Signals and systems class, HSE, Spring 2015. A. Ossadtchi, Ph.D.

Lecture 5

Dicrete Time Fourier Transform

From Discrete Fourier Transform to Discrete Time Fourier Transform

In what we have considered so far our sequences were of finite length and we were able to build a solid understanding of the principles that form the foundation of the transform theory and to explore the Discrete Fourier Transform and its basic properties. I have consciously chosen to exhibit these fundamentals using the finite length sequences where we don't have to worry about the convergence issues of our transforms. Such a presentation allowed us to establish the direct link between the operations in time and frequency domain and basic linear algebra operations.

The restriction we had when working with the DFT(the transform defined for finite length sequences) lies in the fact that for a sequence of N samples long there is only a very specific basis comprising N sequences with very specific spectrum of discrete frequency values $\omega_k = \frac{2\pi k}{N}$ exists. This step over the frequency axis is then equals to $\Delta\omega = \frac{2\pi}{N}$. In real-life situation, especially when dealing with short sequences this may lead to difficulty in discovering harmonic components whose frequencies lie far away from the frequency grid defined by this $\Delta\omega = \frac{2\pi}{N}$ step that is in turn determined by the length N of the analyzed sequence.

A possible solution that would allow us to increase the frequency resolution is to increase the length of our sequence. Since it is not always possible to record more data and ensure the stationarity of the recorded data, one recipe is to add zeros to the end of the sequence. For example, if we augment our original N-samples long sequence with a sequence of N zeros, we will get the frequency step reduced by the factor of two as the total length of the sequence to apply the DFT to will become equal to 2N. Since in practice of data analysis we are always dealing with finite length sequences, this zero-padding technique is frequently used to increase the frequency resolution of DFT.

We can then consider the case when $N \to \infty$ and then $\Delta\omega = \frac{2\pi}{N} \to 0$ meaning that the discrete frequency index now becomes a continuous variable. In this case we get so called Discrete Time Fourier Transform (DTFT) . The expression for it may look as the limit of $S[k] = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} s[n] e^{-j\frac{2\pi k}{N}n}$ with $N \to \infty$ with all the associated convergence problems that we don't face in real life as our sequences always have finite number of non-zero samples.

Strictly speaking, the value of this sum for $N \to \infty$ will depend on the speed of growth of the sum as compared to the speed of growth of $\sqrt{(N)}$. Here we are running in trouble and start facing issues of convergence I so much wanted to avoid in this class.

Therefore, we will take another route here. Since conceptually the pair of Discrete Time Fourier transforms means exactly the same as the pair of Discrete Fourier Transforms I will simply write out it

for you and then prove that from the DTFT coefficients you can get back the original sequence if you apply the synthesis equation, which unfortunately, will have the integral instead of the sum.
In the analysis equation, with $\Delta\omega=\frac{2\pi}{N}\rightarrow 0$ as $N\rightarrow\infty$ we , instead of the integer index k corresponding to some $\omega_k=\frac{2\pi}{N}k$, use continuous variable ω to index the coefficients that will now become a continuous function of this “index” variable. Again, due to 2π periodicity of $e^{j\omega n}$ we will consider only the values of this index in the $[0,2\pi]$ range or which is equivalent $[-\pi,\pi]$ range.

Then, the analysis equation will read
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$$S(\omega)=\sum_{n=-\infty}^{\infty} s[n]e^{-j\omega n} \qquad (1)$$

Pay attention that we removed the normalization coefficient. You will see it in a new disguise in the synthesis equation. Clearly, given that we have summation in the unbounded range this summation will exist only for a limited (very limited) class of functions. This is not of concern in practice, as the sequences are always of finite length. However, in order to interpret the results of the DTFT we need to remember that the fact of finite length sequence when we are using the DTFT is modeled as the product of the corresponding infinitely long sequence with a rectangular window of zeros and ones, with the number of “ones” equal to the actual sequence length.

Clearly, if you apply equation 1 to a finite length sequence, then clearly, with the accuracy of the normalization term you will receive exactly the same values as you get with DFT for frequency values $\omega_k=\frac{2\pi}{N}k$, in other words $S(\omega)|_{\omega=\frac{2\pi}{N}k}=\sum_{n=-\infty}^{\infty} s[n]e^{-j\frac{2\pi}{N}kn}=\sqrt{(N)}DFT_k(s[n])$

Now, we will look into into the synthesis equation. This time it will take the form of an integral as the summation is to be preformed over the continuous “index” variable ω .

$$s[m]=\alpha\int_{-\pi}^{\pi} S(\omega)e^{j\omega m}d\omega \qquad (2)$$

As you can see I left the normalizing constant there to determine its value by the direct substitution of the analysis equation (1) into the synthesis equation (2) as follows

$$s[m_0]=\alpha\int_{-\pi}^{\pi}\left[\sum_{n=-\infty}^{\infty} s[n]e^{-j\omega n}\right]e^{j\omega m_0}d\omega=\alpha\sum_{n=-\infty}^{\infty} s[n]\int_{-\pi}^{\pi} e^{j\omega(m_0-n)}d\omega$$

As you can see in the above we have changed the order of integration and summation. Theoretically, we could have done so only in case the summation converges. However, since in practice the number of non-zero elements in our sequences is finite this sum will always converge and thus this exchange of integration order is valid from the practical point of view. Also, note that we have used different time indices (*m* and *n*) in the sum and in the integral. We did so to separate the time index variable *m* for which we are trying to synthesize the value (s[*m*]) and the index *n* used in summation for calculation of the DTFT coefficients.

Clearly, the integral $\int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega$ will have different values depending on *m* and *n*. When $m=n$ we have $\int_{-\pi}^{\pi} e^{j\omega 0} d\omega = \int_{-\pi}^{\pi} 1\, d\omega = 2\pi$ and in case when $m \neq n$ we get identical zero because during the integration over the period the sine and cosine corresponding to the imaginary and real partdwill give the balanced amount of positive and negative area under the curve values. Remember? The first lecture!

So, in other words we can write using the delta pulse sequence notation

$$s[m] = \alpha \int_{-\pi}^{\pi} \left[\sum_{n=-\infty}^{\infty} s[n] e^{-j\omega n} \right] e^{j\omega m} d\omega = \alpha \sum_{n=-\infty}^{\infty} s[n] \int_{-\pi}^{\pi} e^{j\omega(m-n)} d\omega =$$
$$\alpha 2\pi \sum_{n=-\infty}^{\infty} s[n] \delta(m-n) \sim \alpha 2\pi s[m]$$

clearly the equality will hold if $\alpha = \frac{1}{2\pi}$. Therefore, for the synthesis equation of DTFT we will have

$$s[m] = \frac{1}{2\pi} \int_{-\pi}^{\pi} S(\omega) e^{j\omega m} d\omega \qquad (3)$$

Interestingly, since the analysis equation did not have any normalizing coefficient the value of $\alpha = 2\pi$ corresponds to the square of the norm of the basis function $e^{j\omega n}$ used in the transform.

<u>Properties</u>

Parseval's theorem

$$\sum_{n=-\infty}^{\infty} |s[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |S(\omega)|^2 d\omega$$

Function $P(\omega) = |S(\omega)|^2$ is called energy density function as it describe the distribution of energy over the frequency range. You can think of it as the you used to think about the probability density function in the theory of probabilities. The idea stays the same that the energy does not change in regardless of the basis we use to represent our signal. I want tio reiterate here that you should think of ω as of the indexing variable and always remember that continuous ω is the result of getting infinitely dense grid of frequency values $\omega_k = \frac{2\pi}{N} k$ resulting from considering infinitely long sequences with length $N \rightarrow \infty$. No magic at all, except for some mystery that prescribes to use $\sqrt{(2\pi)}$ as the norm of infinitely long $e^{j\omega n}$ which, I agree, is not intuitive at all! This, however, should not bother you too much as we proved that the pair of transforms works (given the sum converges). In order to really understand this we would need to look into the theory of special functions which goes way beyond the goals of this class.

<u>DTFT of shifted sequence</u>

$$DTFT(s[n-d]) = \sum_{n=-\infty}^{\infty} s[n-d] e^{-j\omega n} = e^{-j\omega d} \sum_{n=-\infty}^{\infty} s[n-d] e^{-j\omega(n-d)} = e^{-j\omega d} S(\omega)$$

<u>DTFT of convolution of two sequences in time</u>

$$DTFT\left(\sum_{k=-\infty}^{\infty} x[n-k]h[k]\right)=\sum_{k=-\infty}^{\infty} h[k]\,DTFT(x[n-k])=$$

$$\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\,DTFT(x[n])=$$

$$DTFT(x[n])\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}=DTFT(x[n])\,DTFT(h[k])=X(e^{j\omega})H(e^{j\omega})$$

<u>DTFT of a product of two sequences</u>

$$DTFT(x[n]y[n])(\omega)=\frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega-\theta)\,Y(\theta)\,d\theta$$

Pay attention to the duality. The DTFT of convolution is the product of the DTFT coefficients and the DTFT of a product is the convolution of DTFT coefficients.

Examples

Compute DTFT of $x[n]=a^n u[n]$. Taking into consideration the fact that $x[n]=0 \;\; \forall \;\; n<0$ we write $X(e^{j\omega})=\sum_{n=0}^{\infty} a^n e^{-j\omega n}=\sum_{n=0}^{\infty}(ae^{-j\omega})^n=\dfrac{1}{1-ae^{-j\omega}}$

To make sense of this expression we should understand that DTFT coefficients are in general complex. Let's try to depict it. Since our results is complex , let's look at its magnitude first. Let's look at the square of magnitude that corresponds to the energy density function introduced above. The energy density function is $P(\omega)=X(e^{j\omega})X^*(e^{j\omega})=\dfrac{1}{1-ae^{-j\omega}}\dfrac{1}{1-ae^{j\omega}}=\dfrac{1}{1-2\mathrm{a}\cos(\omega)+a^2}$

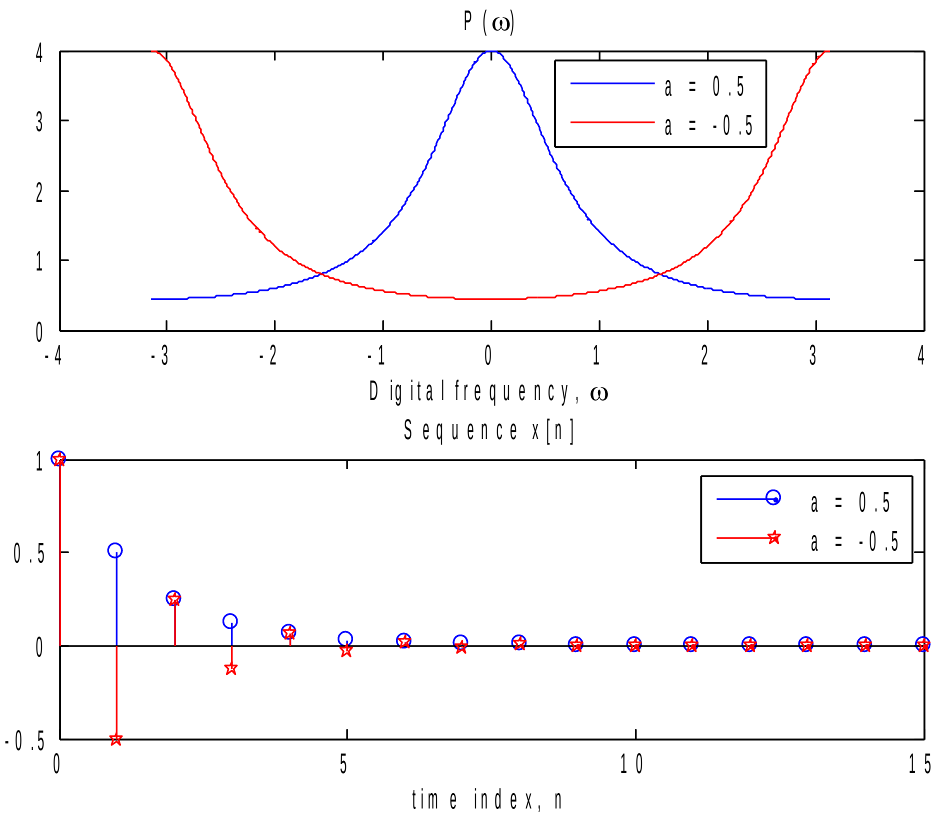


Figure 1: Illustration of the sequence and its energy concentration function for the right sided exponential sequence

The signal $x[n]$ is shown in the lower panel for two values of a. The red stems correspond to negative values of *a* resulting in the sequence with values of alternating signs. The blue ones correspond to the sequence with positive a and thus appears to be a smooth sequence.

The top graph shows energy density function of the sequence. The portion of the plot corresponding to the positive values of ω in the range $[0,\pi]$ correspond to evenly increasing frequency of oscillations of the complex exponents. As we can see for the blue (smooth) sequence most energy is concentrated in the low frequency range. The situation changes to the opposite for the red (sign changing) sequence that clearly manifests dominant concentration of energy in the high frequency range.

DFT of a rectangular pulse

Consider a sequence $x[n]=u[n]-u[n-N]$. The sequence is plotted for N = 10 in figure

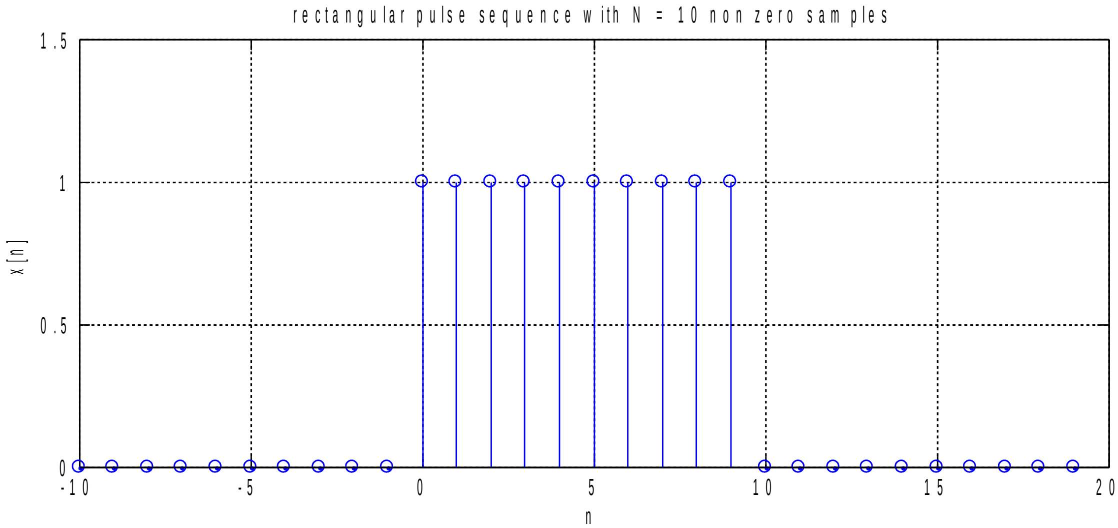
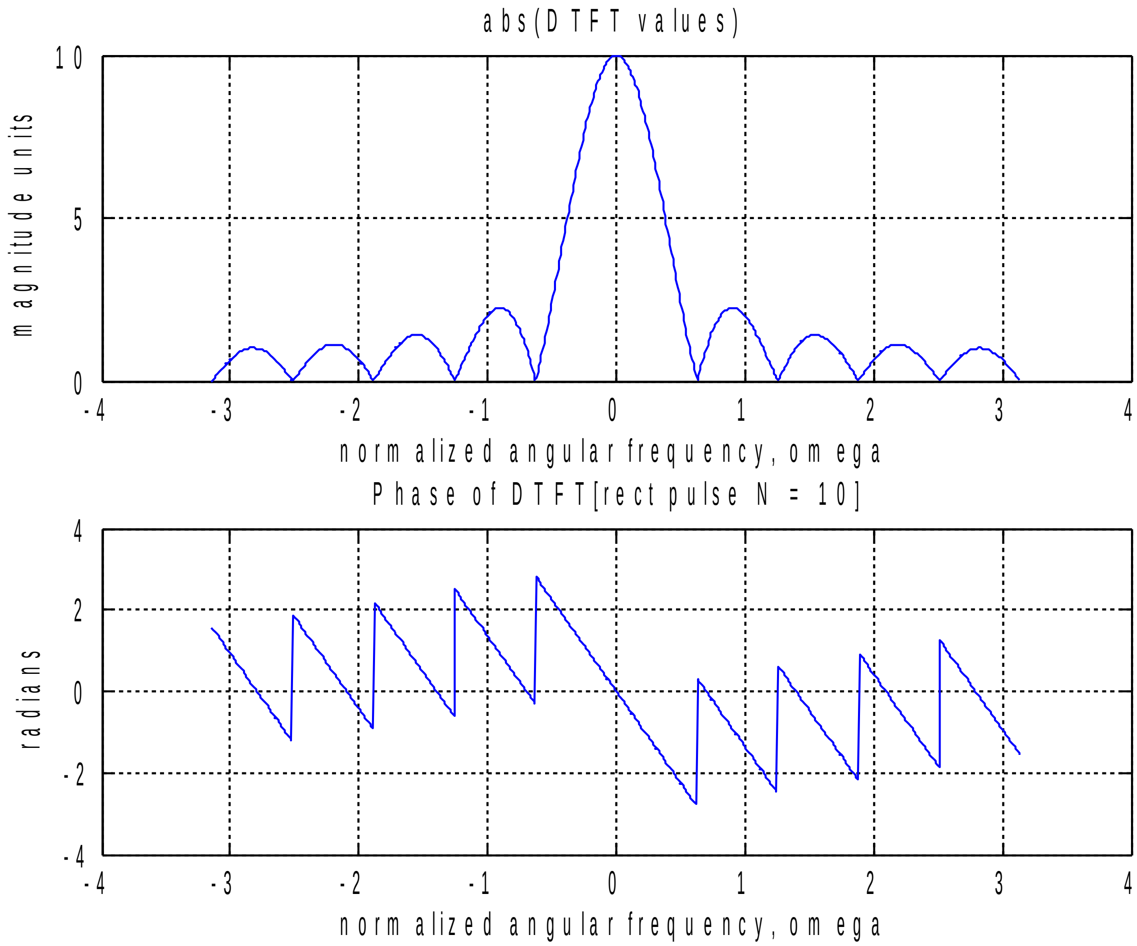


Figure 2: Rectangular sequence

Let's compute its DTFT.

$$X(e^{j\omega})=\sum_{n=0}^{N} 1\, e^{-j\omega n}=\sum_{n=0}^{N}(e^{-j\omega})^n=\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}=\frac{e^{-j\omega N/2}}{e^{-j\omega/2}}\frac{e^{j\omega N/2}-e^{-j\omega N/2}}{e^{j\omega/2}-e^{-j\omega/2}}=e^{-j\omega\frac{N-1}{2}}\frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

Clearly, after some manipulations we were able to split the phases and the magnitude values of the DTFT values. The phase is non-zero reflecting the fact that the sequence we are working with is real and that it is not symmetric around *n = 0*. We can draw the magnitude and the phase separately


abs(DTFT values)
magnitude units
normalized angular frequency, omega
Phase of DTFT[rect pulse N = 10]
radians
normalized angular frequency, omega