Functions and their equivalence classes

- Assume *D* and *R* finite sets, *K* commutative ring, $\boldsymbol{w}: \boldsymbol{R} \to \boldsymbol{K}$ - weight function on the set *R*. For each function *f* from the set $\boldsymbol{\mathcal{F}} = \{\boldsymbol{f} : \boldsymbol{D} \to \boldsymbol{R}\}$ let's define its weight $\boldsymbol{w}$, assuming

$$\boldsymbol{w}(\boldsymbol{f}) = \prod_{\boldsymbol{d} \in \boldsymbol{D}} \boldsymbol{w}(\boldsymbol{f}(\boldsymbol{d}))$$

- Functions $\boldsymbol{f_1}$ and $\boldsymbol{f_2}$ are equivalent if there is such element *g* of the group *G* that $\boldsymbol{f_1}(\boldsymbol{d}) = \boldsymbol{f_2}(\boldsymbol{gd})$ for each $\boldsymbol{d} \in \boldsymbol{D}$.
- The set $\boldsymbol{\mathcal{F}}$ decomposes into equivalence classes $\boldsymbol{F_1}, \dots, \boldsymbol{F_k}$ and as the weights of the equivalent functions from one class are the same we can talk about the weight of an equivalence class of functions from $\boldsymbol{\mathcal{F}}$ (denote it as *W(F)*).

Functions and their equivalence classes, example

- Coloring faces of the cube
- The faces will be colored in black and white. In this case the set D consists of 6 cube faces and the set R consists of black and white colors.
- Cube which faces are colored in black and white we will consider as function from D to R which associates each face to each color.
- Group G which acts on the set D will be considered before group of cube rotations and two functions will be equivalent if corresponding to them colored cubes can be transformed into each other with the help of rotations of the group G.

Functions and their equivalence classes, example

- For example, all the cubes with one black and five white faces are equivalent to each other.
- As a ring K is considered to be the ring of polynomials of variables x and y with integer coefficients.
- White color will have the weight x, black color – y.
- Thus the weight of colored cube and the weight of corresponding function will be polynomial of 6th degree of variables x and y.
- If we are interested in a number of different cubes with three black and three white faces we need to find the number of equivalence classes which weight is x^3y^3 .
- It can be done with the Polya theorem.

Fundamental theorem, Polya theorem

Assume that on the set *D* group *G* acts, on the set *R* weight function w with values in commutative ring *K* is defined.

- Polya theorem

Then weight sum of equivalence classes of *F* functions from *D* to *R* is

$$\sum_{F} W(F) = P_G\left(\sum_{r\in R} w(r), \sum_{r\in R} (w(r))^2, \dots, \sum_{r\in R} (w(r))^k, \dots\right)$$

P_G is a cycle index of the group.

Polya theorem, proof

- Proof
  - Consider element *g* from group *G* under the action of which the set *D* is decomposed into $\boldsymbol{k_1}$ cycles of the length one, $\boldsymbol{k_2}$ cycles of the length two and etc.
  - Assume that cycles with the length one are formed by first $\boldsymbol{k_1}$ elements of the set *D*, cycles with the length two are formed by the next $\boldsymbol{2k_2}$ elements that each cycle has the form $(\boldsymbol{d_i d_{i+1}})$.
  - The last $\boldsymbol{sk_s}$ elements of the set *D* form $\boldsymbol{k_s}$ cycles of the form $(\boldsymbol{d_j d_{j+1} \dots d_{j+s-1}})$.

Polya theorem, proof

- The vector of values $\boldsymbol{v}(\boldsymbol{f})$ of any function f, which is defined on the set *D*, takes values in *R* and which under the action of element *g* transforms into itself looks in a certain way.
- On the first $\boldsymbol{k_1}$ places are randomly arranged any elements of the set *R*.
- The next $\boldsymbol{2k_2}$ places are filled with $\boldsymbol{k_2}$ pairs of the same elements from *R*.
- It is necessary and sufficient for the equality

$$\boldsymbol{f(d) = f(g(d)), d \in \{d_{k_1+1}, \dots, d_{k_1+2k_2}\}}.$$

- The next $\boldsymbol{3k_3}$ places are filled with $\boldsymbol{k_3}$ triples of the same elements from *R* and etc.
- The last $\boldsymbol{sk_s}$ discharges of the vector $\boldsymbol{v}(\boldsymbol{f})$ is the sequence which consists of $\boldsymbol{k_s}$ blocks of length *s*, each of which consists of the same elements.

Polya theorem, proof

§ All such vectors can be obtained by opening brackets in multiplication assuming that multiplication is noncommutative.

$$\left(\sum_{r\in R} r\right)^{k_1}\left(\sum_{r\in R} rr\right)^{k_2}\cdots\left(\sum_{r\in R} r\cdots r\right)^{k_s} \qquad (2)$$

Thus

$$\sum_{f=g(f)} v(f)=\left(\sum_{r\in R} r\right)^{k_1}\left(\sum_{r\in R} rr\right)^{k_2}\cdots\left(\sum_{r\in R} r\cdots r\right)^{k_s} \qquad (3)$$

§ For example, if $s=2, k_1=1, k_2=2, R=\{x,y\}$, then

$$(x+y)(xx+yy)(xx+yy)=xxxxx+xxxyy+xyyxx+ +xyyyy+yxxxx+yxxyy+yyyxx+yyyyy \qquad (4)$$

Polya theorem, proof

- Calculate the weight sum of all functions which under the action of element g transform into itself.
- In (3) let's substitute each element r with its weight $w(r)$.

$$\sum_{f=g(f)} w(v(f)) = \left(\sum_{r\in R} w(r)\right)^{k_1} \left(\sum_{r\in R} (w(r))^2\right)^{k_2} \cdots \left(\sum_{r\in R} (w(r))^s\right)^{k_s} \qquad (5)$$

- Assume that the weight of functions left by element g in place have values $w_1, \dots, w_m$.
- Then the weight sum can be represented as

$$\sum_{w_i} w_i \psi_i(g) \qquad (6)$$

$\psi_i(g)$ is the number of functions with the weight w_i .

Polya theorem, proof

- Assume $\boldsymbol{w(x) = t, w(y) = s}$ and calculate the sum of all function weights which vectors of values are listed on the right side of (4). It can be seen that

$$\big(w(x) + w(y)\big)\big(w(xx) + w(yy)\big)\big(w(xx) + w(yy)\big) =$$
$$= (t + s)(t^2 + s^2)(t^2 + s^2) =$$
$$= t^5 + t^4 s + 2t^3 s^2 + 2t^2 s^3 + ts^4 + s^5$$

- where the coefficient of the monomial $\boldsymbol{t^i s^j}$ is equal to the number of functions which weight is $\boldsymbol{t^i s^j}$.

Polya theorem, proof

- In equality (5) note that multiplication in its right part is the index of the element g, where variables z_k are substituted with

$$\sum_{r \in R} \left(w(r) \right)^k$$

- Hence the sum of weights of all functions which under the action of element g transform into itself is equal to

$$I_g \left(\sum_{r \in R} w(r), \sum_{r \in R} \left(w(r) \right)^2, \dots, \sum_{r \in R} \left(w(r) \right)^s \right) \tag{7}$$

Polya theorem, proof

- Calculating the sum (7) by all of the elements of group G and dividing the result by the order of the group G with the use of (6)

$$P_G\left(\sum_{r\in R} w(r), \sum_{r\in R} \left(w(r)\right)^2, \ldots, \sum_{r\in R} \left(w(r)\right)^k, \ldots\right) =$$

$$= \frac{1}{|G|}\sum_{g\in G}\sum_{w_i} w_i \psi_i(g) = \sum_{w_i} w_i \left(\frac{1}{|G|}\sum_{g\in G} \psi_i(g)\right)$$

Polya theorem, proof

- From the Burnside lemma follows that with fixed value of the weight w the sum

$$\sum_{g \in G} \psi \frac{1}{|G|_i}(g)$$

equals to the number of equivalence classes which arose on the set of functions with weight w_i as a result of the action of the group G on the set D.

- Hence the left part of the last equality equals to the sum of weights of all the equivalence classes.

QED

Polya theorem, example

- With the use of the theorem above find the number of different bicolor cubes with three black and three white faces.

$$P_G = \frac{1}{24}(z_1^6 + 6z_1^2 z_4 + 3z_1^2 z_2^2 + 6z_2^3 + 8z_3^2)$$

- Into found before (1) cycle index of the rotation group of three-dimensional substitute $\boldsymbol{x^i + y^i}$ with $\boldsymbol{z_i}$.

$$\frac{1}{24}((x+y)^6 + 6(x+y)^2(x^4+y^4) + 3(x+y)^2(x^2+y^2)^2 + \\ +6(x^2+y^2)^3 + 8(x^3+y^3)^2) \tag{8}$$

Now find coefficient that will stand beside the monomial $\boldsymbol{x^3 y^3}$ after disclosure of brackets and reduction of similar terms.

Polya theorem, example

- Into first summand $(x+y)^6$ monomial x^3y^3 included with coefficient 20,
- in the second and fourth summands there are no such monomial as they contain only even variables x and y,
- into the third summand monomial x^3y^3 included with the coefficient 12,
- into fifth with coefficient 16.
- Hence coefficient of x^3y^3 in (8) is

$$\frac{1}{24}(20+0+12+0+16)=2$$

- Thus, the faces of three-dimensional cube can be colored in two different ways under condition that three faces will be colored in white and the other three in black.

Consequence, example

§Consequence

The number of equivalence classes is

$P_G(|R|, |R|, \dots, |R|, \dots)$, P_G – is a cycle index of the group.

§Example

Find the number of different ways to color faces of the cube in three colors.

$$P_G = \frac{1}{24}(z_1^6 + 6z_1^2 z_4 + 3z_1^2 z_2^2 + 6z_2^3 + 8z_3^2)$$

Into cycle index of the rotation group each variable z_i substitute with number 3.

$$P_G(3,3,3,3) = \frac{1}{24}(3^6 + 6 \cdot 3^3 + 3 \cdot 3^4 + 6 \cdot 3^3 + 8 \cdot 3^2) =$$
$$= 57$$