Signals and systems class, HSE, Spring 2015. A. Ossadtchi, Ph.D.

Lecture 3

Transforms: basic principles

Introduction

Remember when I demonstrated you that any N-point sequence can be thought of as a vector in the N-dimensional space? This is our starting point for today's lecture. Without loss of generality we will be considering now only the right-handed sequences, i.e. only those that have non-zero entries only for *n*>=0. The *k*-th coordinate of such a vector in the most basic basis set of vectors $\boldsymbol{e}:\{\boldsymbol{e}_k=\delta_N[n-k], k=0,\ldots,N-1\}$ is simply the value of the sequence corresponding to the *k*-th sample.

Therefore, when plotting the sequence in the conventional way as a function of time index *n* we are basically drawing the coordinates $s_k,\ k\in[0,N-1]$ of the sequence in the most trivial basis set $\boldsymbol{e}:\{\boldsymbol{e}_k=\delta_N[n-k], k=0,\ldots,N-1\}$ and therefore this sequence can be represented as

$$s[n]=s_0\delta_N[n]+s_1\delta_N[n-1]+\ldots+s_{N-1}\delta_N[n-N+1]=\sum_{k=0}^{N-1}s_k\delta_N[n-k] \quad (1)$$

or if we think of this sequence as a vector

$$\boldsymbol{s}=s_0\boldsymbol{e}_0+s_1\boldsymbol{e}_1+\ldots+s_{N-1}\boldsymbol{e}_{N-1} \quad (2)$$

with s_k being the actual values of the sequence. C-programmers should appreciate the fact that we index our samples starting with $k=0$!

Note, that when drawing, we follow the order prescribed by the basis set, so that the the earliest sample goes first, i.e. the same way as the vectors in the basis set $\boldsymbol{e}:\{\boldsymbol{e}_k=\delta_N[n-k], k=0,\ldots,N-1\}$ are ordered.

Exercise 3.1
In your homework, you became aware of the folding operation that basically corresponds to flipping the order in which you read off sequence samples. Consider a right-handed sequence with *N* samples, starting from *n* = 0. Let's introduce the folding transform. The image of our original sequence in the folding transform space will be some other sequence (although it is still the same sequence and we just look at it from a different angle). In particular, this image is going to be calculated by simply flipping the order in which we read off the samples.

You don't know this yet, but most of the transforms can be represented by the change of basis and the folding transform is not an exception. Thus, instead of the original basis we will use a new basis set $\boldsymbol{q}$ and call this basis change operation Folding transform. What is this new basis? Express it using the basis vectors from $\boldsymbol{e}:\{\boldsymbol{e}_k=\delta_N[n-k], k=0,\ldots,N-1\}$

So, lets compute the coordinates c_k of vector $\boldsymbol{s}=s_0\boldsymbol{e}_0+s_1\boldsymbol{e}_1+\ldots+s_{N-1}\boldsymbol{e}_{N-1}$ in some other basis $\boldsymbol{b}:\{\boldsymbol{b}_k,\ k\in[0,\ldots,N-1]\}$. To find the coordinates we basically need to look for such a set of coefficients $c_k,\ k\in[0,N-1]$ that the following holds

$$c_0 \boldsymbol{b_0} + c_1 \boldsymbol{b_1} + \ldots + c_{N-1} \boldsymbol{b_{N-1}} = s_0 \boldsymbol{e_0} + s_1 \boldsymbol{e_1} + \ldots + s_{N-1} \boldsymbol{e_{N-1}} \tag{3}$$

Basically, the equation states, that regardless of how we represent our vector $\boldsymbol{s}$ the results should be the same. Why? Imagine we drew a vector on the blackboard. It exists. Then, we added to the plot a coordinate system (two axes) . We can represent this vector as a linear combination of the two unit length vectors and we can also represent it as a sum of any other two vectors. Regardless of which pair of vectors we use use the resulting vector should coincide with the one we've drawn.

To get a full grip of linear algebra power let's write this equation in a more compact form. To do so, let's stack the basis vectors $\boldsymbol{e}_k$ into a matrix $\boldsymbol{E}=[\boldsymbol{e}_{0,} \boldsymbol{e}_{1,} \ldots, \boldsymbol{e}_{N-1}]$ and vectors $\boldsymbol{b}_k$ into a matrix $\boldsymbol{B}=[\boldsymbol{b}_{0,} \boldsymbol{b}_{1,} \ldots, \boldsymbol{b}_{N-1}]$. Also, let's vector $\boldsymbol{s}=[s_{0,} s_{1,} \ldots, s_{N-1}]$ be a vector of expansion coefficients in the original basis and $\boldsymbol{c}=[c_{0,} c_{1,} \ldots, c_{N-1}]$ be the vector of coefficients in the new basis then the equality we require can be written simply as the following matrix equation

$$\boldsymbol{B}\,\boldsymbol{c}=\boldsymbol{E}\,\boldsymbol{s} \tag{4}$$

The way we find vector of coefficients $\boldsymbol{c}$ of vector representation in the new basis $\boldsymbol{b}$ depends on the matrix $\boldsymbol{B}$. If this is a full-rank matrix then the equation has a unique solution and in general, in order compute vector of coefficients $\boldsymbol{c}$ we simply left multiply both sides of equation 4 by the inverse of $\boldsymbol{B}$. In other words

$$\boldsymbol{c}=\boldsymbol{B}^{-1}\boldsymbol{E}\,\boldsymbol{s}=\{since \quad \boldsymbol{E}=\boldsymbol{I}\}=\boldsymbol{B}^{-1}\boldsymbol{s} \tag{5}$$

Clearly, since $\boldsymbol{E}$ is identity the inverse transform is then calculated as

$$\boldsymbol{s}=\boldsymbol{B}\,\boldsymbol{c} \tag{6}$$

Exercise 3.2 (continuation of 3.1)
Take the basis set vectors $\boldsymbol{q}$ you created for the folding transform in exercise 3.1 and stack these vectors into the transform matrix **Q.** What is $\boldsymbol{Q}^{-1}$? Now, take an arbitrary signal, say $s[n]=n(u[n]-u[n-5])$. Verify using equation (5) that the image $S[n]$ of sequence s [n] in the folding transform space coincides with the folded sequence s[n] . Note here, that we introduced this folding transform as an example of a transform. The image S[n] of sequence s[n] is NOT another sequence but another representation of it. Imaging, that we needed this transform to show this original sequence to someone who reads from right to left. So we decided that this person would understand us better if show to him/her not the original sequence but its image in the Folding transform space.

<u>Orthonormal basis case</u>

Orthonormality of the basis set $\boldsymbol{e}$ means that all the vectors in this basis set are mutually orthogonal, i.e. orthogonal projection of one vector onto another is zero ($\boldsymbol{e}_i^T \boldsymbol{e}_j=0, i\neq j$ and of all the vectors are of unit length $\boldsymbol{e}_i^T \boldsymbol{e}_j=1, i=j$) .

Exercise 3.2
Check that $\boldsymbol{e}:\{\boldsymbol{e}_k=\delta_N[n-k], k=0,\ldots, N-1\}$ is an orthonormal basis , i.e. that $\boldsymbol{e}_i^T \boldsymbol{e}_j=0, i\neq j$ and $\boldsymbol{e}_i^T \boldsymbol{e}_j=1, i=j$.

So far we requested only that matrix $\boldsymbol{B}$ is a full rank matrix. This ensures that we can uniquely calculate the transform coefficients $\boldsymbol{c}$ and exactly reconstruct the original signal $\boldsymbol{s}$ based on the transform coefficients . However, quite often, for the practical transforms the transform basis vectors are mutually orthogonal and have unit norm, i.e. $\boldsymbol{b}:\{\boldsymbol{b}_k,\ \boldsymbol{b}_k^H \boldsymbol{b}_l=\delta[k-l], k,l\in[0,...,N-1]\}$. Note that we used the pulse sequence notation to compactly express the condition. This results into matrix $\boldsymbol{B}$ whose columns are orthonormal and whose inverse coincides with the conjugate transpose, i.e. $\boldsymbol{B}^{-1}=conj(\boldsymbol{B})^T=\boldsymbol{B}^H$. This operation of conjugation followed by the transpose is called hermitian transpose. For real matrices hermitian transpose is equivalent to the plain transpose.

This case allows for particularly interesting interpretation of the transform coefficients. Assume that we are starting with the coefficients $\boldsymbol{s}$ of representation of our sequence in the trivial basis and $\boldsymbol{E}=\boldsymbol{I}$. Then, the transform coefficients for some orthonormal transform matrix $\boldsymbol{B}$ are $\boldsymbol{c}=\boldsymbol{B}^{-1}\boldsymbol{E}\,\boldsymbol{s}=\boldsymbol{B}^{-1}\boldsymbol{s}=\boldsymbol{B}^H\boldsymbol{s}=[\boldsymbol{b}_0\boldsymbol{b}_1...\boldsymbol{b}_{N-1}]^H\boldsymbol{s}$. In other words $c_i=\boldsymbol{b}_i^H\boldsymbol{s}=\|\boldsymbol{s}\|\cos(angle(\boldsymbol{b}_i,\boldsymbol{s}))$. Basically, the transform coefficient c_i reflect the similarity of shapes of the *i*-th transform basis vector and the signal $\boldsymbol{s}$. When the shapes are identical then the transform coefficient $c_i=\|\boldsymbol{s}\|\cos(0)=\|\boldsymbol{s}\|$. Since we are dealing with the orthonormal basis and thus $\boldsymbol{b}:\{\boldsymbol{b}_k,\ \boldsymbol{b}_k^H \boldsymbol{b}_l=\delta[k-l], k,l\in[0,...,N-1]\}$ we can also say that if the shape of the signal exactly coincides with that of some transform basis vector then, all the other transform coefficients are zero.

Parseval's relation

Imagine that out signal has only N = 2 samples and thus can be represented as a vector $\boldsymbol{s}=[s_1,s_2]$ in the 2-D coordinate system. Clearly, the square of length of this vector is (according to Pythagorean theorem) is $\|\boldsymbol{s}\|^2=s_1^2+s_2^2$. Extrapolating this to N dimensions for a sequence with N samples we get $\|\boldsymbol{s}\|^2=\sum_{i=0}^{N-1} s_i^2$. Note, that s_i are the coefficients of decomposition of the signal vector in the orthonormal basis (the trivial basis in this case) and it is only due to the fact that the basis vectors are orthogonal we could use the Pythagorean right triangle theorem. Now, consider a set of coefficients $\boldsymbol{c}$ representing our signal in some other orthogonal basis $\boldsymbol{b}:\{\boldsymbol{b}_k,\ \boldsymbol{b}_k^H \boldsymbol{b}_l=\delta[k-l], k,l\in[0,...,N-1]\}$. Since the basis vectors of this new basis are also orthonormal, we can say that the length of our signal vector is also $\|\boldsymbol{s}\|^2=\sum_{i=0}^{N-1} c_i^2$ and therefore $\|\boldsymbol{s}\|^2=\sum_{i=0}^{N-1} c_i^2=\sum_{i=0}^{N-1} s_i^2=\|\boldsymbol{c}\|^2$

$$\sum_{i=0}^{N-1} c_i^2=\sum_{i=0}^{N-1} s_i^2 \tag{7}$$

We could also show the same relation analytically as follows
$\boldsymbol{s}=\boldsymbol{B}\,\boldsymbol{c},\ \sum_{i=0}^{N-1} s_i^2=\boldsymbol{s}^{\boldsymbol{H}}\boldsymbol{s}=\boldsymbol{c}^{\boldsymbol{H}}\boldsymbol{B}^{\boldsymbol{H}}\boldsymbol{B}\,\boldsymbol{c}=\boldsymbol{c}^{\boldsymbol{H}}\boldsymbol{c}=\sum_{i=0}^{N-1} c_i^2$

The interpretation of this theorem states that the energy in the signal, that is the sum of the squares of signal values or the square of the length of the vector representing the N-point sequence in the N-dimensional space, does not dependent on the basis in which the signal is represented, as long as the basis is orthonormal. Refer to figure 1 for a two dimensional illustration of Parseval's theorem.

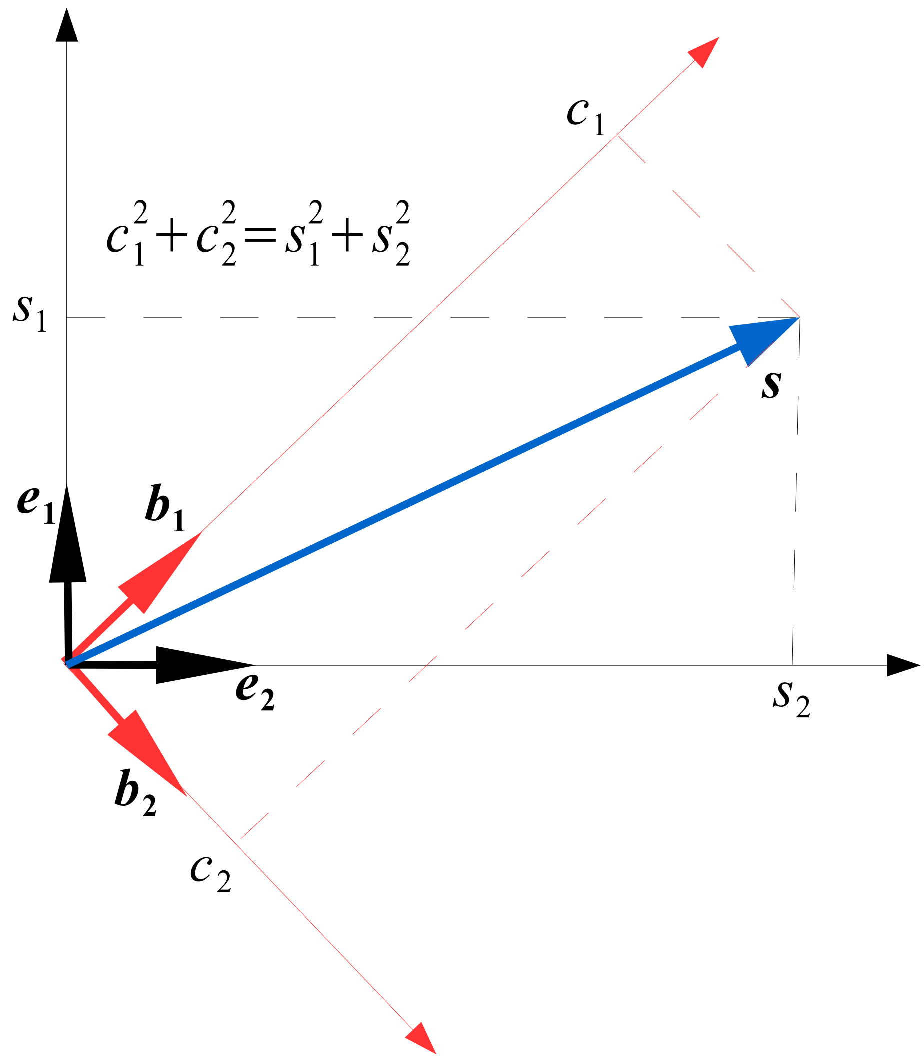

*Figure 1: Two dimensional illustration of Parseval's theorem. The coefficients of decomposition of the signal vector **s** in the orthonormal basis (the trivial basis in this case) are* $[s_{1,} s_2]$ *and the coefficients of decomposition in another basis (red axis) are* $[c_{1,} c_2]$ *. Since these coefficients represent one and the same vector(**s**, blue arrow) and because both of the basis sets are formed by the orthogonal unit length vectors(* $[\boldsymbol{e}_{1,} \boldsymbol{e}_2]$ *and* $[\boldsymbol{b}_{1,} \boldsymbol{b}_2]$ *) we can use the Pythagorean right triangle theorem and write* $c_1^2+c_2^2=s_1^2+s_2^2$ *. Note that in case we had to deal with orthogonal but non unit length vectors we would have to apply the corresponding scaling to the coordinates of our vector **s** to make axis scales equal before we write Parseval's theorem.*

Linearity

We will call an operation L linear when the result of applying the operation to a signal that can be represented as a weighted sum of other signals is equal to the weighted sum(with the same weight) of results of application of L to individual signals. In a more concise and clearer form this statement can be written as

$$L\{a\,\boldsymbol{x}+b\,\boldsymbol{y}\}=aL\{\boldsymbol{x}\}+bL\{\boldsymbol{y}\} \tag{8}$$

This is for the first time in this class you meet a definition of linearity, Note, that it is different from the notion of a linear function, as an operation performing a linear transformation on a signal x according to $a\,\boldsymbol{x}+b$ is a non-linear according to equation (8).

Exercise 3.3
Prove the last statement and show that an operation $L\{\mathbf{x}\}=a\,\mathbf{x}+b,\ \ b\neq 0$ is not a linear operation. To do so, consider two signals $\mathbf{s}_1$ and $\mathbf{s}_2$ and using direct substitution show that $L\{c_1\mathbf{s}_1+c_2\mathbf{s}_2\}\neq c_1 L\{\mathbf{s}_1\}+c_2 L\{\mathbf{s}_2\}$ for the *L* defined above.

<u>Summary</u>

We have considered the idea of transforms in a very simple linear algebra based setting. Given that you have basic background in LA the material presented above should be crystal clear. We considered the idea of transforms in a limited setting of discrete finite length sequences which allowed us to use linear algebra to illustrate the basic principles of transforms. Additionally, in this setting due to finiteness of the length of sequences we did not face convergence issues we would stumble upon should we use the classical presentation. Also, the discrete setting allowed us to use vector based scalar product operation that you all got used to during your LA class.

This discrete finite length presentation of the common principles that lie behind linear transforms leads us directly to the introduction of Discrete Fourier Transform (4), a method to represent a finite length sequence as a linear combination of discrete complex exponential sequences.

Note, that in a more classical presentation that you can find in the numerous Signals and Systems books the DFT would be the last transform considered in the family of Fourier transform methods. It would be presented as the sampled version of the Discrete Time Fourier transform that is in turn derived from the classical continuous time Fourier transform of non-periodic sequence that is considered as a generalization of Fourier Series decomposition ideas to the class of non-periodic signals.

We have consciously chosen this “folded” way of presentation as it allows to develop your intuition behind linear transforms capitalizing on your understanding of the very basic linear algebra concepts.