Post's iterative systems

- An algebra A containing just two elements, which we shall designate as **0** and **1**, constitutes what Post has called a twovalued iterative system.
- Post has enumerated all such algebras, and we repeat below what is essentially his enumeration.
- However, in accordance with Theorem 1, we list only one out of each set of equivalent algebras.
- Also, we omit those systems with only constant functions, which are vacuously axiomatizable.
- Finally, we define the **dual** of a function f to be the function obtained from f under the interchange of the two elements **0** and **1** of A.
- The dual of an algebra A is the algebra whose functions are precisely the duals of those of A. Since an algebra is isomorphic to its dual, we include in our list only one out of each pair of duals.

Post's iterative systems

A two-valued algebra is fully described by listing a set of primitive functions. For this purpose we employ the following notation :

0 and **1** for the two (dual) constant functions;

Nx for the self-dual function of complementation (or negation) ;

$x \vee y$ for the union (maximum) function, and $x \wedge y$, or simply xy, for the dual intersection (minimum) function;

$x \equiv y$ (equivalence) and its dual $x + y$ (symmetric difference);

$x \supset y$ (conditional) and its dual $x - y$ (set difference: xNy);

$x + y + z$, self-dual;

$(x, y, z) = x(y \vee z), [x, y, z] = x(y \equiv z)$, and, for each $n \geq 3$,

$d_n(x_1, \ldots, x_n) = x_2x_3 \ldots x_n \vee x_1x_3 \ldots x_n \vee \cdots \vee x_1 \ldots x_{n-2}x_{n-1}$;

we shall not require a notation for the duals of these functions.

Post's iterative systems

In listing the two-element algebras, we first give the name of the algebra (a capital letter with subscript) in Post's classification; next, a set $(f_1, \dots f_n)$ of primitive functions; and thirdly (in certain cases) a fuller equivalent set of primitive functions.

For future convenience, we divide our list into five sections.

Ia. $O_4 = (N),\ O_9 = (N, 0)$;

$S_1(\vee),\ S_4 = (\vee, 0),\ S_3 = (\vee, 1),\ S_6 = (\vee, 0, 1)$;

$A_4 = (\vee, \wedge),\ A_2 = (\vee, \wedge, 0),\ A_1(\vee, \wedge, 0, 1)$;

$L_3 = (+) = (+, 0),\ L_1 = (+, N) = (+, N, 0, 1)$;

$C_3 = (-, \vee) = (+, \wedge, 0),\ C_1 = (-, N) = (+, \wedge, 0, 1)$.

Ib. $L_4 = (x + y + z),\ L_5 = (x + y + z, N)$.

II. $F_4 = (\supset)$, and $F_4^n = (\supset, d_n)$ for each $n \geq 3$.

Post's iterative systems

III. $F_6 = \big((x,y,z)\big) = \big((x,y,z),xy\big);$

$F_7 = \big((x,y,z),0\big) = ((x,y,z),xy,0)\,;$

$F_5 = ([x,y,z]) = ([x,y,z],(x,y,z),xy)\,;$

$C_4 = ([x,y,z],x \vee y) = ([x,y,z],(x,y,z),xy,x \vee y)\,;$

$F_6^n = \big((x,y,z),d_n\big) = ((x,y,z),d_n,xy)$ **for each** $n \geq 3$**;**

$F_7^n = \big((x,y,z),d_n,0\big) = ((x,y,z),d_n,0,xy)$ **for each** $n \geq 3$**;**

(Note that, for $n > 3, F_6^n = (d_n)$ ***and*** $F_7^n = (d_n,0).)$

$F_5^n = ([x,y,z],d_n = ([x,y,z),(x,y,z),d_n,xy)$ **for each** $n \geq 3$**.**

IV. $D_2 = (d_3),\ D_1 = (d_3,x+y+z),\ D_3 = (d_3,x+y+z,N).$

Systems I

- For each of the systems **Ia** a complete set of axioms can be chosen by inspection from the various familiar sets of axioms for Boolean algebras and Boolean rings.
- Completeness can be proved by showing that the chosen set of axioms serves to reduce every expression to a prescribed normal form, and that distinct normal forms represent distinct functions.
- The same method applies to systems **Ib**.

Systems I

For example, if we temporarily abbreviate $x + y + z$ to xyz, system L_5 has the following set of axioms:

$$NNx = x,\ N(xyz) = (Nx)yz,\ \ xyy = x,$$

$$xyz = yxz,\ \ xyz = xzy,\ \ xy(zuv) = (xyz)uv.$$

Completeness is established by reference to the normal forms

$$a, Na, ab(cd(\ \dots\ (pqr)\dots)), N(ab(cd(\ \dots\ (pqr)\dots))),$$

where $a, b, \dots, p, q, r$ are distinct variables in alphabetical order.

Systems II

- The axiomatizability of these systems, which all contain the conditional, follows from Corollary 2.2.
- Alternatively, for the dual systems, which contain $x — y$ and xy, a proof paralleling that for systems III can be given, in terms of representations by maximal dual ideals.

Systems III

Observe that all of the systems III contain the connectives (x, y, z) and xy.

Theorem 3. Let the algebra A with primitive connectives (x, y, z) and xy satisfy the axioms

$$xx = x, \quad xy = yx, \quad x(yz) = (xy)z, \quad (x, y, y) = xy,$$

$$(x, x, y) = x, \quad (x, y, z) = (x, z, y), \quad (x, y, z) = (x, xy, z), \quad \mathfrak{A}$$

$$w(x, y, z) = (wx, y, z), \text{ and } w(x, y, z) = (x, wy, wz).$$

Then there exists a one-to-one mapping : $x \to \bar{x}$, of A into an algebra $\bar{A}$ of sets, such that

$$\overline{(x, y, z)} = \bar{x}(\bar{y} \vee \bar{z}) \text{ and } \overline{(xy)} = \bar{x}\,\bar{y}.$$

To prove this theorem, we first define $x \sqsubset y$ to mean $xy = x$.

Systems III

It then follows that $x \sqsubseteq x$; that $x \sqsubseteq y$ and $y \sqsubseteq x$ imply $x = y$; and that $x \sqsubseteq y$ and $y \sqsubseteq z$ imply $x \sqsubseteq z$.

An **ideal** in A is defined to be any subset $S \neq A$ satisfying

(1) if $x \sqsubseteq y$ and x is in S, then y is in S,

(2) if x and y are in S, then xy is in S.

An atom in A is an ideal S with the further property

(3) if (x, y, z) is in S, then either xy or xz is in S.

Systems III

Lemma. If $x \subset y$ does not hold, then there exists an atom containing x but not y.

- To prove the lemma, we first observe that the set S_0 of all z such that x □ z is an ideal containing x but not y.
- We shall show that every ideal with this property, if it is not already an atom, can be extended to a larger ideal with the same property.
- Since the union of an ascending chain of ideals with this property is clearly an ideal with the same property, it will follow by Zorn's lemma that there exists a maximal ideal with this property, which must therefore be an atom containing x but not y.

Systems III

- Let S be an ideal, but not an atom, containing x but not y.

Then by definition S contains some (u, v, w) while neither uv nor uw is in S.

Suppose there existed p and q in S such that $puv \sqsubset y$ and $quw \sqsubset y$.

It would follow that $ruv \subset y$ and $ruw \subset y$, where $r = pq$ was in S. Hence

$$yr(u, v, w) = (u, yrv, yrw) = (u, uyrv, uyrw) = (u, ruv, ruw) = (u, rv, rw) = r(u, v, w),$$

that is, $r(u, v, w)$ □ y.

Systems III

- Then, since r and (u, v, w) were in S it would follow that $r(u, v, w)$ was in S, and, since $r(u, v, w) \subset y$, that y was in S, contrary to hypothesis.

Thus we may suppose, by symmetry, that puv □ y holds for no p in S.

The set S of all z such that $puv \subset z$ is clearly an ideal properly containing S, and hence x, but not y.

This completes the proof of the lemma.

Systems III

Define $\overline{x}$ to be the set of all atoms that contain x.

From (1) it follows that x □ y implies $\overline{x} \subset \overline{y}$.

- The lemma shows that if not x □ y, then not $\overline{x} \subset \overline{y}$. Since the mapping $x \rightarrow \overline{x}$ preserves inclusion, it is one-to-one.
- That $\overline{(xy)} = \overline{x}\,\overline{y}$ follows from (1) and (2), if one notes that xy □ x and $xy \subset y$.

It remains to show that $\overline{(x, y, z)} = \overline{x}\,(\overline{y} \vee \overline{z})$.

- First, let (x, y, z) be in S, an atom. Then

$(x, y, z) = (x, y, z)(x, y, z) = ((x, y, z)x, y, z) = ((x, y, z), y, z$

whence by (3) either $(x, y, z)y = (xy, y, z) = (xy, xy, xyz)xy$ is in S, or else $(x, y, z)z = xz$ is in S, and in either case S is in $\overline{(xy)}$ □ $\overline{(xz)} = \overline{x}\,\overline{y} \vee \overline{x}\,\overline{z} = \overline{x}(\overline{y} \vee \overline{z})$.

Systems III

- Conversely, if S is in $\bar{x}(\bar{y} \vee \bar{z})$ we may suppose, by symmetry, that S is in $\bar{x}\,\bar{y} = \overline{(xy)}$;

then $xy(x, y, z) \doteq (xy, xy, xyz) \doteq xy$ implies that

xy □ (x, y, z), so that xy in S implies that (x, y, z) is in S, that is, that S is in $\overline{(x, y, z)}$.

This completes the proof of the theorem.

Systems III

Theorem 4. The axioms $\mathfrak{A}$ form $\boldsymbol{a}$ complete set for the algebra $\boldsymbol{F}_6$.

Proof.

Let $\boldsymbol{A}$ be the free denumerably generated algebra with primitives $\boldsymbol{xy}$ and $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ subject to axioms $\mathfrak{A}$, and let $\overline{\boldsymbol{A}}$ be the isomorphic algebra of sets.

Every identity of the two-element algebra $\boldsymbol{F}_6$ holds also in $\overline{\boldsymbol{A}}$, as a subalgebra of a direct product of replicas of $\boldsymbol{F}_6$. Thus every identity of $\boldsymbol{F}_6$ holds in the free algebra $\boldsymbol{A}$, and so is a consequence of the axioms $\mathfrak{A}$.

Systems III

Theorem 5. Each of the algebras $\boldsymbol{F_7, F_5, C_4, F_6^n, F_7^n}$ and $\boldsymbol{F_5^n}$ is axiomatizable.

Proof. Each of these algebras can be obtained by adjoining further primitives to $\boldsymbol{F_6}$.

To extend the result obtained for $\boldsymbol{F_6}$ it must be shown in each case that adjoining a finite number of new axioms to the set $\mathfrak{A}$ will ensure that the new primitives are properly represented in $\overline{\boldsymbol{A}}$.

- For the algebra $\boldsymbol{F_7}$, with the additional primitive **0**, it suffices to adjoin the single additional axiom $\mathfrak{A}_7$: $\mathbf{0}\boldsymbol{x} \doteq \mathbf{0}$.

That $\overline{\mathbf{0}}$ is indeed the empty set in $\overline{\boldsymbol{A}}$ follows from the fact that **0** in $\boldsymbol{S}$, for an atom $\boldsymbol{S}$, would imply by (1) that all $\boldsymbol{y}$ were in $\boldsymbol{S}$, contrary to the requirement $\boldsymbol{S} \neq \boldsymbol{A}$.

Systems III

- For F_5, with additional primitive $[x, y, z]$, we adjoin the additional axioms $\mathfrak{A}_5$:

$$[x, y, z] = [x, z, y], \qquad x[x, y, z] = [x, y, z],$$
$$y[x, y, z] = xyz,$$
$$(x, (x, y, z), [x, y, z]) = x.$$

- Suppose S is in $\overline{[x, y, z]}$; then $[x, y, z]$ in S and $[x, y, z] \square x$ implies x is in S.
- If neither y nor z is in S, then S is in $\overline{x}(\overline{y} \equiv \overline{z})$ as required.

Otherwise we may suppose that y is in S, whence $y\,[x, y, z] = xyz$ is in S, so also z is in S, and again S is in $\overline{x}(\overline{y} \equiv \overline{z})$.

- For the converse, suppose that S is in $\overline{x}(\overline{y} \equiv \overline{z})$.
- If S is in $\overline{(xyz)}$, it follows from $xyz[x, y, z] = xz(xyz) = xyz$ that $xyz \subset [x, y, z]$ and so $[x, y, z]$ is in S as required.

Systems III

- Otherwise x is in S but neither y nor z is in. S
- By (3), that $x = (x, (x, y, z), [x, y, z])$ is in S implies that either $x(x, y, z) = (x, y, z)$ or $x[x, y, z] = [x, y, z]$ is in S.
- Since, by (3), (x, y, z) in S would imply that either y or z were in S, it must be that $[x, y, z]$ is in S.
- For C_4, with additional primitive $x \square y$, adjoin the further axioms $\mathfrak{A}_4$:

$$x \vee y = y \vee x, \qquad x(x \bar{\vee} y) = x, \qquad (x \square y, x, y) = x \square y.$$

If $x \vee y = (x \vee y, x, y)$ is in S, it follows by (3) that either $(x \vee y)x = x$ or $(x \vee y)y = y$ is in S, so S is in $\bar{x} \vee \bar{y}$.

- Conversely, if either x or y is in S, it follows from $x \sqsubset x \veebar y$ and $y \sqsubset x \square y$ that $x \square y$ is in S.

F_6^n, for $n \geq 3$, contains the additional primitive d_n.

Systems III

- Abbreviate

$(x, y_1, \dots, y_n) = (x, (x, \dots (x, (x, y_1, y_2), y_3), \dots, y_{n-1}), y_n)$

and write x^i for $x_1 \dots x_{i-1} x_{(i+1)} \dots x_n,$ and d_n for $d_n(x_1 \dots, x_n)$.

Adjoin the following finite set of further axioms:

$\mathfrak{S}_n$: axioms expressing that $d_n(x_1, \dots, x_n)$ is invariant under any permutation of its arguments;

$\mathfrak{D}_n$: $d_n(x_1, \dots, x_n) = (d_n(x_1, \dots, x_n), x^1, \dots, x^n),$

$\mathfrak{D}'_n$: $x^1 d_n(x_1, \dots, x_n) \doteq x^1.$

From $\mathfrak{D}'_n$ with $\mathfrak{S}_n$ it follows that d_n is in S whenever any x^i is in S.

Systems III

- For the converse, suppose that d_n is in S.
- Since $d_n = (d_n, x^1, \dots, x^n)$, by (3) either

$d_n(d_n, x^1, \dots, x^{n-1}) = (d_n, x^1, \dots, x^{n-1})$ is in S or else $d_n x^n = x^n$ is in S.

- If x^n is in S, then S is in $\overline{x}^1 \vee \cdots \vee \overline{x}^n$ as required.

Otherwise from $(d_n, x^1, \dots, x^{n-1})$ in S we conclude by (3) again that either $(d_n, x^1, \dots, x^{n-2})$ or x^{n-1} is in S.

Continuing thus, either some one of $x^n, \dots, x^3$ is in S, or else (d_n, x^1, x^2) is in S, whence either $d_n x^1 = x^1$ or $d_n x^2 = x^2$ is in S.

In any case, S is in $\overline{x}^1 \vee \cdots \vee \overline{x}^n$ as required.

Finally, for F_7^n it evidently suffices to adjoin the axiom $\mathfrak{A}_7$ to those for F_6^n; and for F_5^n to adjoin the axioms $\mathfrak{A}_5$ to those for F_6^n.

Systems IV

Theorem 6. The algebra $\boldsymbol{D_2}$ is axiomatizable.

$\boldsymbol{D_2}$ is defined by the single primitive $\boldsymbol{d(x, y, z) = xy \square xz \square yz}$.

- Let $\boldsymbol{A}$ be the free algebra on a denumerable set of generators $\boldsymbol{a, x, y, \ldots}$ subject to the same set of identities as $\boldsymbol{D_2}$. Fixing the generator $\boldsymbol{a}$, introduce the definitions

(Δ) $\boldsymbol{x \wedge_a y = d(a, x, y)}$, $\boldsymbol{(x, y, z)_a = x \square_a d(x, y, z)}$,
$\boldsymbol{0a = a}$.

- Let $\boldsymbol{A_a}$ a be the algebra with the same elements as $\boldsymbol{A}$, but with primitive operations

$\boldsymbol{d(x, y, z), x \square_a y, (x, y, z)_a}$, and $\boldsymbol{0_a}$.

Systems IV

- Let A_0 be the free algebra of type F_7^3, with primitives $d(x, y, z)$, $x \square y$, (x, y, z), and $\mathbf{0}$, on the generators $x, y, \ldots$

Then the mapping $x \Rightarrow x \square N_a$ of the underlying Boolean algebras clearly establishes an isomorphism of A_0 onto A_a.

- Let $\mathfrak{A}_0$ be a finite set of axioms for F_7^3, and so for A_0.
- Let $\mathfrak{A}_a$ be the corresponding axioms for the isomorphic algebra A_a.
- Using **(Δ)** to eliminate defined operations, we obtain from $\mathfrak{A}_a$ a set of equations $\mathfrak{A}$ expressed in the variables

  $a, x, y, \ldots$ and the primitive d, of the algebra A.

Systems IV

- If ϕ is any expression of A, substituting 0_a for a yields an expression ϕ_a in the notation of A_a.
- If ϕ_0 is the expression of A_0 corresponding to ϕ_a under the isomorphism of A_0 onto A_a, we see that formally ϕ_0 is obtained by substituting 0 for a in ϕ.

- In the full notation of Boolean algebra, let ϕ_1 be the dual of ϕ_0; since d is self-dual, we see that ϕ_1 is equivalent to the formal result of substituting 1 for a in ϕ , whence we have the identity

$$\phi = \phi_1 a \, \square \, \phi_0 N a$$

Systems IV

- Now suppose $\phi \doteq \psi$ is one of the equations of $\mathfrak{A}$. This means that $\phi_a \doteq \psi_a$ was one of the axioms $\mathfrak{A}_0$ of A_0, whence $\phi_0 \doteq \psi_0$ and its dual $\phi_1 = \psi_1$, are Boolean identities. From (H) it follows that $\phi = \psi$ p is a Boolean identity. This shows that all the equations $\mathfrak{A}$ are true in A.

- For the converse, let $\phi = \psi$ be any true equation in the notation of A. Then, setting $a = 0$, the equation $\phi_0 = \psi_0$ is true in A0, and hence a consequence of the axioms $\mathfrak{A}_0$ for A_0.

Systems IV

- Then $\phi_a = \psi_a$ is a consequence of the axioms $\mathfrak{A}_\mathbf{a}$ in the isomorphic algebra A_a. Eliminating the defined operations by (Δ) , it follows that $\phi = \psi$ p is a consequence of the axioms $\mathfrak{A}$ for A.

- This completes the proof that D_2 is axiomatizable. An obvious modification of this argument establishes the axiomatizability of the two remaining systems, D_1 and D_3.