Group action on finite sets

Lections 7-8

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Action of the group on the set

- Group G acts on the set D if each element g in group G is associated with one to one mapping $\varphi(g)$ of the set D into itself that $\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$ for any g_1 and g_2 from G.
- In other words, group G acts on the set D if defined homomorphism φ of the set G into the set of one to one mapping of the set D into itself.
- We will omit the symbol of homomorphism and we will consider elements of group G as transformation of the set D.
- The result of action of the element g of group G on element d of the set D denote as $g(d)$ or gd.

Action of the group on the set

Let group *G* act on the set *D*.

- Stabilizer of the element d_0 from *D* denotes the set $St(d_0) = \{g \in G |\ g(d_0) = d_0\}$.
- Orbit of the element d_0 from *D* denotes the set

  $Or(d_o) = \{d \in D | d = g(d_0), g \in G\}$.
- The number of elements of orbit is its length. Stabilizer of any element *d* is a subset in group *G*.

Action of the group on the set

- **Lemma 1**

If a finite group G acts on the finite set D then for any d from D

$$|\boldsymbol{Or}(\boldsymbol{d})| \cdot |\boldsymbol{St}(\boldsymbol{d})| = |\boldsymbol{G}| .$$

- **Proof**

§ We will show that the length of the orbit of some element d from D is equal to the number of cosets of group G by the subgroup $\boldsymbol{St}(\boldsymbol{d})$.

§ If $\boldsymbol{g_1}$ and $\boldsymbol{g_2}$ are in the same coset of group G by the subgroup $\boldsymbol{St}(\boldsymbol{d})$ then $\boldsymbol{g_2 = g_1 s}$, where $\boldsymbol{s} \in \boldsymbol{St}(\boldsymbol{d})$. So,

$$\boldsymbol{g_2(d) = g_1 s(d) = g_1\big(s(d)\big) = g_1(d)} ,$$

e.g. elements from the same coset of group G by the subgroup $\boldsymbol{St}(\boldsymbol{d})$ transform d into the same element of the set D.

Action of the group on the set

- If $g_1(d) = g_2(d)$ then
  $d = g_1^{-1} g_1(d) = g_1^{-1}(g_1(d)) = g_1^{-1}(g_2(d)) = g_1^{-1} g_2(d)$,
  and hence $g_1^{-1} g_2 \in St(d)$.
- But then in $St(d)$ such element *s* will exist that $g_2 = g_1 s$ and hence *g1* and *g2* are in the same coset of group G by the subgroup $St(d)$.
- As the number of cosets of group G by the subgroup $St(d)$ is $|G|/|St(d)|$, the length of the orbit of element *d* is equal to the number of cosets of the group G by the subgroup $St(d)$, then

$$|Or(d)| \cdot |St(d)| = |G| .$$

QED

Burnside lemma

- Let group G act on the finite set D. Elements d_1 and d_2 from D are equivalent if $d_1 = gd_2$ for some g from G.
  - Set D under the action of group G decomposes into equivalence classes, which consist of pairwise equivalent elements.
- Burnside lemma

Let group G act on the finite set D. Then for the number of equivalence classes N, generated on the set D by action of group G, equality exists

$$N = \frac{1}{|G|} \sum_{g \in G} \psi(g)$$

$\psi(g)$ - number of elements d of the set D that satisfy $gd = d$

.

Burnside lemma, proof

- Consider the function

$$\psi(d, g) = \begin{cases} 1, if\ gd = d; \\ 0, if\ gd \neq d. \end{cases}$$

- With the use of Lemma 1

$$\sum_{g \in G} \psi(g) = \sum_{g \in G} \sum_{d \in D} \psi(d, g) = \sum_{d \in D} \sum_{g \in G} \psi(d, g) =$$

$$= \sum_{Or_i} \sum_{d \in Or_i} \sum_{g \in G} \psi(d, g) = \sum_{Or_i} \sum_{d \in Or_i} |St(d)| = \sum_{Or_i} |G| = N|G|$$

Divide left and right parts of this equality on $|G|$

$$N = \frac{1}{|G|} \sum_{g \in G} \psi(g)$$

QED

Cycle index

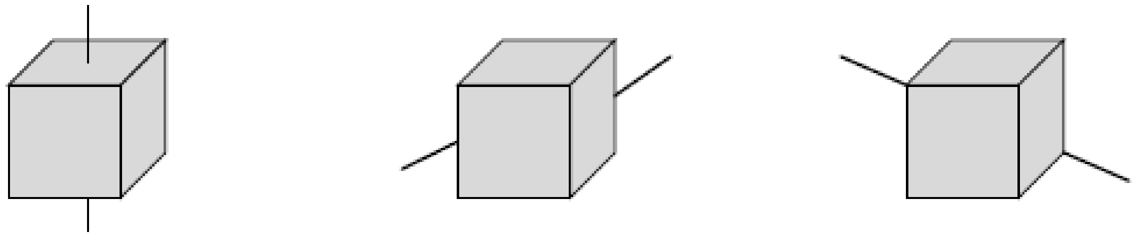
- If element *g* of group *G* acting on the set *D* divides its set into k_i orbit of length $i, i = 1, \ldots, s$ then cycle index I_g of the element *g* is $z_1^{k_1} z_2^{k_2} \cdots z_s^{k_s}$.
- Cycle index of group *G* is

$$P_G(z_1, z_2, \ldots, z_k, \ldots) = \frac{1}{|G|} \sum_{g \in G} I_g(z_1, z_2, \ldots, z_k, \ldots)$$

Cycle index, example

- The problem of coloring the faces of three-dimensional cube.
  - In this task group G of the rotation of the cube acts on the set of its faces.
  - Note that any edge with ordered vertices with the help of rotation can be transformed into any other edge and direct image of the edge can be oriented in two different ways.
  - As the direct image of some edge uniquely defines rotation and there are twelve edges in the cube then group G consists of 24 elements.

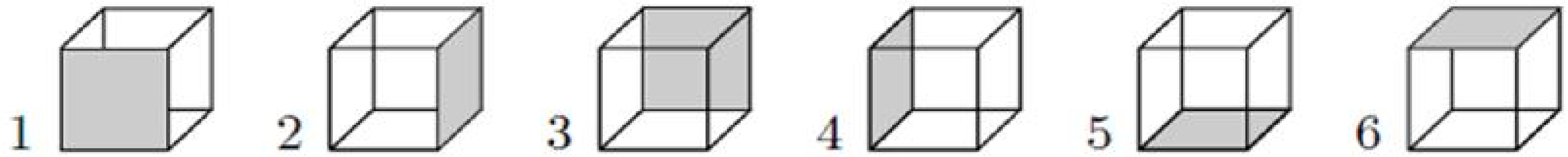
Cycle index, example



- You can rotate the cube around the axes that pass through the centers of opposite faces at angles 90°, 180° and 270° – overall 3·3=9 different rotations.
- Around the axes pass through the centers of opposite edges at angles 180° – overall 6 different rotations.
- Around the axes pass through the opposite vertexes at angles 120° and 240°– overall 4·2=8 different rotations.
- The total number of rotations taking into account the identity rotation is 24.

Cycle index, example

§ Enumerate the elements of the group *G* and write their indexes.



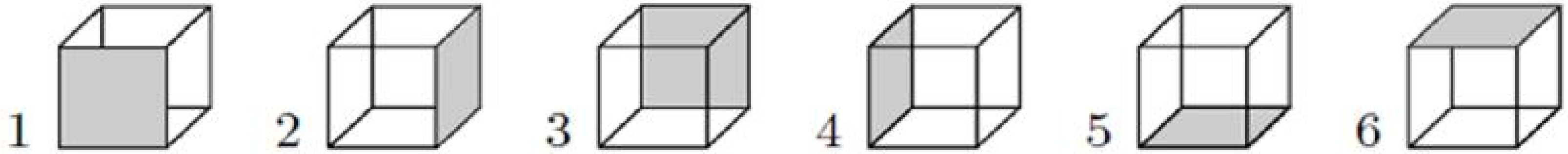
1. Neutral element *e* leaves all the faces of the cube on the site. Hence $I_e = z_1^6$

2. Rotation at 90° and 270° around axes that passes through the centers of the opposite faces.

Consider rotation around the axis that passes through the centers of 5 and 6 faces at 90° clockwise.

There are two cycles with the length one (5) and (6) on the set of faces and there is one cycle with the length four (1234). Other rotations operate in a similar way and hence for each of 6 similar rotations cycle index is $z_1^2 z_4$.

Cycle index, example



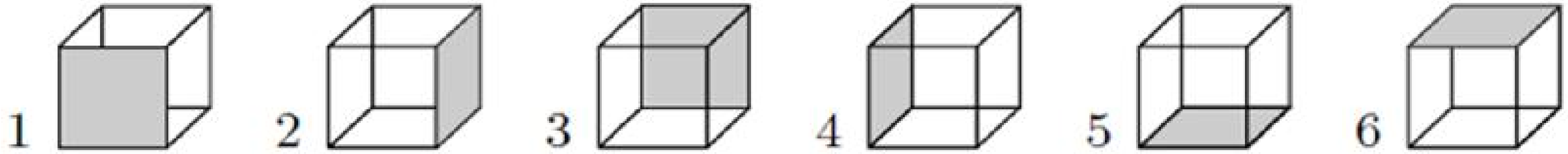
3. Rotations at 180° around the axes that pass through the centers of the opposite faces.

There are two cycles with the length one (5) and (6) and two cycles with the length two (13) and (24). Hence for each of three such rotations cycle index is $z_1^2 z_2^2$.

4. Rotations at 180 around the axes that pass through the centers of the opposite edges.

There are three cycles with the length two (14) (23) and (56). Thus for each of 6 such rotations cycle index is z_2^3 .

Cycle index, example



5. Rotations at 120° and 240° around the axes that pass through the opposite vertexes.

There are two cycles with the length three (146) and (253). Thus for each of 8 such rotations cycle index is z_3^2 .

§ Putting it all together we obtain cycle index of the group of rotations of three-dimensional cube with the action of this group on the set of cube faces.

$$P_G = \frac{1}{24}(z_1^6 + 6z_1^2 z_4 + 3z_1^2 z_2^2 + 6z_2^3 + 8z_3^2) \qquad (1)$$