Boolean logic

Lecture 12

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Propositions

- A proposition is a statement that is either true or false. Whichever of these (true or false) is the case is called the **truth value** of the proposition.

  ‘Canberra is the capital of Australia’

  ‘There are 8 day in a week.’

- The first and third of these propositions are true, and the second and fourth are false.
- The following sentences are not propositions:

  ‘Where are you going?’

  ‘Come here.’

  ‘This sentence is false.’

Propositions

- Propositions are conventionally symbolized using the letters $\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}, \ldots$. Any of these may be used to symbolize specific propositions, e.g.

  $\boldsymbol{p}$: Manchester is in Scotland,

  $\boldsymbol{q}$: Mammoths are extinct.

The previous propositions are **simple propositions** since they make only a single statement.

Logical connectives and truth tables

- Simple propositions can be combined to form more complicated propositions called **compound propositions**.
- The devices which are used to link pairs of propositions are called **logical connectives** and the truth value of any compound proposition is completely determined by the truth values of its component simple propositions, and the particular connective, or connectives, used to link them.

  > ‘If Brian and Angela are not both happy, then either Brian is not happy or Angela is not happy.’

- The sentence about Brian and Angela is an example of a **compound proposition**. It is built up from the **atomic** propositions ‘Brian is happy’ and ‘Angela is happy’ using the words **and, or, not and if-then**. These words are known as **connectives.**

Logical connectives and truth tables

Connective	Symbol
And (conjunction)	∧
Or (disjunction)	∨
Xor (exclusive disjunction)	⊕
Not (negation)	¬ (−)
If-then (implication)	→
If-and-only-if (equivalence)	↔ (≡)
the Sheffer stroke	↑ (|)
the Peirce arrow	↓ (⊥)

Logical connectives and truth tables

The truth table for conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

exclusive disjunction

p	q	p⊕q
T	T	F
T	F	T
F	T	T
F	F	F

Logical connectives and truth tables

The truth table for negation

p	$\neg p$
T	F
F	T

implication

p	q	p→q
T	T	T
T	F	F
F	T	T
F	F	T

equivalence

p	q	p↔q
T	T	T
T	F	F
F	T	F
F	F	T

Logical connectives and truth tables

The truth table for Sheffer stroke

p	q	$p \uparrow q$
F	F	T
F	T	T
T	F	T
T	T	F

$$p \uparrow q = \overline{p \wedge q}$$

The truth table for Peirce arrow

p	q	$p \downarrow q$
F	F	T
F	T	F
T	F	F
T	T	F

$$p \downarrow q = \overline{p \vee q}$$

Compound propositions

Example 1

Express the proposition **‘Either my program runs and it contains no bugs, or my program contains bugs’** in symbolic form.

Solution:

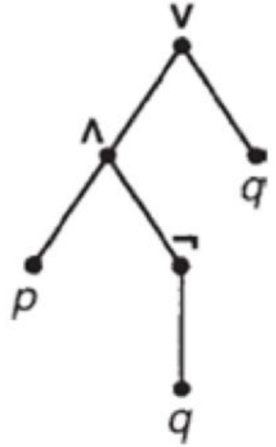
Let $\boldsymbol{p}$ denote the statement **‘My program runs’**

Let $\boldsymbol{q}$ denote the statement **‘My program contains bugs’**

Then the proposition can be written in symbolic form as follows:

$(\boldsymbol{p} \quad \neg \boldsymbol{q}) \vee$

The structure of the expression $(\boldsymbol{p} \quad \neg \boldsymbol{q}) \vee$ can be depicted using an expression tree.



Compound propositions

The truth value of $(p \quad \neg q)\vee$ for each possible combination of truth values of p and q can be found by constructing a **truth table.**

Example 2

Construct the truth table for the expression $(p\wedge\neg q)\vee$

Solution

p	q	$\neg q$	$p\wedge\neg q$	$(p\wedge\neg q)\vee q$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	T
F	F	T	F	F

Compound propositions

- A **tautology** is a compound proposition which is true no matter what the truth values of its simple components.
- A **contradiction** is a compound proposition which is false no matter what the truth values of its simple components.

Example 3 Show that $(p \wedge \overline{q}) \wedge (\overline{p} \vee q)$ is a contradiction.

Solution

p	q	$\overline{q}$	$p \wedge \overline{q}$	$\overline{p}$	$\overline{p} \vee q$	$(p \wedge \overline{q}) \wedge (\overline{p} \vee q)$
T	T	F	F	F	T	F
T	F	T	T	F	F	F
F	T	F	F	T	T	F
F	F	T	F	T	T	F

The last column shows that $(p \wedge \overline{q}) \wedge (\overline{p} \vee q)$ is always false, no matter what the truth values of p and q.
Hence $(p \wedge \overline{q}) \wedge (\overline{p} \vee q)$ is a contradiction.

Disjunctive normal form (DNF)

- In Boolean logic, a **disjunctive normal form (DNF)** is a standardization (or normalization) of a logical formula which is a disjunction of conjunctive clauses.

p	q	r	?
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

We circle each of the output labeled ‘true’. Considering the input **(T-T-T)** of the topmost circled T, we create the following normal form: $p \wedge q$

Here the input is **T-F-F**, therefore the normal form is $p \wedge \neg q \wedge \neg r$.

Here the input is **F-F-T** so the normal form is $\neg p \wedge \neg q \wedge r$ There are only three true outputs, therefore there will be only three normal forms.

We join these with the disjunctive ‘or’ resulting in the ‘**disjunctive normal form**’: $(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$

Logical equivalence

Two expressions (composed of the same variables) are **logically equivalent** is they have the same truth values for every combination of the truth values of the variables.

Example 4 Show that $\overline{p} \vee \overline{q}$ and $\overline{p \wedge q}$ are logically equivalent, i.e. that $(\overline{p} \vee \overline{q}) \equiv (\overline{p \wedge q})$.

Solution

p	q	$\overline{p}$	$\overline{q}$	$\overline{p} \vee \overline{q}$	$p \wedge q$	$\overline{p \wedge q}$
T	T	F	F	F	T	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

Comparing the columns for $\overline{p} \vee \overline{q}$ and $\overline{p \wedge q}$ we note that the true values are the same. Each is true except in the case where p and q are both true. Hence $\overline{p} \vee \overline{q}$ and $\overline{p \wedge q}$ are logically equivalent propositions.

Logical equivalence

There is distinction between the connective **if-and-only-if** and the concept of **logical equivalence**.

When we write $\boldsymbol{p} \leftrightarrow \boldsymbol{q}$, we are writing a **single** logical expression. Logical equivalence, on the other hand, is a relationship between **two** logical expressions. The two concepts are related in the following way: two expressions $\boldsymbol{A}$ and $\boldsymbol{B}$ are logically equivalent if and only if the expression $\boldsymbol{A} \leftrightarrow \boldsymbol{B}$ is tautology.

- If $\boldsymbol{P} \equiv \boldsymbol{Q}$ then $\boldsymbol{P} \leftrightarrow \boldsymbol{Q}$ is tautology.
- The converse is also the case, i.e. if $\boldsymbol{P} \leftrightarrow \boldsymbol{Q}$ is a tautology, then $\boldsymbol{P} \equiv \boldsymbol{Q}$.

Logical equivalence

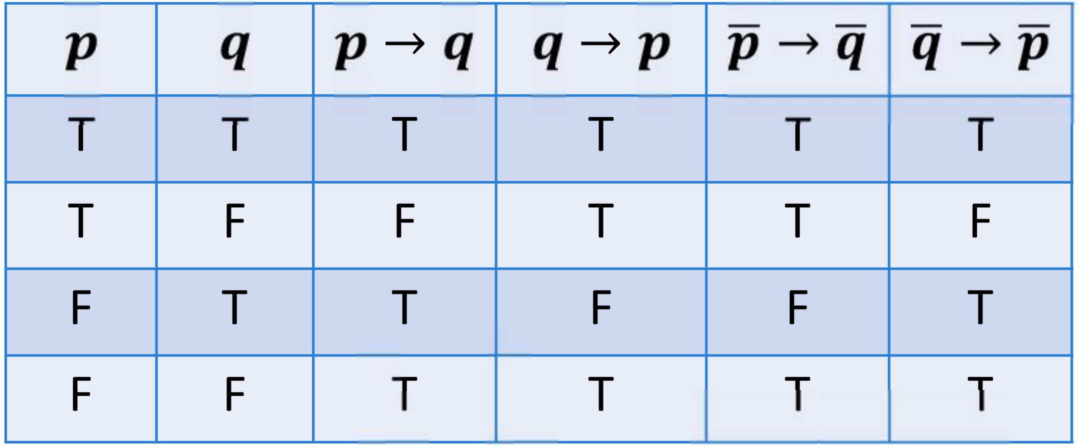
Given the conditional proposition $\boldsymbol{p} \rightarrow \boldsymbol{q}$, we define the following:

- the **converse** of $\boldsymbol{p} \rightarrow \boldsymbol{q}$: $\boldsymbol{q} \rightarrow \boldsymbol{p}$
- the **inverse** of $\boldsymbol{p} \rightarrow \boldsymbol{q}$: $\overline{\boldsymbol{p}} \rightarrow \overline{\boldsymbol{q}}$
- the **contrapositive** of $\boldsymbol{p} \rightarrow \boldsymbol{q}$: $\overline{\boldsymbol{q}} \rightarrow \overline{\boldsymbol{p}}$.

The following truth table give values of the conditional together with those for its converse, inverse and contrapositive.

Logical equivalence

The truth table gives values of the conditional together with those for its converse, inverse and contrapositive.



p	q	$p \to q$	$q \to p$	$\overline{p} \to \overline{q}$	$\overline{q} \to \overline{p}$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

- From the table we note the following useful result: a conditional proposition $p \to q$ and its contrapositive $\overline{q} \to \overline{p}$ are logically equivalent, i.e. $(p \to q) \equiv (\overline{q} \to \overline{p})$.
- Note that a conditional proposition is not logically equivalent to either its converse or inverse. However, the converse and inverse of a proposition are logically equivalent to each other.

Laws of logic

$p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$		Equivalence law
$p \rightarrow q \equiv \neg p \lor q$		Implication law
$\neg\neg p \equiv p$		Double negation law
$p \land p \equiv p$	$p \lor p \equiv p$	Idempotent laws
$p \land q \equiv q \land p$	$p \lor q \equiv q \lor p$	Commutative laws
$(p \land q) \land r \equiv p \land (q \land r)$	$(p \lor q) \lor r \equiv p \lor (q \lor r)$	Associative laws
$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$	Distributive laws
$\neg(p \land q) \equiv \neg p \lor \neg q$	$\neg(p \lor q) \equiv \neg p \land \neg q$	de Morgan’s laws
$p \land T \equiv p$	$p \lor F \equiv p$	Identity laws
$p \land F \equiv F$	$p \lor T \equiv T$	Annihilation laws
$p \land \neg p \equiv F$	$p \lor \neg p \equiv T$	Inverse laws
$p \land (p \lor q) \equiv p$	$p \lor (p \land q) \equiv p$	Absorption laws

Laws of logic

Example Use a truth table to verify the first de Morgan’s law:

$\neg(\boldsymbol{p} \land \boldsymbol{q}) \equiv \neg \boldsymbol{p} \lor \neg \boldsymbol{q}$

Solution

Note that the law can be paraphrased as follows: ‘If it is not the case that *p* and *q* are both true, then that is the same as saying that at least one of $\boldsymbol{p}$ or $\boldsymbol{q}$ is false.’

Laws of logic

The truth table

$\boldsymbol{p}$	$\boldsymbol{q}$	$\boldsymbol{p} \wedge \boldsymbol{q}$	$\neg(\boldsymbol{p} \wedge \boldsymbol{q})$	$\neg\boldsymbol{p}$	$\neg\boldsymbol{q}$	$\neg\boldsymbol{p} \vee \neg\boldsymbol{q}$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

The column for $\neg(\boldsymbol{p} \wedge \boldsymbol{q})$ and $\neg\boldsymbol{p} \vee \neg\boldsymbol{q}$ are identical, and therefore the two expressions are logically equivalent.

Laws of logic

Example Use the laws of logic to simplify the expression:

$p \vee \neg(\neg p \rightarrow q)$

$p \vee \neg(\neg p \rightarrow q) \equiv$	$p \vee \neg(\neg\neg p \vee q)$	Implication law (with $\neg p$ in place of **p**)
	$p \vee \neg(p \vee q)$	Double negation law
	$p \vee (\neg p \wedge \neg q)$	Second de Morgan law
	$(p \vee \neg p) \wedge (p \vee \neg q)$	Second distributive law (with $\neg p$ and $\neg q$ in place of q and r respectively)
	$\mathbf{T} \wedge (p \vee \neg q)$	Second inverse law
	$(p \vee \neg q) \wedge T$	First communicative law (with T and $(p \vee \neg q)$ in place of p and q respectively)
	$p \vee \neg q$	First identity law (with $(p \vee \neg q)$ in place of p)

Laws of logic

Example Use the laws of logic to show that

$[(p \rightarrow q) \wedge \neg q] \rightarrow \neg p$ is a tautology.

$\neg[(\neg p \vee q) \wedge \neg q] \vee \neg p$	Implication law (twice)
$\neg[\neg q \wedge (\neg p \vee q)] \vee \neg p$	First communicative law
$\neg[(\neg q \wedge \neg p) \vee (\neg q \wedge q)] \vee \neg p$	First distributive law
$\neg[(\neg q \wedge \neg p) \vee (q \wedge \neg q)] \vee \neg p$	First commutative law
$\neg[(\neg q \wedge \neg p) \vee F] \vee \neg p$	First inverse law
$\neg(\neg q \wedge \neg p) \vee \neg p$	Second identity law
$(\neg\neg q \vee \neg\neg p) \vee \neg p$	First de Morgan law
$(q \vee p) \vee \neg p$	Double negation law
$q \vee (p \vee \neg p)$	Second associative law
$q \vee T$	Second inverse law
T	Second annihilation law

Predicate logic

A **predicate** is a statement containing one or more variables. If values are assigned to all the variables in a predicate, the resulting statement is a proposition.

For example, **‘** $<$ **5** is a predicate, where *x* is a variable denoting any real number. If we substitute a real number for x, we obtain a proposition; **for example**, **‘3 < 5’** and **‘6 < 5’** are propositions with truth values **T** and **F** respectively.

- The expressions ‘for all’ and ‘there exists’ are called **quantifiers**. The process of applying a quantifier to a variable is called **quantifying** the variable. A variable which has been quantified is said to be **bound**.
- **For example**, the variable x in **‘There exists an x such that $x < 5$’** is bound by the quantifier **‘there exists’**. A variable that appears in a predicate but is not bound is said to be **free**.

Predicate logic

A predicate can contain more than one variable;

- a predicate P with two variables, x and y for example, can be written $P(x, y)$.
- In general, a predicate with n variables, $x_1, x_2, \dots x_n$, can be written $P(x_1, x_2, \dots x_n)$.

The quantifiers **'for all'** and **'there exists'** are denoted by the symbols ∀ and ∃ respectively. With this notation, expressions containing predicates and quantifiers can be written symbolically.

- The symbol ∀ is called the **universal quantifier.**
- The symbol ∃ is called the **existential quantifier.**

Predicate logic

Example In the specification of a system for booking theatre seats, $\boldsymbol{B(p,s)}$ denotes the predicate 'person *p* has booked seat *s*'. Write the following sentences in symbolic form:

a) Seat $\boldsymbol{s}$ has been booked.
b) Person $\boldsymbol{p}$ has booked a (that is, at least one) seat.
c) All the seats are booked.
d) No seat is booked by more than one person.

Solution

a) $\exists \boldsymbol{p} \boldsymbol{B}(\boldsymbol{p},\boldsymbol{s})$

b) $\exists \boldsymbol{s} \boldsymbol{B}(\boldsymbol{p},\boldsymbol{s})$

c) $\forall \boldsymbol{s} \exists \boldsymbol{p} \boldsymbol{B}(\boldsymbol{p},\boldsymbol{s})$

d) If no seat is booked by more than one person, then $\boldsymbol{B(p,s)}$ and $\boldsymbol{B(q,s)}$ cannot both be true unless *p* and *q* denote the same person. In symbols: $\forall \boldsymbol{s} \forall \boldsymbol{p} \forall \boldsymbol{q} \{[\boldsymbol{B}(\boldsymbol{p},\boldsymbol{s}) \wedge \boldsymbol{B}(\boldsymbol{q},\boldsymbol{s})] \rightarrow (\boldsymbol{p} = \boldsymbol{q})\}$

Predicate logic

Applying **not** to a proposition is called **negating** the proposition.

- $\neg[\forall x P(x)] \equiv \exists x[\neg P(x)]$
- $\neg[\exists x P(x)] \equiv \forall x[\neg P(x)]$

Example Write down the negation of the following proposition:

‘For every number x there is a number y such that $y < x$’.

Solution Write the negation in symbols and simplify it using the laws of logic:

$\forall x \exists y(y < x)] \equiv$	$\exists x\{\neg[\exists y(y < x)]$
	$\exists x \forall y[\neg(y < x)]$
	$\exists x \forall y(y \geq x)$

Write the answer as an English sentence:

‘There is a number x such that, for every number $y, y \geq x$’.