Graph coloring, vertex coloring

Vertex colorings

- A (proper) vertex k-coloring (or k-coloring) of a simple graph G is a function

$$f: VG \rightarrow \{1, \ldots, k\}$$

such that adjacent vertices are assigned different numbers.

- Quite often the set $\{1, \ldots, k\}$ is regarded as a set of colors.
- A coloring of a graph is a k-coloring for some integer k.
- An improper coloring of a graph permits two adjacent vertices to be colored the same.
- A graph is k-vertex colorable (or k-colorable) if it has a vertex k-coloring.

Graph coloring, vertex coloring

- The vertex chromatic number or (chromatic number) $\chi(G)$ of a graph G is the minimum number k such that G is k-vertex colorable;
- that is, $\chi(G)$ is the smallest number of colors needed to color the vertices of G so that no adjacent vertices have the same color.
- A graph G is k-chromatic if $\chi(G) = k$.
- A graph G is chromatically k-critical if G is k-chromatic and if $\chi(G - e) = k - 1$ for each edge of G.
- An obstruction to k-coloring is a chromatically $(k + 1)$ critical graph, when that graph is regarded as a subgraph of other graphs, and thereby prevents them from having chromatic number k.

Graph coloring, vertex coloring

- A (complete) obstruction set for k-coloring is a set of chromatically $(k+1)$- critical graphs such that every graph that is not k-colorable contains at least one of them as a subgraph.
- An elementary contraction of a simple graph G on the edge e, denoted $G \downarrow e$ (or $G \cdot e$),is obtained by replacing the edge e and its two endpoints by one vertex adjacent to all the other vertices to which the endpoints were adjacent.
- A graph G is (combinatorially) contractible to a subgraph H if H can be obtained from G by a sequence of elementary contractions.
- The chromatic polynomial of the graph G is the function $\pi G(t)$ whose value at the integer t is the number of different functions $VG \rightarrow \{1, \ldots, t\}$ that are proper colorings of G.

Graph coloring, vertex coloring

- $\chi(G) = 1$ if and only if the graph G is edgeless.
- $\chi(G) = 2$ if and only if the graph is bipartite and its edgeset is nonempty.
- The four color theorem : If G is planar,then $\chi(G) \leq 4$.

That is every planar graph has a proper coloring of its vertices with 4 or fewer colors.

- $\chi(G) \leq diam(G)$,where the diameter $diam(G)$ is the length of a longest path in G.

Graph coloring, vertex coloring

Algorithm 1: Greedy coloring algorithm

input: a graph $\boldsymbol{G}$ with vertex list $\boldsymbol{v_1, v_2, \ldots, v_n}$

c:=0 { Initialize color at “color 0”}

while some vertex still has no color

$c := c + 1$ {Get the next unused color}

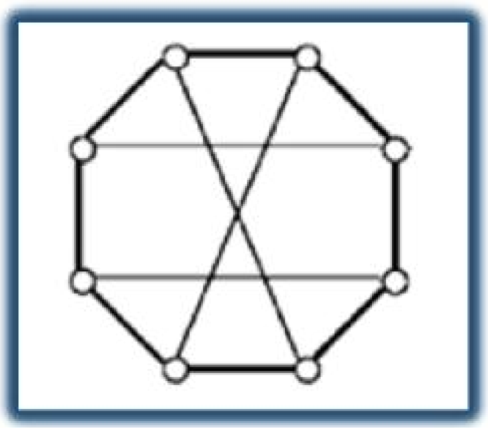
for $i := 1$ to n {Assign the new color to as many vertices as possible}

if v_i is uncolored and no neighbor of v_i has color c

then assign color c to v_i

Graph coloring, vertex coloring

- The greedy coloring algorithm produces a vertex coloring of a graph G,whose vertices are ordered. (It is called "greedy" because once a color is assigned,it is never changed.)
- The number of colors it assigns depends on the vertex ordering, and it is not necessarily the minimum possible.
- At least one ordering of the vertices of a graph G yields $\chi(G)$ under the greedy algorithm.



Example

Applying the greedy coloring algorithm, with the vertices considered in cyclic order around the 8-cycle, yields a 3-coloring. Since this graph contains an odd cycle (a 5-cycle), it can't be 2-colored. Thus, χ=3.

Graph coloring, vertex coloring

Brooks' theorem: In a connected graph in which every vertex has at most Δ neighbors, the vertices can be colored with only Δ colors, except for two cases, complete graphs and cycle graphs of odd length, which require $\Delta + 1$ colors.

Proof

- If the graph is not biconnected, its biconnected components may be colored separately and then the colorings combined.
- If the graph has a vertex v with degree less than Δ, then a greedy coloring algorithm that colors vertices farther from v before closer ones uses at most Δ colors.
- Therefore, the most difficult case of the proof concerns biconnected Δ-regular graphs with $\Delta \geq 3$.

Graph coloring, vertex coloring

- In this case, one can find a spanning tree such that two nonadjacent neighbors u and w of the root v are leaves in the tree.
- A greedy coloring starting from u and w and processing the remaining vertices of the spanning tree in bottom-up order, ending at v, uses at most Δ colors.
- For, when every vertex other than v is colored, it has an uncolored parent, so its already-colored neighbors cannot use up all the free colors, while at v the two neighbors u and w have equal colors so again a free color remains for v itself.

Graph coloring, edge coloring

- An edge coloring of a graph is an assignment of colors to its edges such that adjacent edges receive different colors.
- A graph $\boldsymbol{G}$ is $\boldsymbol{k}$-edge colorable if there is an edge coloring of $\boldsymbol{G}$ using at most $\boldsymbol{k}$ colors.
- The edge chromatic number $\chi'(\boldsymbol{G})$ of a graph $\boldsymbol{G}$ is the minimum $\boldsymbol{k}$ such that $\boldsymbol{G}$ is k-edge colorable.
- If $\chi'(\boldsymbol{G}) = \boldsymbol{k}$, then $\boldsymbol{G}$ is edge $\boldsymbol{k}$-chromatic.
- Chromatic index is a synonym for edge chromatic number.
- A graph is edge-chromatically k-critical if it is edge $\boldsymbol{k}$-chromatic and $\chi'(\boldsymbol{G}-\boldsymbol{e}) = \mathrm{X}'(\boldsymbol{G}) - \mathbf{1}$ for every edge $\boldsymbol{e}$ of $\boldsymbol{G}$.

Graph coloring, edge coloring

- For a graph G, the line graph $L(G)$ has as vertices the edges of G, with two vertices adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G.

Vizing's theorem

$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$ for any simple graph G
where $\Delta(G)$ is the maximum degree of vertices in graph G.

Proof

- The inequality $\Delta(G) \leq \chi'(G)$ being trivial, we show $\chi'(G) \leq \Delta(G) + 1$.
- To prove this inductively, it suffices to show for any simple graph G:
- Let v be a vertex such that v and all its neighbours have degree at most k, while at most one neighbour has degree precisely k. Then if $G - v$ is k-edge-colourable, also G is k-edge-colourable. (1)

Graph coloring, edge coloring

- We prove (1) by induction on k.
- We can assume that each neighbour u of v has degree $k-1$, except for one of degree k, since otherwise we can add a new vertex w and an edge uw without violating the conditions in (1).
- We can do this till all neighbours of v have degree $k-1$, except for one having degree k.
- Consider any k-edge-colouring of $G-v$.
- For $i=1,\ldots,k,$ let X_i be the set of neighbours of v that are missed by colour i.
- So all but one neighbour of v is in precisely two of the X_i , and one neighbour is in precisely one X_i.

Graph coloring, edge coloring

- Hence

$$\sum_{i=1}^{k} |X_i| = 2\,deg(v) - 1 < 2k$$

(2)

- We can assume that we have chosen the colouring such that

$$\sum_{i=1}^{k} |X_i|^2$$

is minimized.

- Then for all $i, j = 1, \dots, k$:

$$||X_i| - |X_j|| \leq 2.$$

(3)

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- For if, say, $|X_1| > |X_2| + 2$, consider the subgraph H made by all edges of colours 1 and 2.
- Each component of H is a path or circuit. At least one component of H contains more vertices in X_1 than in X_2.
- This component is a path P starting in X_1 and not ending in X_2 .
- Exchanging colours 1 and 2 on P reduces $|X_1|^2 + |X_2|^2$, contradicting our minimality assumption. This proves (3).
- This implies that there exists an i with $|X_i| = 1$, since otherwise by (2) and (3) each $|X_i|$ is 0 or 2, while their sum is odd, a contradiction.
- So we can assume $|X_k| = 1$, say $X_k := \{u\}$.

Graph coloring, edge coloring

- Let G′ be the graph obtained from G by deleting edge vu and deleting all edges of colour k. So $G' - v$ is $(k-1)$-edge-coloured.
- Moreover, in G' , vertex v and all its neighbours have degree at most $k-1$, and at most one neighbour has degree $k\ -1$.
- So by the induction hypothesis, G' is $(k-1)$-edge-colourable.
- Restoring colour k, and giving edge vu colour k, gives a k-edge-colouring of G.

Graph coloring, edge coloring

- Every edge coloring of a graph G can be interpreted as a vertex coloring of the associated line graph $L(G)$.
  Thus, $\chi'(G) = \chi(L(G))$.
- $\chi'(K_{m,n}) = \max\{m, n\}$, if $m, n \geq 1$.
- If G is bipartite, then $\chi'(G) = \Delta\max(G)$.
- $\chi'(K_n) = n$ if n is odd; $\chi'(K_n) = n - 1$ if n is even.

Graph coloring, edge coloring

Algorithm 2: Greedy edge-coloring algorithm

input: a graph $\boldsymbol{G}$ with edge list $\boldsymbol{e_1, e_2, \ldots, e_n}$

c:=0 { Initialize color at “color 0”}

while some edge still has no color

c:=c+1 {Get the next unused color}

for i:=1 to n

{Assign the new color to as many edges as possible}

if e_i is uncolored and no neighbor of e_i has color c then assign color c to e_i

- The greedy edge-coloring algorithm (Algorithm 2) produces an edge-coloring of a graph $\boldsymbol{G}$,whose vertices are ordered.
- The number of colors it assigns depordering, and it is not necessarily the minimum possible. ends on the vertex

Graph coloring, edge coloring

Example 1.

- The following three graphs are all edge 3-chromatic.
- None of them is edge-chromatically 3-critical.
- Since each edge graph has a vertex of degree three, no 2-edge-coloring is possible.

Graph coloring, edge coloring

Example 2.

- The following graph is 5-edge-chromatic.
- Since there are 14 edges, a 4-edge-coloring would have to give the same color to four of them.
- For this edge-coloring to be proper, these four edges would have to have no endpoints in common.
- That is impossible, because the graph has only seven vertices.

Graph coloring

- The following table gives the chromatic numbers and edge-chromatic numbers of the graphs in some common families.

Graph G	$\chi(G)$	$\chi_1(G)$
Path graph $P_n, n \geq 3$	2	2
Cycle graph $C_n, n\ even, n \geq 2$	2	2
Cycle graph $C_n, n\ odd, n \geq 3$	3	3
Wheel $W_n, n\ even, n \geq 4$	3	n
Wheel $W_n, n\ odd, n \geq 3$	4	n
Complete graph $K_n, n\ even, n \geq 2$	n	n-1
Complete graph $K_n, n\ odd, n \geq 3$	n	n
Complete bipartite graph $K_{m,n}, m, n \geq 1$	2	max{m, n}
Bipartite G, at least one edge	2	Δmax(G)