Theorems

- Matching (independent) edge set in a graph is a set of edges without common vertices.
- A perfect matching is a matching which matches all vertices of the graph. That is, every vertex of the graph is incident to exactly one edge of the matching.
- An edge cover of a graph G is a set of edges C such that each vertex in G is incident with at least one edge in C. The set C is said to cover the vertices of G.
- Vertex cover is a subset of vertices such that in all edges at least one endpoint is in this subset.
- Shadow $S(A)$ of A is a subset of vertices, each of which is adjacent to at least one vertex of A.

Halls' theorem

Halls' theorem:

Given a bipartite graph, there is a perfect matching iff $|N(s)| \geq |S| \ \forall S \subseteq X$, where X is one of the sets of points of the graph.

Proof

This proof is with induction.

- We take n the amount of vertices in X.
- For $n = 1$ it is trivial, because we just have one vertex in each part, so we can match those vertices with one edge.
- S is empty or contains one point and in both cases the amount of neighbors the same.

Halls' theorem

- Now we assume that the statement is true for all graphs on $\boldsymbol{k}$ vertices in $\boldsymbol{X}$. We want to prove that it is also true for $\boldsymbol{k+1}$ vertices in $\boldsymbol{X}$.
- Take vertex $\boldsymbol{v}$ in $\boldsymbol{X}$, then there is at least one edge leaving $\boldsymbol{v}$ otherwise there won't be a perfect matching possible at all.

Now we consider 2 cases.

- Case 1: Suppose $\forall \boldsymbol{S} \subseteq \boldsymbol{X} - \{\boldsymbol{v}\}$: $|\boldsymbol{N}(\boldsymbol{S})| \geq |\boldsymbol{S}| + \boldsymbol{1}$.
- In this case we have slack so we can match $\boldsymbol{v}$ with that one edge and then there is still a perfect matching for the other vertices, because we then have a graph of size $\boldsymbol{k}$ where $|\boldsymbol{N}(\boldsymbol{S})| \geq |\boldsymbol{S}|$ holds, and we assumed that the statement is true.
- The $+\boldsymbol{1}$ disappears because we have matched $\boldsymbol{v}$ to some vertex and in the worst case that was a neighbor of all other vertices in $\boldsymbol{X}$ as well. Qed

Halls' theorem

- Case 2: In this case there exists some subset $S \subseteq X - v$ such that $|N(S)| = |S|$.
- Pick a minimum cardinality set satisfying this property. By induction and the minimality of S we know that S can be matched to N(S).
- Now look at $X - S$, for this specific S, take a set $S' \subseteq X - S$. Then this S' needs to have neighbors outside $N(S)$, otherwise $S \cup S'$ doesn't suffice the Hall's condition.
- It has to have $|S'|$ outside $N(S)$. If we then look at the rest, by induction those points also have a matching in $X - N(S)$.
- So combining this matchings we have a full matching. This proves the statement. QED

Halls' theorem

Example

We have a usual deck of 52 cards.

- Divide deck into 13 arbitrary piles of 4 cards.
- Prove that it is always possible to get exactly the set $A, 2, 3, \dots, J, Q, K$ by picking one card from each pile.

We can transform this problem into a bipartite graph.

- On one side 13 vertices representing the piles and on the other side 13 vertices – one for A, one for 2, etc.
- Each vertex has degree 4 and we want a perfect matching. Take a set S, a subset of the piles, 4·|S| edges are leaving this set, but the receiving vertices have also degree 4, so $|N(S)| \geq \frac{number\ of\ edges}{4} = \frac{4|S|}{4} = |S|$, otherwise there is a vertex with degree greater than 4.
- So by Halls' theorem a perfect matching exists.

Menger's theorem

Definitions

- Paths are said to be edge disjoint if they share no edges.
- s-t paths are said to be vertex disjoint if they share no vertices other then s and t.
- An antichain A of a poset P is a subset of elements of P such that for all $x, y \in A, x \nleq y$ and $y \nleq x$.
- A chain cover of a poset P is a collection of chains whose union is P.

Menger's theorem

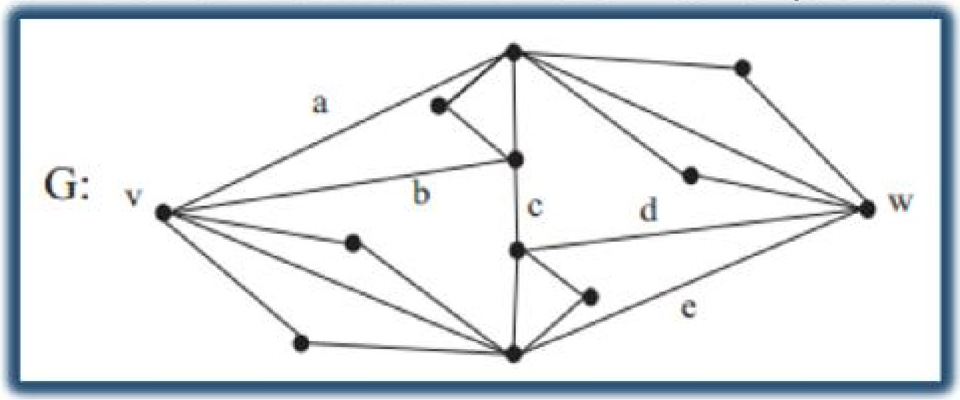
Vertex-connectivity

- Let v and w be two non-adjacent vertices in a graph G.
- A set S of vertices is a $v - w$ separating set if v and w lie in different components of G – S; that is, if every $v - w$ path contains a vertex in S.
- The minimum order of a $v - w$ separating set is called the $v - w$ connectivity and is denoted by $k(v, w)$.
- For any two vertices v and w, a collection of $v - w$ paths is called internally disjoint if the paths are pairwise disjoint except for the vertices v and w.
- The maximum number of internally disjoint $v - w$ paths is denoted by $\mu(v, w)$.
- Since each path in such a set must contain a different vertex from every $v - w$ separating set, it is clear that $\mu(v, w) \leq k(v, w)$.

Menger's theorem

Menger's Theorem

If v and w are non-adjacent vertices in a graph G, then the maximum number of internally disjoint $v-w$ paths equals the minimum number of vertices in a $v-w$ separating set.



Proof

- Let v and w be a pair of non-adjacent vertices in a graph G. As observed earlier $\mu(v, w) \leq k(v, w)$ since a $v-w$ separator must contain at least one vertex from each of the paths in any collection of internally disjoint $v-w$ paths.

Menger's theorem

- We now show that $\mu(v, w) \geq k(v, w)$.
- Let $k = k(v, w)$. Then no set of fewer than k vertices separates v and w.
- We proceed to show, by induction on k, that if $k(v, w) \geq k$, then $\mu(v, w) \geq k$.
- If $k = 1$, then there is a v– w path.
- Assume thus that $k \geq 1$ and that if $k(v, w) \geq k$ that $\mu(v, w) \geq k$.
- Assume further that v and w are non-adjacent vertices in G with $k(v, w) \geq k + 1$.
- By the induction hypothesis, there are k internally disjoint v– w paths $P_1, P_2, \ldots, P_k$.

Menger's theorem

- Since the collection of vertices that follow v on these paths (there are k of these) do not separate v and w, there is a v– w path P whose initial edge is not on any P_i .
- Let x be the first vertex after v on P that belongs to some P_i.
- Let P_{k+1} be the v– x subpath of P.
- Assume that $P_1, P_2, \ldots, P_{k+1}$ have been chosen in such a way that the distance from x to w in $G - v$ is a minimum.
- If $x = w$, then we have the desired collection of $k + 1$ internally disjoint paths.
- Assume therefore that $x \neq w$.
- Again, by the induction hypothesis, there are k internally disjoint v– w paths $Q_1, Q_2, \ldots, Q_k$ in $G - x$.

Menger's theorem

- Assume that these paths have been chosen so that a minimum number of edges not on any of the paths P_i are used.
- Let H be the graph consisting of the paths $Q_1, Q_2, \ldots, Q_k$ together with the vertex x.
- Choose some P_j for $1 \leq j \leq k+1$, whose initial edge is not in H.
- Let y be the first vertex on P_j after v which is in H.
- If $y = w$, then we have the desired collection of $k+1$ internally disjoint v– w paths.
- So assume $y \neq w$.
- If $y = x$, then let R be the shortest x– w path in $G - v$.

Menger's theorem

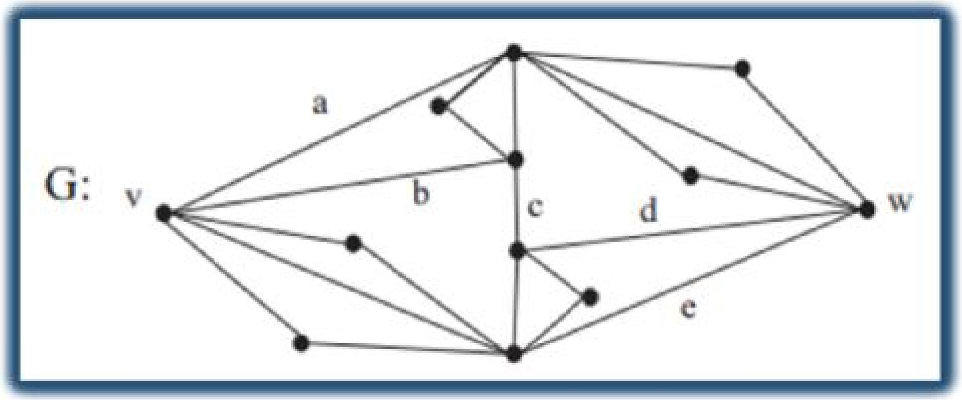
- Let z be the first vertex on R that is on some Q_i .
- Then the distance in $G - v$ from z to w is less than the distance from x to w.
- This contradicts our choice of $P_1, P_2, \ldots, P_{k+1}$. So $y \neq x$.
- If y is on some Q_i for $1 \leq i \leq k$, then the v– y subpath of Q_t has an edge in B.
- Otherwise, two paths from among $P_1, P_2, \ldots, P_{k+1}$ intersect at a vertex other than v, w or x.
- If we replace the v– y subpath of Q_i by the v– y subpath of P_j , we get a collection of k internally disjoint v– w paths in $G - x$ that uses fewer edges from B than $Q_1, Q_2, \ldots, Q_k$ do, which is a contradiction.

Menger's theorem

Edge-connectivity

- The maximum number of edge-disjoint v– w paths in G is denoted by $\nu(v, w)$. Since each such path must contain an edge from every v– w edge-separating set, $\nu(v, w) \leq \lambda(v, w)$.
- Theorem For any vertices v and w in a graph G, $\nu(v, w) = \lambda(v, w)$.
- One may well ask whether there always exists a system of $\nu(v, w)$ edge-disjoint paths that contains a system of $\mu(v, w)$ internally disjoint v– w paths.

Menger's theorem



For the graph G,
$\mu(v, w) = 3$ and
$\nu(v, w) = 5$,

- but no set of three internally disjoint v– w paths is contained in a set of five edge-disjoint v– w paths.
- To see this, note that every set of three internally disjoint v– w paths contains all five edges a, b, c, d, e of a minimal v– w edge-separating set and thus cannot be extended to five edgedisjoint v– w paths.
- If v and w are not adjacent, then both $\deg v$ and $\deg w$ may exceed $\kappa(v, w)$ by an arbitrarily large amount.

Menger's theorem

- Theorem: Every non-null graph has adjacent vertices $\boldsymbol{v}$ and $\boldsymbol{w}$ for which $\mu(\boldsymbol{v},\boldsymbol{w}) = \min\{\deg \boldsymbol{v}, \deg \boldsymbol{w}\}$.
- An immediate consequence of the above theorem is that there exist vertices $\boldsymbol{v}$ and $\boldsymbol{w}$ such that

$$\mu(\boldsymbol{v},\boldsymbol{w}) = \boldsymbol{\nu}(\boldsymbol{v},\boldsymbol{w}) = \boldsymbol{\lambda}(\boldsymbol{v},\boldsymbol{w}) = \min\{\deg \boldsymbol{v}, \deg \boldsymbol{w}\}.$$

- Note that this theorem is not true for multigraphs, since a multigraph formed from a cycle by doubling every edge does not satisfy the theorem.
- However, it is true that every multigraph $\boldsymbol{M}$ has adjacent vertices v and w for which $\boldsymbol{\nu}(\boldsymbol{v},\boldsymbol{w}) = \min\{\deg \boldsymbol{v}, \deg \boldsymbol{w}\}$.

Menger's theorem

- The edge-connectivity $\lambda(G)$ of a non-trivial graph G is the smallest number of edges whose deletion produces a disconnected graph, while that of the trivial graph is defined to be 0.
- It is not difficult to see that

$$\lambda(G) = \min\{\lambda(v, w) : v, w \in V(G)\}.$$

A graph G is l-edge-connected if $\lambda(G) \geq l$.

Dilworth's theorem

Dilworth's theorem: In a finite partial order, the size of a maximum antichain is equal to the minimum number of chains needed to cover its elements.

Proof

- Let $\boldsymbol{P}$ be a finite partially ordered set.
- The theorem holds trivially if $\boldsymbol{P}$ is empty. So, assume that $\boldsymbol{P}$ has at least one element, and let $\boldsymbol{a}$ be a maximal element of $\boldsymbol{P}$.
- By induction, we assume that for some integer $\boldsymbol{k}$ the partially ordered set $\boldsymbol{P' = P \setminus \{a\}}$ can be covered by $\boldsymbol{k}$ disjoint chains $\boldsymbol{C_1, \ldots, C_k}$ and has at least one antichain $\boldsymbol{A_0}$ of size $\boldsymbol{k}$.

Dilworth's theorem

- Clearly, $A_0 \cap C_i \neq 0$ for $i = 1, 2, \dots, k$.
- For $i = 1, 2, \dots, k$, let x_i be the maximal element in C_i that belongs to an antichain of size k in P', and set

$$A = \{x_1, x_2, \dots, x_k\}.$$

- We claim that A is an antichain.
- Let A_i be an antichain of size k that contains x_i.
- Fix arbitrary distinct indices i and j. Then $A_i \cap C_j \neq 0$.
- Let $y \in A_i \cap C_j$. Then $y \leq x_j$, by the definition of x_j.
- This implies that $x_i \ngeq x_j$, since $x_i \ngeq y$. By interchanging the roles of i and j in this argument we also have $x_j \ngeq x_i$. This verifies that A is an antichain.

Dilworth's theorem

- We now return to $\boldsymbol{P}$. Suppose first that $\boldsymbol{a} \geq \boldsymbol{x_i}$ for some $\boldsymbol{i} \in \{\mathbf{1}, \mathbf{2}, \dots, \boldsymbol{k}\}$.
- Let $\boldsymbol{K}$ be the chain $\{\boldsymbol{a}\} \cup \{\boldsymbol{z} \in \boldsymbol{C_i}: \boldsymbol{z} \leq \boldsymbol{x_i}\}$.

Then by the choice of $\boldsymbol{x_i}$, $\boldsymbol{P} \backslash \boldsymbol{K}$ does not have an antichain of size $\boldsymbol{k}$.

- Induction then implies that $\boldsymbol{P} \backslash \boldsymbol{K}$ can be covered by $\boldsymbol{k} - \mathbf{1}$ disjoint chains since $\boldsymbol{A} \backslash \{\boldsymbol{x_i}\}$ is an antichain of size $\boldsymbol{k} - \mathbf{1}$ in $\boldsymbol{P} \backslash \boldsymbol{K}$.
- Thus, $\boldsymbol{P}$ can be covered by $\boldsymbol{k}$ disjoint chains, as required.
- Next, if $\boldsymbol{a} \ngeq \boldsymbol{x_i}$ for each $\boldsymbol{i} \in \{\mathbf{1}, \mathbf{2}, \dots, \boldsymbol{k}\}$, then $\boldsymbol{A} \cup \{\boldsymbol{a}\}$ is an antichain of size $\boldsymbol{k} + \mathbf{1}$ in $\boldsymbol{P}$ (since a is maximal in $\boldsymbol{P}$).
- Now $\boldsymbol{P}$ can be covered by the $\boldsymbol{k} + \mathbf{1}$ chains $\{\boldsymbol{a}\}$, $\boldsymbol{C_1}, \boldsymbol{C_2}, \dots, \boldsymbol{C_k}$, completing the proof.