R.C. Lyndon's theorem

Lecture 13

Contents

IDENTITIES IN TWO-VALUED CALCULI
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Introduction.

Classical two-valued logic is ordinarily treated as a **deductive system**, with certain **propositions** (or prepositional functions) given as axioms, from which all true propositions are derivable by substitution and **modus ponens**.

Essentially the same calculus may be treated as an **algebraic system (Boolean algebra),** in which the axioms are now certain **equations** (or identities), from which all true equations are derivable by substitution and the usual rules for equality.

Let $f_1, \dots, f_n$ be any set of Boolean functions; then the algebraic theory having $f_1, \dots, f_n$ as primitive operations possesses a finite complete set of axioms.

Preliminaries

Preliminaries.

- In view of the elementary nature of our proofs, we permit ourselves a certain informality of expression.

- By an algebra, A, we mean a certain set of **elements**, a_i, together with a set of **primitive functions** f_j (of various numbers of arguments) defined for all sets of arguments from A and assuming values in A.

- Any function compounded out of the primitive functions in the usual sense will be said to belong to, and two such functions will be called equal if their values agree for all sets of arguments.

- It will be convenient further to identify a "reducible" function, that does not effectively depend upon certain of its arguments, with the corresponding function of the remaining arguments.

Preliminaries

- Thus, in Boolean algebra, the reducible function $f(x) = x \vee \jmath$ of a single argument will be identified with the constant function 1.

- A complete set of **axioms** for A is a set of (true) equations

$$\phi_1(x_1, \ldots, x_{n_1}) = \psi_1(x_1, \ldots, x_{m_1}), \ldots,$$
$$\phi_k(x_1, \ldots, x_{n_k}) = \psi_k(x_1, \ldots, x_{n\,k}), \ldots,$$

where the x_i are indeterminates and the ϕ_i and ψ_k functions belonging to A, from which all identically true equations of the same form are derivable by means of the following rules:

(I) reflexivity and symmetry of equality;

(II) uniform substitution for a variable x_i in any established equation;

(III) given $\phi = \psi$, substitution of ϕ for ψ at any occurrence in an established equation.

Preliminaries

- An algebra is **axiomatizable** if it possesses a finite complete set of axioms.

- Two algebras A and A' will be called **equivalent** if they possess the same elements and the same functions.

- The primitives of one must then be definable in terms of those of the other, whence we have the following theorem.

Theorem 1. If A and A' are equivalent, then A is axiomatizable if and only if A' is.

For the present purpose it is thus not necessary to distinguish between equivalent algebras.

Thus we avoid certain trivial complications. In particular, every algebra is equivalent to an algebra none of whose primitives is a reducible function, nor the identity function $f(x) = x$.

Connection with deductive systems

- In an algebra A, let a certain element 1 be "designated" as "true," and let a certain function of two arguments be called the "conditional" and written multiplicatively as xy.

- Under certain restrictions, A, as a "logical matrix," will define a logical calculus $L(A)$, with the rule of inference:

If x and xy are theorems, then y is a theorem.

Connection with deductive systems

Theorem 2. Suppose that $L(A)$, as a deductive system, has a complete set of axioms : $\alpha_1, \dots, \alpha_n$.

Suppose further that in A the conditional function and the single designated element satisfy the following identities:

$$xx = 1 \qquad (1)$$

$$1x = x \qquad (2)$$

$$(xy)y = (yx)x, \quad \textbf{(3)}$$

where x and y represent arbitrary elements of A.

Then A, as an algebraic system, possesses a complete set of axioms consisting of (I), (2), and (3) together with the equations $\alpha_1 = 1, \dots, \alpha_n = 1$.

Connection with deductive systems

Proof.

Clearly the equations listed are true in A.

- If f is an axiom of L, then $f = 1$ among the axioms listed for A.

- If f is a theorem of L, and $f = 1$ has been derived from the axioms for A by the rules I, II, and III, then $f' = 1$ follows in A for any f' obtained from f by uniform substitution.

- Suppose f and f_g are theorems of L, and that $f = 1$ and $f_g = 1$ have been derived from the axioms for A.

- Then substitution (in accordance with III) gives $1_g = 1$, whence from (2) it follows that $g = 1$ is derivable from the axioms for A. This proves, recursively, that if f is a theorem of L, then $f = 1$ follows from the axioms for A.

Connection with deductive systems

- Suppose now that $f = g$ is true in A. By (1) it follows that $fg = 1$ and $gf = 1$ are true equations, whence fg and gf must be theorems of L.

- Therefore $fg = 1$ and $gf = 1$ are derivable from the axioms for A.

- But (3) gives $(gf)f = (fg)g$, whence by substitution $1f = 1g$, and applying (2) now gives $f = g$.

- Thus $f = g$ is derivable from the axioms for A.

Connection with deductive systems

■ **Remark.**

That the three equations (1), (2), (3) in A cannot be replaced by the six corresponding conditionals in $L(A)$ is shown by the three-valued calculus with xy defined by the accompanying table; for in this calculus the six conditionals hold while equation (2) fails.

xy	1	2	3
1	1	3	3
2	1	1	1
3	1	1	1

Connection with deductive systems

- Note that the class of "designated" elements and the selected "conditional" function play no special role in A as an algebra.

- Thus the correspondence between logic and algebra is not unique in either sense.

- This ambiguity may be exploited to deduce, from Rosser and Turquette's deductive axiomatization of the Lukasiewicz-Tarski n -valued calculi (with an arbitrary number of designated values), the following result:

Connection with deductive systems

Corollary 2.1.

The algebra defined by the matrix for each of the Lukasiewicz-Tarski n -valued logics is axiomatizable.

Henkin has shown that every two-valued logic containing the classical conditional function is deductively axiomatizable.

By Theorem 2 we conclude:

Corollary 2.2.

Every two-element algebra containing the classical conditional function is (algebraically) axiomatizable.