IDENTITIES IN FINITE ALGEBRAS

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- A finite algebraic system $\mathcal{A}$ will be constructed, with the property that the set of all laws of $\mathcal{A}$ is not a consequence of any finite subset of these laws.
- $\mathcal{A}$ is of order seven, with a binary (non-associative) operation written as multiplication, and a **0**-ary operation (or constant element) **0**. Denoting the elements of $\mathcal{A}$ by $\mathbf{0}, e, b_i, b_2, c, d_1, d_2$, the multiplication is defined by setting

$$ce = c, \qquad cb_j = d_j, \qquad d_je = d_j, \qquad d_jb_k = d_j$$

$(j, k = 1, 2)$ and setting all other products equal to **0.**

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Lemma 1. The following laws hold in $\boldsymbol{A}$:

$(A123)$: $\quad 0x = 0, xO = 0, x(yz) = 0;$

(B_n): $\quad \Big(\big(\ldots((x_1x_2)x_3)\ldots\big)x_n\Big)x_1 = 0 \ (n = 1, 2, \ldots);$

(C_n): $\quad \Big(\big(\ldots((x_1x_2)x_3)\ldots\big)x_n\Big)x_2 = \big(\ldots((x_1x_2)x_3)\ldots\big)x_n$

$(n = 1, 2, \ldots)$.

Moreover, every law of $\mathcal{A}$ that contains no more than n indeterminates is a consequence of $A123$ and of B_m, C_m

for $m \leq n$.

Proof.

Verification that $A123, B_n$ and C_n hold identically in $\mathcal{A}$ is routine.

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- For the second assertion, let $\boldsymbol{\phi_\alpha = \phi_\beta}$ be any law of $\mathcal{A}$ containing at most $\boldsymbol{n}$ indeterminates.
- By $\boldsymbol{A}$ we may suppose that each member is either $\mathbf{0}$ or a left-associated product, which we abbreviate as

$$\boldsymbol{\phi_\alpha = x_{\alpha_1} x_{\alpha_2} \dots x_{\alpha_\mu}, \quad \phi_\beta = x_{\beta_1} x_{\beta_2} \dots x_{\beta_\mu}}$$

- By $\boldsymbol{B_n}$ and $\boldsymbol{C_n}$ for $\boldsymbol{m \leq n}$, we can eliminate any repetition among the first $\boldsymbol{n+1}$ arguments of $\boldsymbol{\phi_\alpha}$ or of $\boldsymbol{\phi_\beta}$, whence we may suppose that $\boldsymbol{\mu, \nu \leq n}$ and that neither product contains repetitions.
- If $\boldsymbol{\phi_\alpha}$, say, is not $\mathbf{0}$, then setting $\boldsymbol{x_{\alpha_1} = c, x_{\alpha_2} = \cdots = x_{\alpha_\mu} = e}$, and any other $\boldsymbol{x = 0}$, gives $\boldsymbol{\phi_\alpha = c}$; it follows that $\boldsymbol{\phi_\beta}$ is not $\mathbf{0}$, and indeed that $\boldsymbol{x_{\beta_1} = x_{\alpha_1}}$, and that all the indeterminates of $\boldsymbol{\phi_\beta}$ are among those of $\boldsymbol{\phi_\alpha}$.

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- By symmetry, $\mu \doteq \nu$ and the x_{α_i} are a permutation of the x_{β_i} .
- If some x_1 preceded x_2 in ϕ_α, while x_2 preceded x_1 in ϕ_β (therefore $x_{\alpha_i} \neq x_1, x_2$), then setting $x_{\alpha_1} = c, x_1 = b_1, x_2 = b_2$ and all other $x = e$ gives $\phi_\alpha \doteq d_1$ and $\phi_\beta \doteq d_2$.
- Since this contradicts $\phi_\alpha \doteq \phi_\beta$, the x_{β_i} must be the identical permutation of the x_{α_i}, whence ϕ_α and ϕ_β are formally identical.
- Define $\psi_\alpha \doteq x_{\alpha_1} x_{\alpha_2} \ldots x_{\alpha_\mu}$ to have property P_n if ψ_α contains n indeterleast once to the left of the second minates, and in the product ψ_α each of them occurs at occurrence of x_{α_i}.
- **Note** that this implies that if x_{α_k} is a later occurrence of x_{α_1} then $k > n$.

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Lemma 2. If $\psi_\alpha = \psi_\beta$ is a formal consequence of the laws A and B_n , C_n for $m < n$, and if ψ_α has the property P_n, then ψ_β also has the property P_n.

Proof. It is sufficient to consider the case where $\psi_\alpha = \psi_\beta$ results from a single application of A, B_n , or C_n .

Now, no application of A is possible, since ψ_α contains no part **0** or of the form $u(vw)$.

For an application of B_m or C_m we may suppose, by symmetry, that ψ_α contains a part of the form $y_1 y_2 \dots y_m y_d$, where $y_d = y_1$ in the case of B_m and $y_d = y_2$ in the case of C_m.

Clearly $y_1 = x_{\alpha_1} \dots x_{\alpha_k}$ for some $k \geq 1$, and therefore

$$y_2 = x_{\alpha_{k+1}}, \dots, y_d = x_{\alpha_{k+m}} .$$

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For B_n , $y_d = y_1 = x_{\alpha_1} \dots x_{\alpha_k}$ is clearly impossible for $k > 1$; while if $k = 1$, $y_d = x_{\alpha_{n+1}}$ is a later occurrence of $y_1 = x_{\alpha_1}$, impossible by property P_n since $m + 1 \leq n$.

Therefore no B_n , $m < n$, is applicable. Finally, if some C_n is applicable it can only delete an occurrence after the first of some indeterminate other than x_{α_1} , or else an occurrence after the second of x_{α_1}, and hence in either case preserves the property P_n.

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Theorem. If $\boldsymbol{L}$ is any finite set of laws of $\mathcal{A}$, then some one of the laws $\boldsymbol{B_n}$ is not a consequence of $\boldsymbol{L}$.

Proof. The finite set $\boldsymbol{L}$ of laws will contain fewer than $\boldsymbol{n}$ indeterminates, for some finite $\boldsymbol{n}$.

By Lemma 1, any consequence of $\boldsymbol{L}$ is a consequence of $\boldsymbol{A123}$ and $\boldsymbol{B_m}, \boldsymbol{C_m}$ for $\boldsymbol{m < n}$.

Now $\boldsymbol{x_1 x_2 \dots x_n x_1}$, the left member of $\boldsymbol{B_n}$, has property $\boldsymbol{P_n}$, while **0**, the right member, has not.

Therefore, by Lemma 2, $\boldsymbol{B_n}$ is not a consequence of $\boldsymbol{A123}$ and $\boldsymbol{B_m}, \boldsymbol{C_m}$ for $\boldsymbol{m < n}$, whence $\boldsymbol{B_n}$ is not a consequence of $\boldsymbol{L}$.