence from B on sequences $\alpha_1, \dots, \alpha_m$ from A, equation exists

$$w_2(\sigma) = w_{a(\alpha_1)} + \dots + w\_a(\alpha_m)$$

Generating functions of several variables

$$F^{w}_{B\circ A}(x,y) = F^{w}_{B}(x, F^{w_a}_{A}(y))$$

- As $$B = (\lambda \cup 1)(01)^{*}(\lambda \cup 0)$$
- Then $$F^{w}_{B}(x,y) = \frac{(1+x)(1+y)}{1-xy}$$
- Substituting y in this formula by generating function of the set A, we find generating function of the set $B \circ A$

$$F^{w}_{B\circ A}(x,y) = \frac{(1+x)\left(1+\frac{y}{1-y}\right)}{1-\frac{xy}{1-y}} = \frac{1+x}{1-y-xy}$$

Generating functions of several variables

- As we are interested only in the number of sequences of certain length (the number of ones and zeros in this sequence isn't important), so to derive generating function of the set C we identify variables in formula (21). We obtain

$$F_C^w(x,x) = \frac{1+x}{1-x-x^2}$$

which is equal to the one we got using another method.