The Two-Valued Iterative Systems of Mathematical Logic

- By a two-valued truth-function, we may understand simply a function, of which the independent variables range over a domain of two objects, and of which the value of the dependent variable for each set of arguments is taken from the same domain.
- For a given set of n distinct variables as the independent variables, there are exactly 2^{2^n} such functions, each of them being describable by a truth-table which gives the function value for each of the 2^n sets of arguments.
- In the truth-table method, the primitives of the propositional calculus are interpreted by truth-functions; in other words, truth-tables are assigned to them.

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- The assigned functions determine a class of functions which are obtainable from them by repeated use of substitution, that is, by explicit definition, or in Post's terminology by iterative processes:

- If $f(x_{i_1}, \cdots, x_{i_n})$ is a given function of distinct variables $x_{i_1}, \cdots, x_{i_n}$, and $X_1, \cdots, X_n$ are variables and given functions, which in their totality depend on the distinct variables $x_{j_1}, \cdots, x_{j_n}$, then $f(X_1, \cdots, X_n)$ is an iterative process yielding a function of the latter.

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- The functions of this class are the functions which can be represented by formulas of the calculus, on the basis of the assignment of truth-tables to the primitives.

- The class of these functions is called the system of functions generated by the assigned functions as generators.

- It is an iteratively closed system of functions, that is, a non-vacuous class of functions closed under substitution.

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- For the propositional calculus of Principia Mathematica, with the primitives “negation” and “disjunction” and the truth-tables which their meanings suggest, the generated system is the complete system C_1 of all two-valued truth-functions. In general, given a finite list of two-valued truth-functions as generators, we shall obtain as generated iterative system a subsystem of C_1.

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Post posed two problems:

(1) What sets of generators give the complete system $\mathbf{C_1}$ (or what primitives can be taken for the classical calculus of propositions)?

(2) For arbitrary generators, what are the systems generated (or what are the non-equivalent sub-languages of the classical calculus of propositions)?

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- The definition of an iteratively closed system of two-valued truth-functions does not depend on the assumption that there is a finite set of generators. From Post's solution of the second of the problems, there appeared the remarkable result that every iteratively closed system of two-valued truth-functions can be generated by a finite set of generators.

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- The number of distinct independent variables of a function is called its order; the order of a set of functions is the greatest order of any function of the set; and the order of a closed system is the least order of any set of generators of the system.

- Post catalogues the distinct closed systems completely, as follows. There are **9** systems of order one, **37** of order two, **20** of order three, and **8** of each higher order.

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- The cataloguing is accomplished by means of iterative conditions, that is, properties of functions such that, whenever the given functions for an iterative process have the property, so must the resulting function.

- To begin with, it is observed that a first order set of generators generates a system comprising only first order functions; and examination of the **15** non-vacuous subsets of the **4** first order functions of a given variable reveals the **9** first order systems.

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- From any function a first order function is obtained whenever we substitute some one variable for all the independent variables; and for any closed system the functions so obtainable must constitute a closed first order system, called the associated first order system. Thus all closed systems are separated into **9** categories.

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- By imposing on the functions further iterative conditions, such as self-duality, in various combinations, the catalog is eventually completed. Eight of the third order systems, together with the systems of higher order, are obtained as eight infinite families of systems $\boldsymbol{F_i^{\mu}}$ $(\boldsymbol{i} = \mathbf{1}, \cdots, \mathbf{8};\ \mu = \mathbf{2}, \mathbf{3}, \cdots)$, where F_i^{μ} is of order $\mu + 1$ and the parameter $\boldsymbol{\mu}$ appears in the iterative conditions.

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- All iterative conditions employed are effectively applicable, so that, for any given function and any named one of the systems, one can ascertain whether or not the function belongs to the system.

- In the presentation, the author follows the suggestion of the referee of the original version of the paper [l.c.] that the truth-tables be replaced by logical expressions in the Jevons notation (1864).

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- In this a function is represented by expansions, each of which consists of a pair of expressions, of which the first is essentially (in the terminology of Hilbert-Ackermann) a disjunctive normal form of the function (with **0** and **1** also allowed), and the second is the like for the negation of the function.

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- With the aim of combining the precision of his own truth-table method with somewhat the logical context of Jevons, Post treats the functions in this connection as membership functions of classes; for such a function $\boldsymbol{f}(\boldsymbol{B}, \boldsymbol{D}, \cdots, \boldsymbol{H})$, the independent variables $\boldsymbol{B}, \boldsymbol{D}, \cdots, \boldsymbol{H}$ range over the subclasses of a given non-vacuous class (the latter being called the universal class), and each function value $\boldsymbol{f}(\boldsymbol{B}, \boldsymbol{D}, \cdots, \boldsymbol{H})$ is such a class, subject to the general condition on the function that the membership or non-membership of an arbitrary member $\boldsymbol{x}$ of the universal class in the class $\boldsymbol{f}(\boldsymbol{B}, \boldsymbol{D}, \cdots, \boldsymbol{H})$ should depend only on the membership or non-membership of $\boldsymbol{x}$ in each of the classes $\boldsymbol{B}, \boldsymbol{D}, \cdots, \boldsymbol{H}$. The membership functions of classes are abstractly isomorphic with the two-valued truth functions. The iterative conditions employed are mostly stated as formal properties of the expansions.

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- From an expansion of a function, one can obtain its truth table, and vice versa; and the conditions translate readily into terms of the truth-tables. The relationships of inclusion among the closed systems are presented graphically by means of an inclusion diagram, with dual symmetry.

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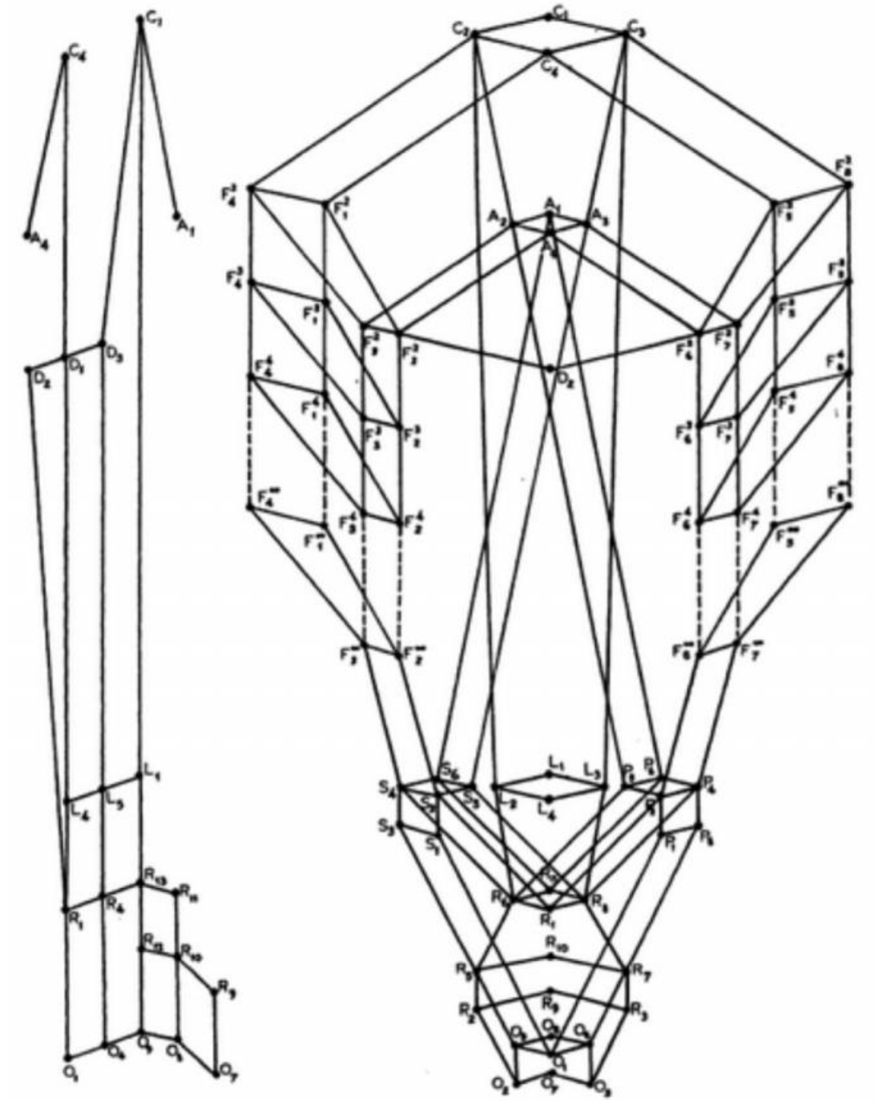
- From this diagram, one can extract much specific information about the closed systems. To solve the first of the original two problems, Post gives:

  (a) necessary and sufficient conditions that a set of functions generate the complete system,

  (b) a classification of such sets subject to the requirement of independence, and

  (c) a catalog of **36** essentially distinct such sets which is complete when a further irredundancy condition is imposed.

The inclusion diagram



The inclusion diagram

- The analysis of the inclusion relation for closed systems thus achieved is presented pictorially in the accompanying inclusion diagram.
- Dual closed systems are represented by points symmetrically placed with reference to a central vertical line, self-dual closed systems by points on this line.
- Due to the inherent difficulties of this requirement of symmetry, only those self-dual closed systems which immediately include, or are immediately included in non-self-dual closed systems are represented in the main diagram, while all self-dual closed systems are separately represented in the supplementary diagram at the left.

The inclusion diagram

- For each pair of closed systems $\boldsymbol{T_0}, \boldsymbol{T_1}$ such that $\boldsymbol{T_1}$ immediately includes $\boldsymbol{T_0}$ we have joined the corresponding points by a straight line. The systems have been so arranged vertically, that this straight line always slopes downward from $\boldsymbol{T_1}$ to $\boldsymbol{T_0}$.
- In the case of two self-dual closed systems $\boldsymbol{T_0}, \boldsymbol{T_1}$ the relationship of immediate inclusion is thus represented only in the supplementary diagram.
- The eight dotted vertical lined serve the double purpose of indicating that the relation of immediate inclusion explicitly represented for systems in the infinite families with $\boldsymbol{\mu} = 2, 3, 4$ be repeated indefinitely, and that the open inclusion chain $\boldsymbol{F_1^2, F_1^3, F_1^4, \ldots, i = 1, 2, \ldots, 8}$, thus resulting immediately includes $\boldsymbol{F_1^\infty}$.

The inclusion diagram

- Now let the supplementary diagram be considered to form part of the main diagram, let all intersections at points other than those representing closed systems be disregarded, and let the diagram be sufficiently extended in the manner indicated by the dotted lines to include all closed systems with $\boldsymbol{\mu}$ not exceeding those of the closed systems about to be mentioned, assuming that the latter belong to the infinite families.
- Then result about inclusion chains has the following diagrammatic equivalent. A closed system $\boldsymbol{T_1}$ properly includes a closed system $\boldsymbol{T_0}$ when and only when there is a broken slopes downward at each point as it run from $\boldsymbol{T_1}$ to $\boldsymbol{T_0}$.
- The inclusion diagram thus completely determines the inclusion relation for closed systems.