Signals and systems class, HSE, Spring 2015. A. Ossadtchi, Ph.D.

Lecture 4

Linear time invariant systems and convolution

Introduction

Harmonic analysis plays pivotal role in understanding of /physical processes happening around us. This analysis allows to represent the observed signals and interpret them from the standpoint of their rythmicity. Interestingly, the rhythmic process surround us and most of the processes are indeded periodic. Human ear can be considered as a device for performing the harmonic analysis of incoming acoustic pressure signal aimed at representing the short time segments of this signal decomposed into harmonic components. The excitation of nervous cells propagated into the brain depend on the spectral composition of the incoming signal and on the way this composition changes over time.

Using computers to perform harmonic analysis of some signal we would probably start from some sort of Fourier analysis and most of you would use the FFT (Fast Fourier Transform) function to …. to do what? To efficiently compute the coefficients of Discrete Fourier Transform that we are going to introduce in this chapter.

Basis set of harmonic functions

In the first lecture we considered complex exponential sequence and expolored their interesting properties. In one of the exercises you were left to experimentally show that half of the set of sequences of the form $x_k[n]=\cos\left(\frac{2\pi k}{N}n\right)$ constitutes a set of mutually orthogonal functions.

Now, consider the following family of complex exponential sequences of length N

$$\tilde{b}_k[n]=e^{j\frac{2\pi k}{N}n}=\cos\left(\frac{2\pi k}{N}n\right)+j\ \sin\left(\frac{2\pi k}{N}n\right),\ \ k\in[0,N-1],\ \ n\in[0,N-1] \qquad (1)$$

We can view this set of sequences as a set of vectors $\tilde{\boldsymbol{b}}_k=[\tilde{b}_k[0]\tilde{b}_k[1]...\tilde{b}_k[N-1]]^T$. These vectors are mutually orthogonal . We can show it analytically as follows

$$\sum_{n=0}^{N-1} e^{j2\pi kn/N} e^{-j2\pi ln/N}=\sum_{n=0}^{N-1} e^{-j2\pi(k-l)n/N}=\begin{pmatrix} N, & k=l \\ 0, & k\neq l\end{pmatrix}$$

Exercise 3.4
Prove the last statement. The first line we can get from the fact that the exponent of 0 is 1. The second line can be computed as the sum of N members of the geometric progression sequence and shown to be zero. If you can not do it analytically, write a simple Matlab code to prove this fact to you.

Note, that the square norm of vectors $\tilde{\boldsymbol{b}}_k=[\tilde{b}_k[0]\tilde{b}_k[1]...\tilde{b}_k[N-1]]^T$ is the same for all k and is equal to N. Therefore, in order to obtain a system of ortho*normal basis* vectors we need to normalize our basis vectors by the norm and thus obtain the following system of basis sequences

$$b_k[n]=\frac{1}{\sqrt{N}}\tilde{b}_k[n]=\frac{1}{\sqrt{N}}e^{j\frac{2\pi k}{N}n}=\frac{1}{\sqrt{N}}\left[\cos\left(\frac{2\pi k}{N}n\right)+j\ \sin\left(\frac{2\pi k}{N}n\right)\right],\quad k,n\in[0,N-1] \tag{2}$$

Analogously from these normalized sequences we can form vectors $\boldsymbol{b}_k=[b_k[0]\,b_k[1]...b_k[N-1]]^T$
Based on the previous exercise we can easily show that $\boldsymbol{b}_k^H\boldsymbol{b}_l=\delta[k-l]$ and therefore the system of vectors $\{\boldsymbol{b}_k\},k\in[0,N-1]$ is an orthonormal basis set. Representing a finite length sequence in this basis is called Discrete Fourier Transform or the DFT.

Computing the DFT coefficients

According to material presented in the previous lecture compute the transform coefficients (DFT in this case) we should first form transform matrix $\boldsymbol{B}$ whose columns are vectors $\{\boldsymbol{b}_k\},k\in[0,N-1]$ and then , since B is an orthonormal matrix and $\boldsymbol{B}^{-1}=\boldsymbol{B}^H$, compute DFT decomposition coefficients vector $\boldsymbol{S}$ of an arbitrary sequence s as $\boldsymbol{S}=\boldsymbol{B}^H\boldsymbol{s}$. Note, that we will often denote Fourier coefficients of a sequence with the corresponding capital letter. Correspondingly, reconstruction of the original sequence from the coefficients in Fourier space can be accomplished as $\boldsymbol{s}=\boldsymbol{B}\,\boldsymbol{S}$.

Explicit expressions for the DFT

We can now write the explicit expressions for the DFT by simply filling matrix **B** with the actual exponents and explicitly writing the compact form $\boldsymbol{S}=\boldsymbol{B}^H\boldsymbol{s}$
It took me time to type this matrix in LaTex like language but it looks as follows and is obtained by substituting values from equation(2) into the matrix so that $B_{nk}=b_k[n]$. The indexing for both n and k runs from 0 to N-1.

$$\boldsymbol{B}=\frac{1}{\sqrt{N}}\begin{bmatrix} 1 & 1 & 1 & ... & 1 & ... & 1 \\ 1 & e^{j2\pi/N} & e^{j2\pi 2/N} & ... & e^{j2\pi k/N} & ... & e^{j2\pi(N-1)/N} \\ \vdots & \vdots & \vdots & ... & \vdots & ... & \vdots \\ 1 & e^{j2\pi n/N} & e^{j2\pi 2n/N} & ... & e^{j2\pi kn/N} & ... & e^{j2\pi n(N-1)/N} \\ \vdots & \vdots & \vdots & ... & \vdots & ... & \vdots \\ 1 & e^{j2\pi(N-1)/N} & e^{j2\pi 2(N-1)/N} & ... & e^{j2\pi k(N-1)/N} & ... & e^{j2\pi(N-1)(N-1)/N} \end{bmatrix}$$

Then according to $\boldsymbol{S}=\boldsymbol{B}^H\boldsymbol{s}$ we can write

$$\boldsymbol{S}=\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ \vdots \\ S_{N-1} \end{bmatrix}=\boldsymbol{B}^H\boldsymbol{s}=\frac{1}{\sqrt{N}}\begin{bmatrix} 1 & 1 & 1 & ... & 1 & ... & 1 \\ 1 & e^{-j2\pi/N} & e^{-j2\pi 2/N} & ... & e^{-j2\pi n/N} & ... & e^{-j2\pi 2(N-1)/N} \\ \vdots & \vdots & \vdots & ... & \vdots & ... & \vdots \\ 1 & e^{-j2\pi k/N} & e^{-j2\pi 2k/N} & ... & e^{-j2\pi kn/N} & ... & e^{-j2\pi k(N-1)/N} \\ \vdots & \vdots & \vdots & ... & \vdots & ... & \vdots \\ 1 & e^{-j2\pi(N-1)/N} & e^{-j2\pi 2(N-1)/N} & ... & e^{-j2\pi n(N-1)/N} & ... & e^{-j2\pi(N-1)(N-1)/N} \end{bmatrix}\begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{N-1} \end{bmatrix}$$

and therefore

$$S[k]=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} s[n]e^{-j\frac{2\pi k}{N}n} \tag{3}$$

Correspondingly

$$\boldsymbol{s}=\begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ s_{N-1} \end{bmatrix}=\boldsymbol{B}\boldsymbol{S}=\frac{1}{\sqrt{N}}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 & \cdots & 1 \\ 1 & e^{j2\pi/N} & e^{j2\pi 2/N} & \cdots & e^{j2\pi k/N} & \cdots & e^{j2\pi 2(N-1)/N} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 1 & e^{j2\pi n/N} & e^{j2\pi 2n/N} & \cdots & e^{j2\pi kn/N} & \cdots & e^{j2\pi n(N-1)/N} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 1 & e^{j2\pi(N-1)/N} & e^{j2\pi 2(N-1)/N} & \cdots & e^{j2\pi k(N-1)/N} & \cdots & e^{j2\pi(N-1)(N-1)/N} \end{bmatrix}\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ \vdots \\ S_{N-1} \end{bmatrix}$$

and then

$$s[n]=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} S[k]e^{j\frac{2\pi k}{N}n} \tag{4}$$

Equations (3,4) form a pair of the DFT transforms. Equation 3 is called analysis equation as it decomposes the original signal and allows to **analyze** its spectral content. Equation 4 is called synthesis equation as it shows how the signal can be **synthesized** from the harmonic components.

In general, DFT coefficients are complex numbers. In case $s[n]$ is a real number then the real part of the k-th the DFT coefficient corresponds to the scalar product of $s[n]$ and the cosine sequence of the k-th frequency, i.e. $\frac{1}{\sqrt{N}}\cos\left(\frac{2\pi k n}{N}\right)$. The imaginary part of the k-th DFT coefficient of a real-valued $s[n]$ is the scalar product of $s[n]$ and the sine sequence of the k-th frequency, i.e. $\frac{1}{\sqrt{N}}\sin\left(\frac{2\pi k n}{N}\right)$.
For a real valued s[n] the first DFT coefficient (corresponding to k=0) is always a real number.

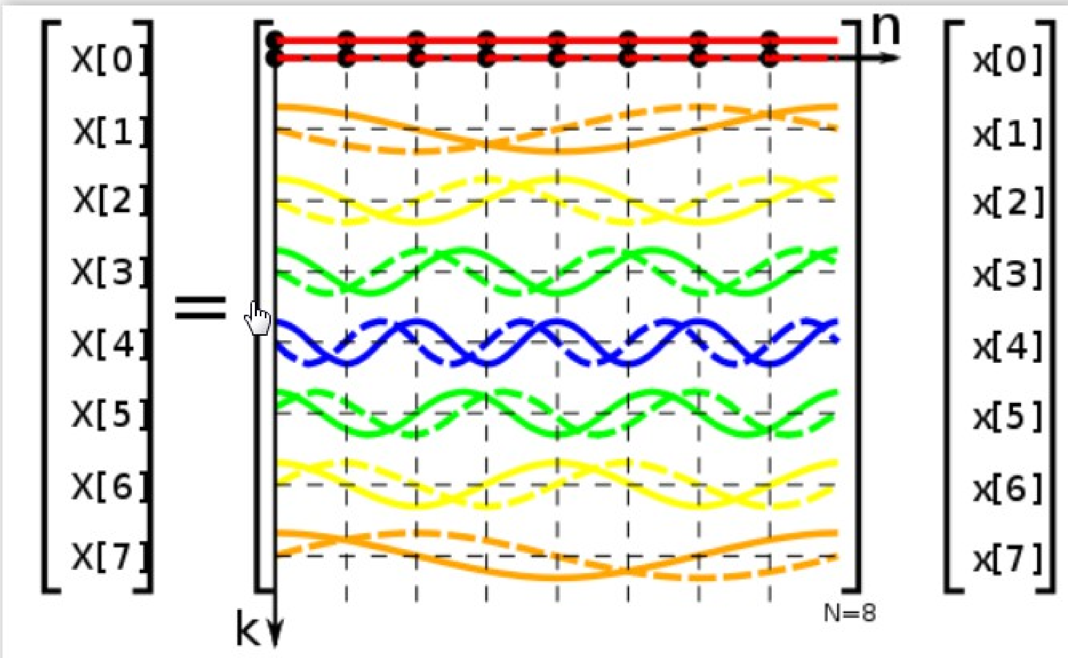

Figure 1: A very nice graphical representation of the DFT matrix $\boldsymbol{B}^H$ *from* [*http://en.wikipedia.org/wiki/DFT_matrix*](http://en.wikipedia.org/wiki/DFT_matrix) *The traces shown are complete sine and cosine profiles whereas for the transform, only the values corresponding to the dashed vertical grid lines are used. Note that the first 3 traces the dashed line(the imaginary part) is -sin() due to the complex conjugation operation. Then, the last 3 traces are simply complex conjugate of the traces 1,2,3.*

Exercise 3.5
Write a simple Matlab program to create a figure similar to the one above. Use subplot command to make several plots on the same figure. You may just plot the actual values corresponding to the vertical dashed grid lines, i.e. only 8 points per basis function. Use red color for the imaginary part and blue color for the real part.

Properties of the DFT

We are dealing here with apparently a very special transform matrix that dictates some interesting properties of this transform.

DC coefficient
First of all, the first DFT coefficient corresponds to the constant basis vector and reflects the DC shift of our signal. If the average value ($m=\frac{1}{N}\sum_{n=0}^{N-1} s[n]$) of our sequence is zero, then the first coefficient $S[0]$ of the DFT will also be zero. To see this write

$$S[0]=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} s[n]e^{-j\frac{2\pi 0}{N}n}=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} s[n]=m\sqrt{N}$$

Conjugate symmetry
Also, recall that the complex sinusoids on both sides of the spectrum are related via complex conjugation. Therefore, in case the analyzed signal is real, i.e. $s[n]=conj(s[n])$ the DFT

coefficients are related via complex conjugations as well. To see this write

$$S[N-k]=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} s[n]e^{-j\frac{2\pi(N-k)}{N}n}=\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} s[n]e^{-j\frac{2\pi N}{N}n}e^{j\frac{2\pi k}{N}n}=$$

$$=conj\left(\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} conj(s[n])e^{-j\frac{2\pi k}{N}n}\right)=since(s[n]=conj(s[n]))=conj(S[k])$$

This result is easily predictable. Think this way. Since the cosine components are the same for the miror indices k and N-k and sine components are in antiphase (this is just another way to say that $b_k[n]=conj(b_{N-k}[n])$) then clearly for real valued s[n] the DFT coefficients(projection coefficients onto these cosine and sine sequences) of mirror frequencies will have identical real part and sign reversed imaginary parts.

This means in turn that absolute values of the DFT coefficients are the same for mirror frequency pairs when $s[n]$ is a real sequence, i.e. $\|S[N-k]\|=\|S[k]\|$.

<u>DFT of a circularly shifted sequence</u>

Now, let us explore the relationship between the DFT of the original and circularly shifted sequence

First, let us introduce variable $W=e^{-j\frac{2\pi}{N}}$ called twiddle factor to simplify the exressions and make our derivation more transparent. Let us now express the hermitian transpose tramfporm matrix using the twiddle factors.

$$\boldsymbol{B}^H=\frac{1}{\sqrt{N}}\begin{bmatrix} 1 & 1 & 1 & \dots & 1 & \dots & 1 \\ 1 & W & W^2 & \dots & W^n & \dots & W^{(N-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 1 & W^k & W^{k2} & \dots & W^{kn} & \dots & W^{k(N-1)} \\ \vdots & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ 1 & W^{(N-1)} & W^{(N-1)2} & \dots & W^{(N-1)n} & \dots & W^{(N-1)(N-1)} \end{bmatrix}$$

Consider the k-th row (omit $\frac{1}{\sqrt{N}}$ for clarity). Its scalar product, according to equation (3 with s[n] represents the k-th DFT coefficient. Let us see what happens if we multiply this k-th row by the k-th power of the twiddle factor $W^k=e^{-j\frac{2\pi k}{N}}$:

$$W^k(1 \;\; W^k \;\; W^{k2} \;\; \dots \;\; W^{kn} \;\; \dots \;\; W^{k(N-1)})=(W^k \;\; W^{2k} \;\; W^{k3} \;\; \dots \;\; W^{kn} \;\; \dots \;\; 1)$$

In the above to get one in the last entry we used the following sequence of trivial manipuilations

$$W^{kn}=e^{j\frac{2\pi}{N}k(N-1)}e^{j\frac{2\pi}{N}k}=e^{j\frac{2\pi}{N}kN}=\cos(2\pi k)+j\;\;\sin(2\pi k)=1$$

Pay attention that this multiplication cyclically shifts backwards the elements of the k-th row of $\boldsymbol{B}^H$ around, so that after the first multiplication the first one becomes the last element and the rest of the elements simply shift bty one index backwards.

What happens if we multiply it by W^{2k} ? Let's see $W^{2k}(1 \;\; W^k \;\; W^{k2} \;\; \dots \;\; W^{kn} \;\; \dots \;\; W^{k(N-1)})=(W^{2k} \;\; W^{3k} \;\; W^{4k} \;\; \dots \;\; W^{k[(n+2)mod\,N]} \;\; \dots \;\; W^k)$. Multiplication by W^{2k} circularly shifts the original vector of twiddle factors by two samples backwards.

Apparently, calculating the scalar product of thus shifted row and the samples $s[n]$ will no longer give us the k-th DFT coefficient of $s[n]$. I will tell you that thus computed values is a DFTA coefficient of a sequence very closely related to s[n], what is this sequence? Write this $W^k s[0]+W^{2k} s[1]+W^{k3} s[2]+ \ ... \ W^{kp} \ s[N-p-1]...+1s[N-1]$. Now order this summation so that the twiddle factor power monotonically grows, to reproduce the order in which they are used in equation (3).

$1s[N-1]+W^k s[0]+W^{2k} s[1]+W^{k3} s[2]+ \ ... \ W^{kp} \ s[N-p-1]=\sum_{n=0}^{N-1} W^{nk} \tilde{s}[n]$ with $\tilde{s}[n]=[s[N-1] \ \ s[0] \ \ s[1]...s[N-2]]$ - cyclically shifted forward original sequence s[n]!

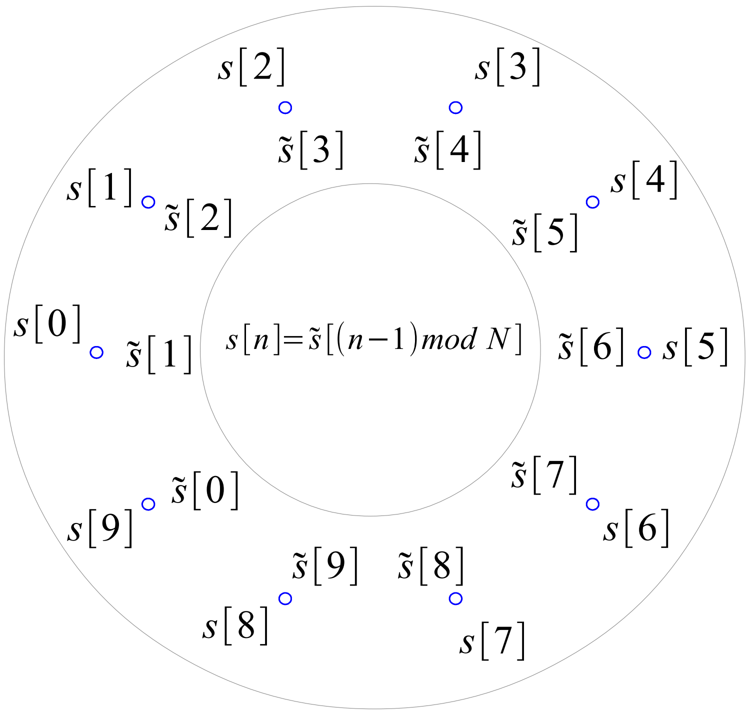

Figure 2: Illustration of circular shift operation by a single sample. Sequence $\tilde{s}[n]$ is shifted backward with respect to the sequence $s[n]$ by one sample. The easiest way to illustrate it is using the circular buffer notion depicted above. What happens, is that we start with drum(circle) with original sequence samples. Rotation of this drum corresponds to the circular shift operation. Basicallyt, when shifting to the left, the first sample of the original sequence wraps around and appears to the be the first element of the circularly shifted sequence.

Basically, we can think of the circular shift as controlled by some specific manipulation of indices. i.e. every time index value exceeds the length of the sequence it wraps around to the beginning of the sequence. This corresponds to calculating modulo N division of the resultant index.

Therefore, we conclude that the DFT a cyclically shifted sequence is related to the DFT of the original sequence as

$$S[k]=DFT_k(s[(n-m)_N])=e^{-j\frac{2\pi k}{N}m} DFT_k(s[n]) \quad (5)$$

where k denotes DFT coefficient index.

From Discrete Fourier Transform to Discrete Time Fourier Transform

In what we have considered so far our sequences were of finite length and we were able to build a solid understanding of the principles that form the foundation of the transform theory and to explore the Discrete Fourier Transform and its basic properties. I have consciously chosen to exhibit these fundamentals using the finite length sequences where we don't have to worry about the convergence issues of our transforms. Such a presentation allowed us to establish the direct link between the operations in time and frequency domain and basic linear algebra operations.

The restriction we had when working with the DFT(the transform defined for finite length sequences) lies in the fact that for a sequence of *N* samples long there is only a very specific basis comprising *N* sequences with very specific spectrum of discrete frequency values $\omega_k = \frac{2\pi k}{N}$ exists. This step over the frequency axis is then equals to $\Delta\omega = \frac{2\pi}{N}$. In real-life situation, especially when dealing with short sequences this may lead to difficulty in discovering harmonic components whose frequencies lie far away from the frequency grid defined by this $\Delta\omega = \frac{2\pi}{N}$ step that is in turn determined by the length *N* of the analyzed sequence.

A possible solution that would allow us to increase the frequency resolution is to increase the length of our sequence. Since it is not always possible to record more data and ensure the stationarity of the recorded data, one recipe is to add zeros to the end of the sequence. For example, if we augment our original *N*-samples long sequence with a sequence of N zeros, we will get the frequency step reduced by the factor of two as the total length of the sequence to apply the DFT to will become equal to 2N. Since in practice of data analysis we are always dealing with finite length sequences, this zero-padding technique is frequently used to increase the frequency resolution of DFT.

We can then consider the case when $N \to \infty$ and then $\Delta\omega = \frac{2\pi}{N} \to 0$ meaning that the discrete frequency index now becomes a continuous variable. In this case we get so called Discrete Time Fourier Transform (DTFT) . The expression for it may look as the limit of $S[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} s[n] e^{-j\frac{2\pi k}{N}n}$ with $N \to \infty$ with all the associated convergence problems that we don't face in real life as our sequences always have finite number of non-zero samples. I

DFT of circular convolution

Circular convolution is defined as $y[n] = h[n] *_c x[n] = \sum_{m=0}^{N} x[(n-m)_N] h[m]$. In what's written the operation $()_N$ denotes calculation of the remainder of the division of the element in brackets by N and results in cycling around the time index axis over which the sequences are defined. Let us apply the DFT to $y[n]$ sequence and write the expression for the k-th DFT coefficient as $Y[k] = DFT(y[n]) = \sum_{n=0}^{N} \sum_{m=0}^{N} x[(n-m)_N] h[m] e^{-j\frac{2\pi k}{N}n}$ We can change the order of summation to get $Y[k] = DFT(y[n]) = \sum_{m=0}^{N} h[m] \sum_{n=0}^{N} x[(n-m)_N] e^{-j\frac{2\pi k}{N}n}$. In this expression we took $h[m]$ out from the brackets as it is constant for different n. We can now see that what we in fact have is the scaled summation of DFTs of the sequences shifted by m samples. According to equation (5) we can replace $\sum_{n=0}^{N} x[(n-m)_N] e^{-j\frac{2\pi k}{N}n}$ with $e^{-j\frac{2\pi k}{N}m} DFT_k(x[n])$ and then obtain

$$Y[k]=DFT_k(y[n])=\sum_{m=0}^{N} h[m] e^{-j\frac{2\pi k}{N}m} DFT_k(x[n])=DFT_k(h[n])DFT_k(x[n])=X[k]H[k] \quad .$$

Therefore, we can say that operation of convolution in the trivial basis corresponds to simple multiplication of the corresponding coordinates of the sequence in the DFT basis. This fact along with the Fast Fourier Transform, a method for express calculation of the DFT coefficients, is often used to compute the convolution efficiently.