Signals and systems class, HSE, Spring 2015. A. Ossadtchi, Ph.D.

Lecture 5

Linear time invariant systems and convolution

Introduction

In these two lectures we will temporarily abandon the topic of Discrete Fourier Transform and will look into the time-domain again. We will meet a very special class of systems called linear time invariant systems (LTI) and will study their properties. We will see how to characterize such systems and how to calculate the output signal for an arbitrary input signal. And you will meet a family of functions that behave in a very special way when passed through the LTI systems. Intriguingly, you have already met this family before! Traditionally, we will establish a link between the covered material basic concepts of linear algebra that I hope will further your understanding.

Linearity

As defined before for an operator, we will call a system *L* linear when the reaction of such a system to the input signal that can be represented as a weighted sum of other signals is equal to the weighted sum(with the same weight) of reactions of L to the indivdual signals. In a more concise and clearer form this statement can be written as

$$L\{a\,\boldsymbol{x}(t)+b\,\boldsymbol{y}(t)\}=aL\{\boldsymbol{x}(t)\}+bL\{\boldsymbol{y}(t)\} \quad (1)$$

This is for the second time in this class you meet a definition of linearity, Note, that a system that adds a DC offset to a signal is NOT a linear system.

Exercise 5.1
Prove the last statement and show that a system that reacts to the input signal as $L\{\boldsymbol{x}\}(t)=a\,\boldsymbol{x}(t)+b,\ \ b\neq 0$ is not a linear system. To do so, consider two signals $\boldsymbol{s}_1(t)$ and $\boldsymbol{s}_2(t)$ and using direct substitution show that $L\{c_1\boldsymbol{s}_1(t)+c_2\boldsymbol{s}_2(t)\}\neq c_1L\{\boldsymbol{s}_1(t)\}+c_2L\{\boldsymbol{s}_2(t)\}$ for the *L* defined above.

Time invariance

This property is somewhat similar to the notion of stationarity of a random process. The idea is that a *time invariant* system does not change its properties as the time goes by. In other words it is *invariant* (not changing) to time. Time invariance can be formalized as
$y[n]=L\{x[n]\}\rightarrow L\{x[n-k]\}=y[n-k]$

Moving average system

Linear systems do a bit more than just multiplication of the input signal by some constant, otherwise they would not be of much use and we probably would not study them in this class. An example of a linear system you can easily understand is a system that computes the moving average of the input signal. In order to compute the *n*-th sample of the output sequence *y*[*n*] we take the *n*-th sample of the input sequence and its immediate neighbors on both sides and average these three numbers or equivalently multiply each of the numbers by 1/3 and sum these scaled quantities. Doing so for each *n* we get output of the moving average system *y*[*n*].

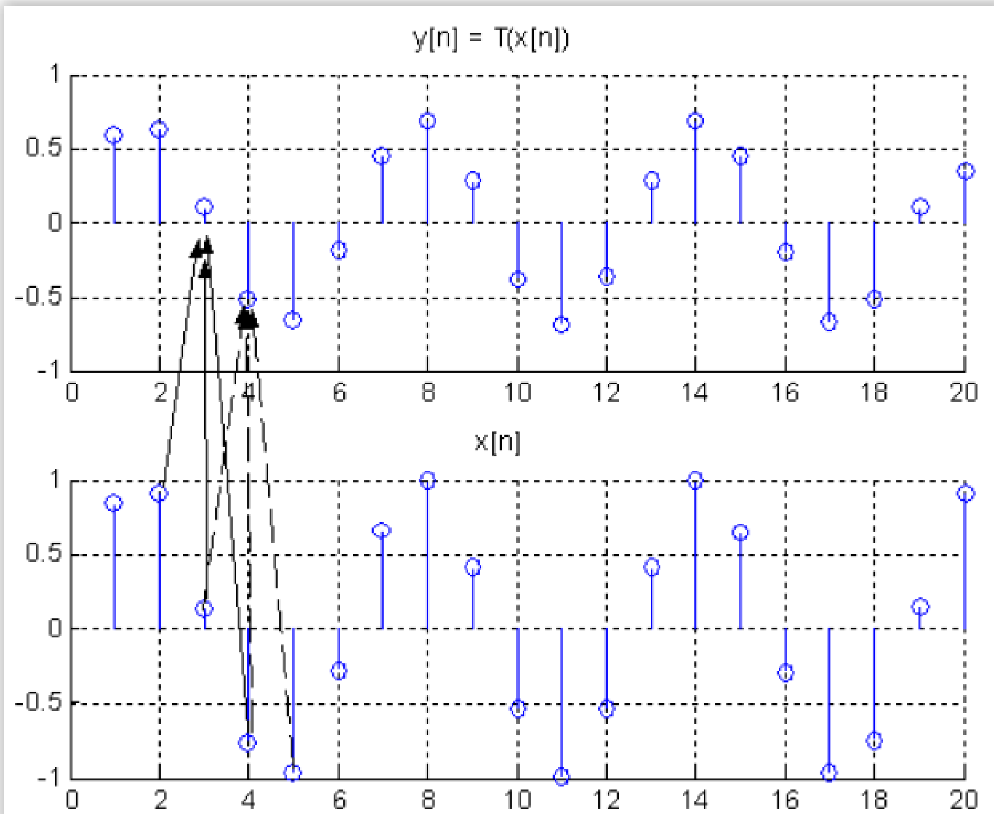

Figure 1: Illustration of the 3-point moving average operation. In order to compute the n-th sample of the output sequence y[n] we take the n-th sample of the input sequence and its immediate neighbors on both sides and averages these three numbers or equivalently multiplies each of the numbers by 1/3 and sums these products. Doing so for each n we get y[n].

So, apparently this operation has reduced the amplitude of our sampled sinusoid as shown in figure 1 but did not really change the shape of it. This is a very interesting and a very useful observation as you will see by the end of the lecture.

As you all already know, any sequence can be represented as a sum of delayed and aproprietly weighted pulse sequences. In other words, $x[n]=\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$. Now, the fact that the system is linear and time invariant leads allows us write the following $y[n]=L\{x[n]\}=L\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\}=\sum_{k=-\infty}^{\infty} x[k]L\{\delta[n-k]\}$.

Let us call the response of our system to a pulse as $h[n]=L\{\delta[n]\}$ then according to the time invariance property we can write $h[n-k]=L\{\delta[n-k]\}$, and then

$$y[n]=L\{x[n]\}=\sum_{k=-\infty}^{\infty} x[k]L\{\delta[n-k]\}=\sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad (2)$$

You should now recognize the convolution operation of sequence x[k] and sequence h[k]. The latter completely characterizes the system as it is only this sequence that we need to compute the output of an LTI system to an arbitrary input sequence x[n].

There is an alternative form of writing equation (2). Since in general, both *n* and *k* change from $-\infty$ to ∞ we can change the variable so that $m=n-k$, then apparently $k=n-m$ and we can write

$$y[n]=L\{x[n]\}=\sum_{k=-\infty}^{\infty} h[m]x[n-m] \quad (3)$$

Sequence $h[n]$ is called pulse response and completely characterizes a linear time invariant system. To see why this function is called pulse response simply compute the output of an LTI system to the delta pulse stimulation.

$$y[n]=L\{\delta[n]\}=\sum_{m=-\infty}^{\infty} h[m]\delta[n-m]=h[n]$$
$$y[0]=...h[-1]\delta[1]+h[0]\delta[0]+h[1]\delta[-1]+...=h[0]$$
$$y[1]=...h[0]\delta[1]+h[1]\delta[0]+h[2]\delta[-1]+...=h[1]$$

Therefore, the reaction of a system to a pulse is this sequence $h[n]$ that completely characterizes our LTI system. Think of a bell, that you hit with a hammer. The bell's vibration produces the sound. Then, you wait till the sound fades away and then if you heat the bell again it will produce exactly the same sound if the parameters of you hit remain the same. The fact that you get one and the same response for the same delta pulse like inputs but applied at different time moments is basically the time invariance property of the system (the bell). Now, if the system is addition linear, you can repeat the same experiment but strike the bell more frequently without fading till the sound produced by the previous hit fades away. In case of linearity your overall response to the train of pulses will simply be the superposition of the responses for each individual hit. If you add to it that for a linear system the amplitude of response is proportional to the strength of the hit then you get precisely expression (2), keeping in mind is that $h[n-k]$ is the delayed by *k* samples pulse response of a system.

We will denote the convolution operation with the start sign *, in other words

$$h[n]*x[n]=\sum_{k=-\infty}^{\infty} h[k]x[n-k]=\sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

<u>Properties of LTI systems</u>

The LTI systems can be casc aded and then the pulse response of the entire cascaded system is a convolution of pulse response of individual systems, in other words if a the output of the first system system with pulse response $h_1[n]$ is fed to the input of the second system with pulse response $h_2[n]$ then the pulse response then the pulse response of the systems represented by thus connected two systems can be computed as a convolution of the two pulse responses

To see this, just present the delta pulse to the input of the first system. As the output you will get the sequence $h_1[n]$ which is the the reaction of the first system to the unit pulse. This sequence is now input to the second system whose output is basically the convolution of the input $h_1[n]$ with this system's own pulse response $h_2[n]$

$$h_{12}[n]=h_1[n]*h_2[n]$$

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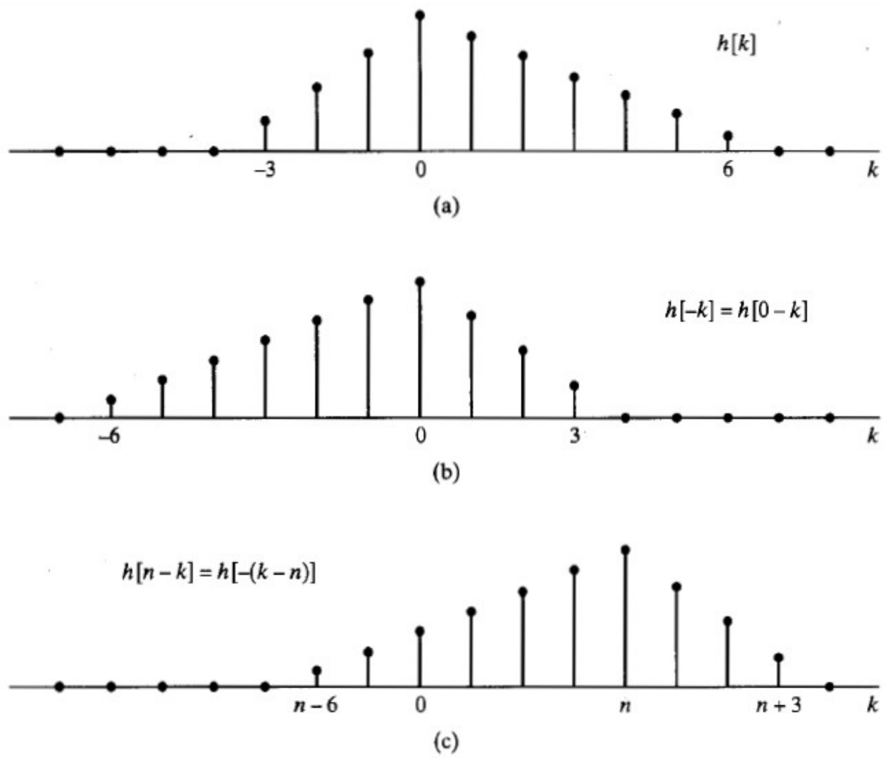

Figure 2.9 Forming the sequence $h[n-k]$. (a) The sequence $h[k]$ as a function of k. (b) The sequence $h[-k]$ as a function of k. (c) The sequence $h[n-k] = h[-(k-n)]$ as a function of k for $n = 4$.

Figure 2: Illustration from Oppenheim and Schafer book, showing the convolution operation for each time moment n is equivalent to the scalar product of time reversed and delayed by n samples pulse response and the input sequence.

Two systems concatenated in parallel will have the same input and their outputs will sum, so if we write $L\{\delta[n]\} = h_1[n]*\delta[n] + h_2[n]*\delta[n] = (h_1[n] + h_2[n])*\delta[n] = (h_1[n] + h_2[n])$

<u>Simplest LTI systems</u>

The simplest system doing something to a signal and a delay system whose pulse response is clearly $h_d[n] = \delta[n-d]$ for a system delaying our signal by d samples.

Moving average system

$$h[n] = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M_2} \delta[n-k]$$

Forward difference

$$h[n] = \delta[n+1] - \delta[n]$$

Backward difference

$$h[n] = \delta[n] - \delta[n-1]$$

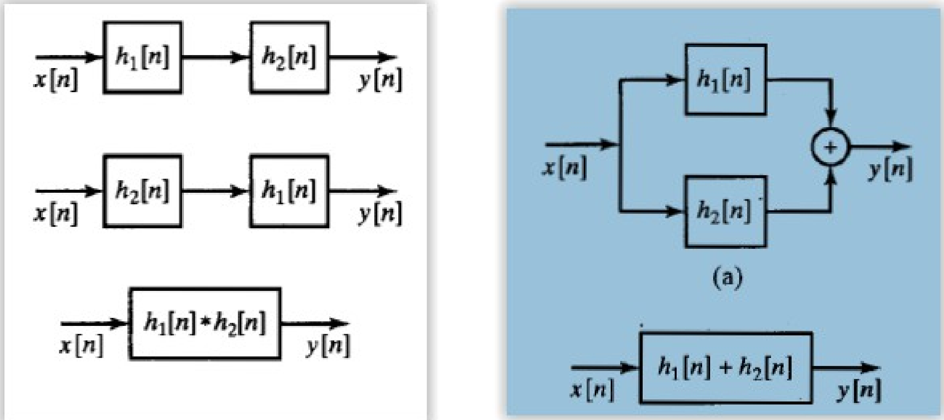

Figure 3: Two different ways to concatenate the systems. Left panel shows cascade concatenation. Such a concatenated system can be replaced by a single system with pulse response being the convolution of pulse responses of individual systems. The right panel shows parallel concatenation and the pulse response of an equivalent system is computed as the sum of pulse responses . Oppenheim and Schafer edition 2.

<u>Causal and non-causal systems</u>

As you can from analysis of the convolution equation, in cases when the pulse response sequence has zero values for $n<0$ then no future values of the signal $x[n]$ are used to calculate the output. Such systems are called causal systems. On the contrary, systems, whose pulse response sequence has all zero values for $n>0$ are called anti-causal. A system whose pulse response has non-zero values for both positive and negative n is called non-causal system.

Clearly, forward difference is an anti-causal system as it requires the data from the current moment and from the future in order to calculate the output value. The backward difference system is causal and the moving average system is non-causal as it uses values from the past and from the future to calculate the output.

<u>Eigenfunctions of LTI systems</u>

Eigenfunctions of a linear operator are special functions that when this operator applied to them do not change shape but only get scaled by some number, possibly a complex number. This number is then called an eigenvalue.

Let us compute the reaction $y[n]$ of an LTI system with pulse response $h[n]$ to a complex exponential sequence $x[n]=e^{j\omega n}$:

$$y[n]=h[n]*e^{j\omega n}=\sum_{k=-M_1}^{k=M_2} h[k]e^{j\omega(n-k)}=e^{j\omega n}\sum_{k=-M_1}^{k=M_2} h[k]e^{-j\omega k}=e^{j\omega n}H(e^{j\omega})$$

As we can see the output is basically a scaled version of the original sequence and the scale factor is a DTFT coefficient corresponding to frequency ω . In general this coefficient $H(\omega)=H(e^{j\omega})$ is a complex number. The magnitude of this number tells about the scaling applied to a harmonic with frequency ω . For example, if ω appears to fall in the system's stop-band than the magnitude of this coefficient will be very low. The argument of $H(\omega)=H(e^{j\omega})$ is the phase angle the harmonics

of this frequency is delayed by.
This fact goes inline with the last result of the previous lecture where we have shown that the DFT of the convolution of two sequence is the product of DFT coefficients of each of these two two sequences.

Convolution as a sliding scalar product

According to the expression for convolution $y[n]=L\{x[n]\}=\sum_{k=-\infty}^{\infty} h[n-k]x[k]$, calculation of the *n*-th sample of the convolution operator is equivalent to a scalar product of two vectors, corresponding to the sequences $x[k]$ and $h[n-k]$. Since in summation we move down the sequences by means of index *k* and this index enters the $h[n-k]$ with negative sign we first need to fold the original sequence and then shift it to the right by *n* samples. Please, refer to figure 2 for a graphical illustration of this process. Then the result of convolution is simply a scalar product of the two vectors comprised of the elements of $x[k]$ and $h[n-k]$.

Circular convolution as a matrix – vector multiplication

Convolution operation can be expressed as a matrix vector multiplication. IN this case, we should crerate a matrix $\boldsymbol{H}=h_{nk}$, whose elements $h_{nk}=h[(n-k)_N]$. Usually, the number of non-zero elements in the convolution kernel is less than the number of samples of the signal $\boldsymbol{x}$ the convolution is applied to. In this case, the size of matrix $\boldsymbol{H}$ is determined by the longer signal . The matrix $\boldsymbol{H}$ created as described will have a very special structure and is called circulant. The eigenvectors of circulant matrices are complex exponent vectors $\boldsymbol{b}_k=[1,e^{j\frac{2\pi k}{N}1}\ldots e^{j\frac{2\pi k}{N}n}\ldots e^{j\frac{2\pi k}{N}(n-1)}]^T$, which are basis vectors of the DFT.

Now you can see that the complex exponent sequence is ubiquitous and the family of complex exponents indexed by frequency play a very important role in both harmonic analysis and the theory of Linear Time Invariant Systems.