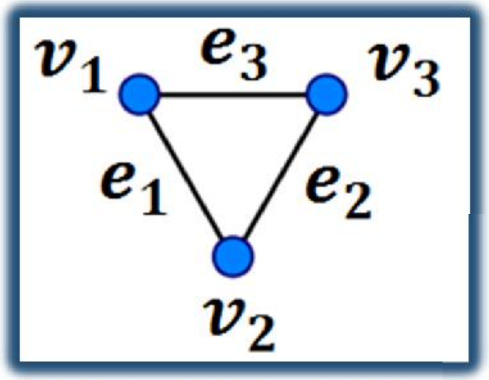
Graphs and trees, basic theorems on graphs and coloring of graphs

Lectures 9-11

Contents

Basic concepts in graph theory

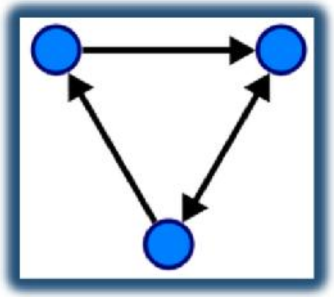


An **undirected graph** (or **graph**) is a pair (V, E) where

$V = \{v_1.v_2, \ldots\}$ is a set of **vertices**,

$E = \{e_1, e_2, \ldots\}$ is a set of **edges** in which each element e_k is disordered pair $\{v_i, v_j\}$.

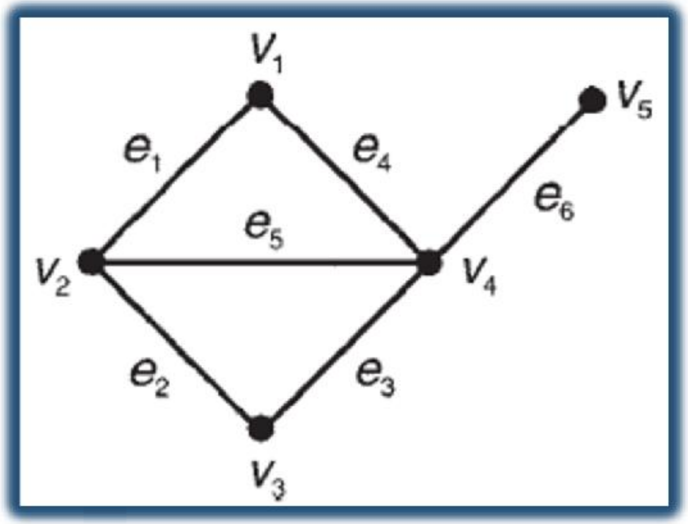
If E contains $k \geq 2$ pairs $\{v_i ; v_j\}$ then the edge $\{v_i; v_j\}$ is called an edge with multiplicity k



A **directed graph** (or **digraph**) is a pair (V, E)

E is a set of ordered pairs (v_i, v_j),
called **arcs**, **directed edges**, or **arrows**.

Basic concepts in graph theory



An edge is said to be **indicent** to the vertices to which it is attached.
For example, the edge e_1 is indicent to the vertices v_1 and v_2.

- Two vertices that are joined by an edge are said to be **adjacent**. For example, v_1 and v_2 are adjacent while v_1 and v_3 are not adjacent.
- If two edges are indicent to one vertex they are called **adjacent**.

Basic concepts in graph theory

- Graph does not have to be in one piece.

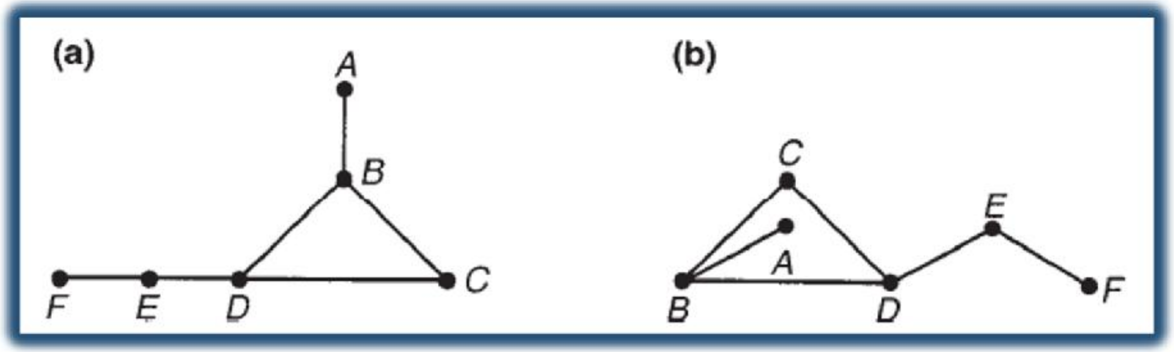
For example, this is a perfectly legitimate graph.

- There is no restriction on the numbers of edges and vertices a graph may have (except it must have at least one vertex).
- It is permissible for a graph to have no edges; such graphs are called **null** graphs.
- It is also permissible for the number of vertices or edges of a graph to be infinite.

Basic concepts in graph theory

- A graph may have loops (an edge from one vertex to the same vertex).

- There are many different ways of drawing the same graph, some of which may look very different from others.



- The only restrictions are that we may not create new edges or vertices, delete any edges or vertices, or reattach edges to different vertices.

Basic concepts in graph theory

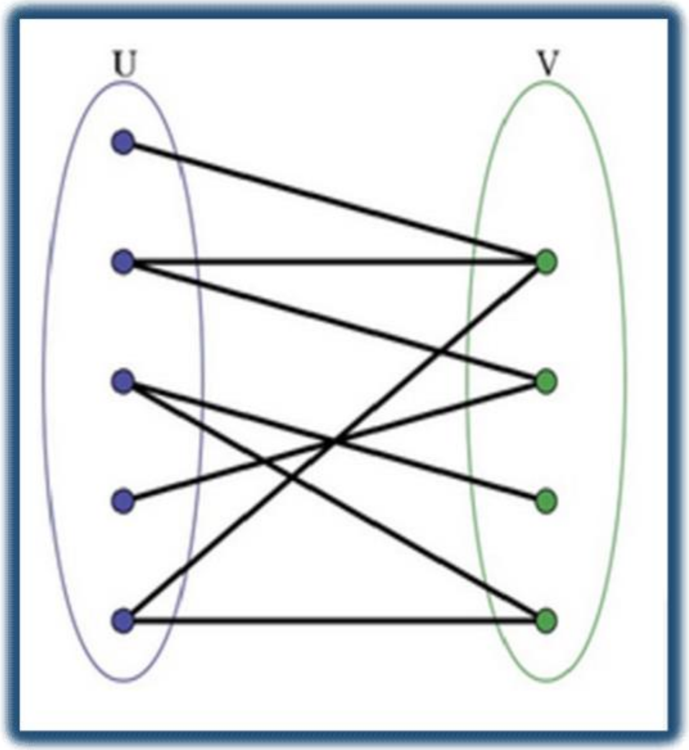
- A **subgraph**, $G_1 = (V_1, E_1)$, of a graph, $G = (V, E)$, is a graph whose vertices are a subset of the vertex set of G, and whose edges are a subset of the edge set of G.

$$V_1 \subseteq V, E_1 \subseteq E$$

- A subgraph is a **spanning subgraph** if it has the same vertex set as G.
- Graph, G, is said to be **induced** (or **full**) if for any pair of vertices there is a chain that connects them.

- A **simple graph** is complete if each vertex of the graph is adjacent to every other vertex
- The **complement** $\overline{G}$ of a simple graph G is the graph with the same vertices as G, such that any two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

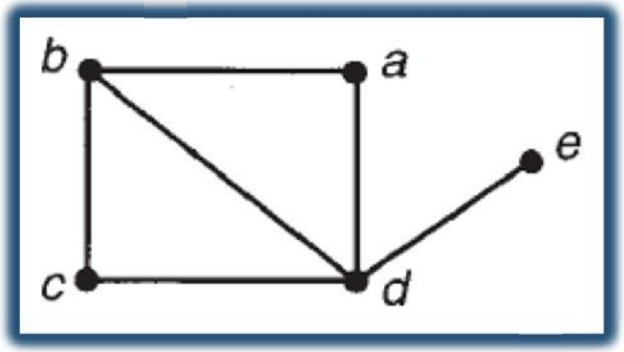
Basic concepts in graph theory



A **Bipartite Graph** is a graph whose vertices can be divided into two independent sets, $\boldsymbol{U}$ and $\boldsymbol{V}$ such that every edge $(\boldsymbol{u}, \boldsymbol{v})$ either connects a vertex from ***U*** to ***V*** or a vertex from ***V*** to ***U***.

Basic concepts in graph theory

- The **Degree** of a vertex of a graph is the number of edges incident to the vertex. The degree of a vertex v is denoted $\deg(v)$. A vertex with degree **0** is said to be **isolated**.



$$\deg(a) = 2$$
$$\deg(b) = 3$$
$$\deg(c) = 2$$
$$\deg(e) = 1$$

Theorem: In any graph, the sum of the degrees of the vertices equals twice the number of edges.

$$\sum_{v \in V} \deg(v) = 2|E|$$

- **Handshaking lemma**: In any graph, the number of vertices with odd degree is even.

Basic concepts in graph theory

Theorem: An undirected graph has an even number of vertices of odd degree.

Proof:

Let V_1 be the vertices of even degree and V_2 be the vertices of odd degree in graph $G = (V, E)$ with m edges. Then even

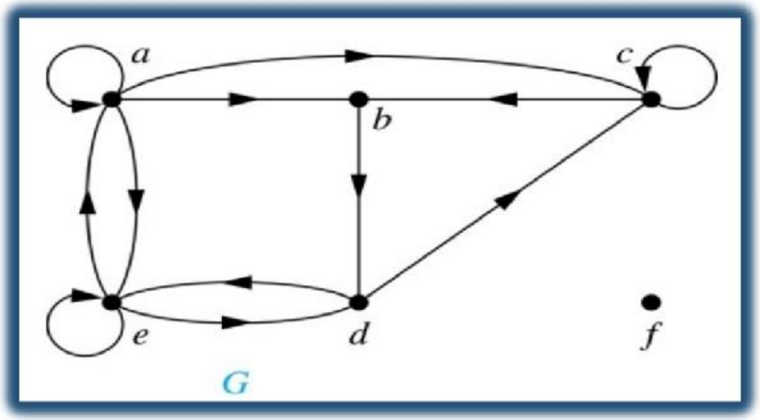
$$2|\boldsymbol{E}| = \sum_{v \in V} \boldsymbol{deg}(v) = \sum_{v \in V_1} \boldsymbol{deg}(v) + \sum_{v \in V_2} \boldsymbol{deg}(v)$$

must be even since $\deg(v)$ is even for each $v \in V_1$

must be even because 2|E| is even and the sum of degrees of vertices of even degree is even. Thus, since this is the sum of degrees of all vertices of odd degree, there must be an even number of them.

Basic concepts in graph theory

- The number of head endpoints adjacent to a vertex is called the **indegree** of the vertex and denoted $deg^{-}(v)$.
- The number of tail endpoints adjacent to a vertex is called the **outdegree** of the vertex and denoted $deg^{+}(v)$.
- **Note** that a loop at a vertex contributes 1 to both indegree and outdegree.



$$deg^{-}(a) = 2$$
$$deg^{-}(b) = 2$$
$$deg^{-}(c) = 3$$
$$deg^{-}(e) = 3$$
$$deg^{-}(f) = 0$$

- Degree of a vertex is a sum of $deg^{+}(v)$ and $deg^{-}(v)$.
- Graph is called **regular** if all degrees of its vertices are equal.

Basic concepts in graph theory

Theorem: Let G = (V, E) be a directed graph. Then:

$$|E| = \sum_{v \in V} deg^{-}(v) = \sum_{v \in V} deg^{+}(v)$$

Proof:

- The first sum counts the number of outgoing edges over all vertices.
- The second sum counts the number of incoming edges over all vertices.
- Both sums must be |E|.

The matrix representation of a graph

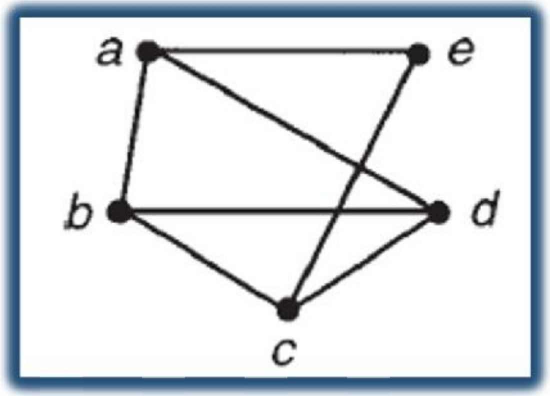
- If G is a graph with **n** vertices, labelled $v_1, v_2, \dots, v_n$, then the **adjacency matrix** of G is an $n \times n$ matrix whose entries are given by the following rule:
- The entry in the i-th row and the **j**-th column is the number of edges from v_i to v_j.
- In particular, if G is a simple graph, then the entry in the i-th row and the j-th column of the adjacency matrix

  is **1** if v_i and v_j are adjacent,

  and **0** if they are not.
- The elements on the principal diagonal of the matrix are all **0**, because simple graphs do not have loops.

The matrix representation of a graph

Example 1: Construct the adjacency matrix for the graph assuming the vertices are given in the order a,b,c,d,e.

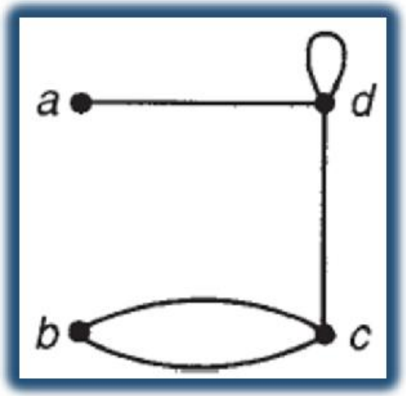


The adjacency matrix for the graph

$$\begin{array}{ccccc} a & b & c & d & e \end{array}$$
$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Example 2: Draw the graph with the following adjacency matrix.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix}$$



The matrix representation of a graph

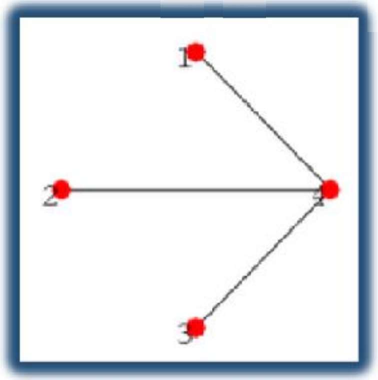
- The adjacency matrix of any graph is symmetric. The entries on and below the principal diagonal of the matrix is called the **lower triangular matrix** representation of the graph.

$$\begin{pmatrix} 0 & & & \\ 0 & 0 & & \\ 0 & 2 & 0 & \\ 1 & 0 & 1 & 1 \end{pmatrix}$$

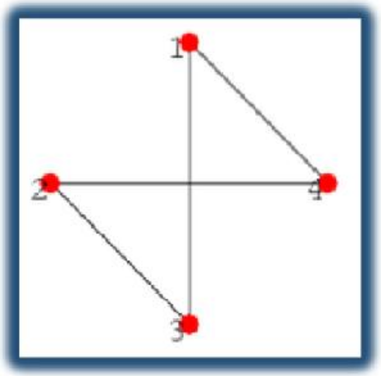
- **Notice** that the adjacency matrix of a graph depends on the order in which the vertices are labelled.
- A graph with **n** vertices could have up to **n!** different adjacency matrices.

The matrix representation of a graph

- The **incidence matrix** of a graph gives the (0,1)-matrix which has a row for each vertex and column for each edge, and $(v, e) = 1$ iff vertex v is incident upon edge e.



$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

Isomorphism of graphs

Definitions

Let G and H be two simple graphs, with vertex sets $V(G)$ and $V(H)$ respectively. An **isomorphism** from G to H is a function $f\colon V(G) \to V(H)$ with the following properties:

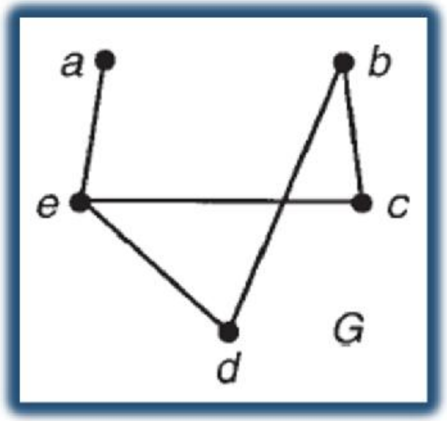
- f is one-to-one and onto.
- For any two vertices u and v of G, if u and v are adjacent in G, then $f(u)$ and $f(v)$ are adjacent in H, and if u and v are not adjacent in G, then $f(u)$ and $f(v)$ are not adjacent in H.
- If there exists an isomorphism from **G** to **H**, then G and H **isomorphic**.

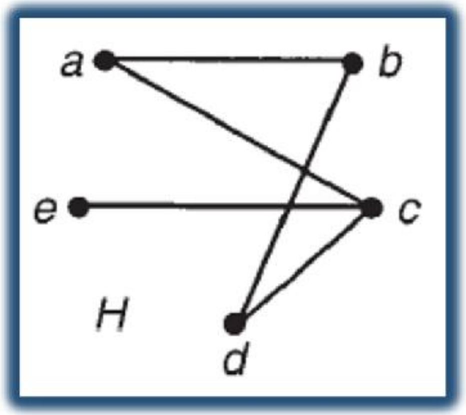
Isomorphism of graphs

Example: Find an isomorphism between the two graphs G and H.

- One isomorphism is: $f: V(G) \to V(H), f(a) = e, f(b) = b,$ $f(c) = d, f(d) = a, f(e) = c$

Therefore the graphs G and H are **isomorphic**.





- A better way of showing that **two graphs are not isomorphic** is to find **a graph-theoretic property** that **one graph** has but the **other does not**.
- **For example**, two graphs with **different numbers** of **edges or vertices** cannot possibly be isomorphic.

Isomorphism of graphs

Example

- Both graphs have **6 vertices** and **7 edges**. Also, both graphs have **4 vertices** with **degree 2** and **2 vertices** with **degree 3**.
- However, the first graph has a sequence def of three vertices with **degree 2**, with d adjacent to e and e adjacent to f, whereas there is **no such sequence of three vertices** with **degree 2** in the **second** graph. Therefore the graphs are **not isomorphic**.

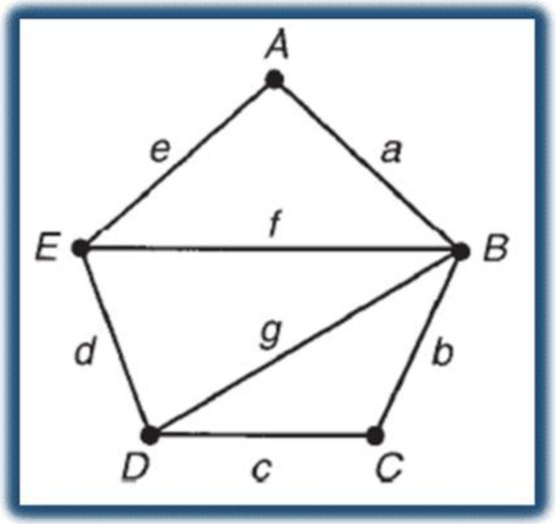
Paths and circuits

A path of **length n** in a graph is a sequence of vertices and edges of the form:

$$v_0, e_1, v_1, e_2, v_2, \dots, e_n, v_n$$

where e_i is an edge joining v_{i-1} and v_i, for all $i \in \{1, 2, \dots, n\}$.

- A path may include repeated edges or repeated vertices. The **length of a path** is the number of edges in the path
- **Any vertex** by itself is both a path and a circuit of **length 0**.



$AaBgDdE$ $(Aagd)$ is a path of length **3.**

$EdDgBbCcDdE$ $(Edgbcd)$ is a **circuit** of length **5**

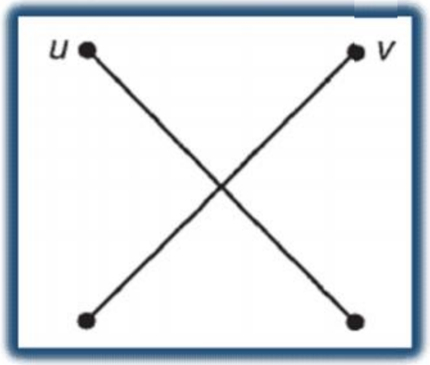
Paths and circuits

- A path which does not contain coinciding edges is called **chain**.
- A path which does not contain coinciding vertices is called **simple chain**.
- Path or chain in a directed graph are called **directed** is all the edges are directed from v_i to v_{i+1}.

Paths and circuits

Definitions

A graph is **connected** if, for any pair of vetices u and v of G, there is a path from u to v.



The graph is **not connected**, because there is no path from u to v.

- We **note** that if there is a path from a vertex u to a vertex v in a graph, then in particular there is a path from u to v that contains no repeated vertices or edges.
- It follows that in a connected graph, each pair of vertices is connected by at least one path that **doesn't contain repeated vertices or edges**.

Paths and circuits

Definitions

- Let G be a connected graph. An **Eulerian path** is a path that includes every edge of G exactly once.
- An **Eulerian circuit** is an Eulerian path for which the first and the last vertices coincide.
- A connected graph that has an Eulerian circuit is called **Eulerian**.
- A connected graph that has an Eulerian path but no Eulerian circuit is called **semi-Eulerian**.

Note that an Eulerian circuit may have repeated vertices.

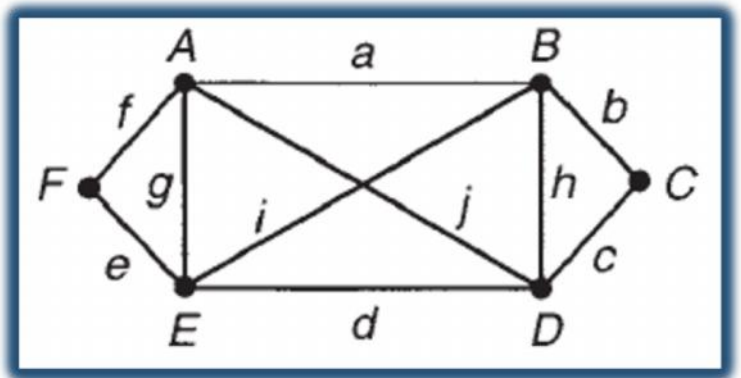
Paths and circuits

Theorem

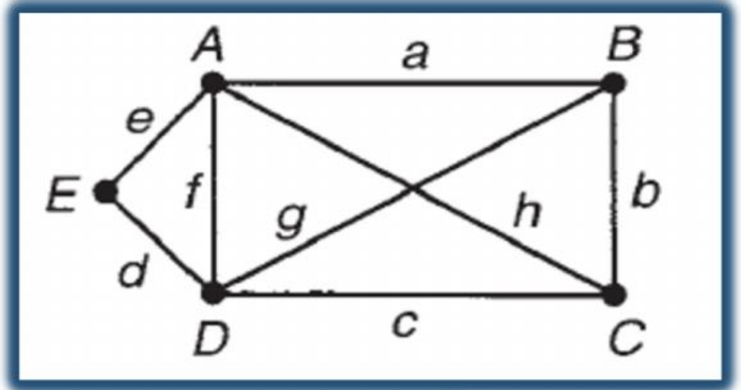
Let G be a connected graph.

- If all the vertices of G have even degree, then G is **Eulerian**.
- If exactly two vertices of G have odd degree, then G is **semi-Eulerian**, and every Eulerian path in G must start at one of the vertices with odd degree and end at the other.
- If **G** has more than two vertices with odd degree, then **G** is **neither Eulerian nor semi-Eulerian**.

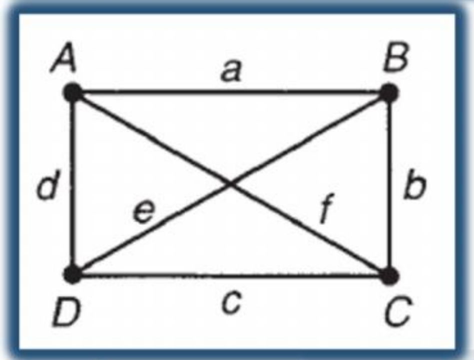
Paths and circuits



All the vertices have even degree, so the graph is Eulerian. An Eulerian circuit $\boldsymbol{Aabcdefgihj}$.



Two vertices, $\boldsymbol{B}$ and $\boldsymbol{C}$, have odd degree, so it is possible to find an Eulerian path from $\boldsymbol{B}$ to $\boldsymbol{C}$. One such path is $\boldsymbol{Baedcbgfh}$.



There are four vertices with odd degree, so this graph does not have Eulerian path.

Paths and circuits

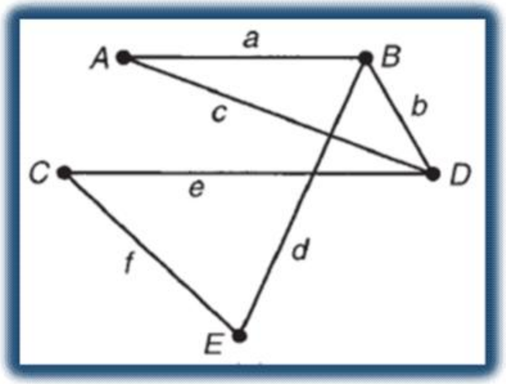
Let G be a connected graph. A **Hamiltonian path** is a path that includes every vertex of G exactly once, except only that the first and the last vertices may coincide.

- A **Hamiltonian circuit** is a Hamiltonian path for which the first and the last vertices coincide.

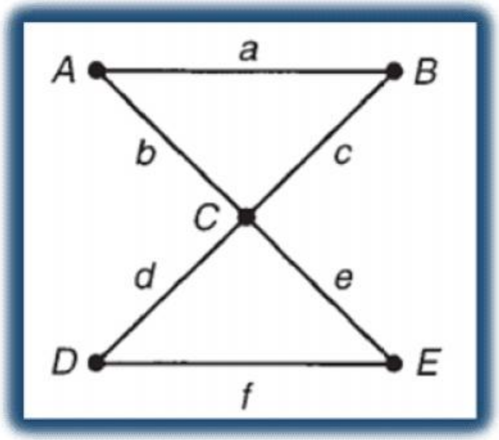
Note that a Hamiltonian path or a Hamiltonian circuit need not use all of the edges of the graph.

Paths and circuits

Example



The graph is Hamiltonian, because it has a Hamiltonian circuit, $\boldsymbol{Acefda}$.



The graph has a Hamiltonian path, $\boldsymbol{Babef}$; however, the graph is **not Hamiltonian**, because any circuit passing through all of the vertices would have to pass through vertex C **twice**.

Paths and circuits

Definition

A **cycle** is a path in a graph with the following properties:

- It includes at least one edge.
- There are no repeated edges.
- The first and last vertices coincide, but there are no other repeated vertices.

A cycle with n edges is called n-cycle.

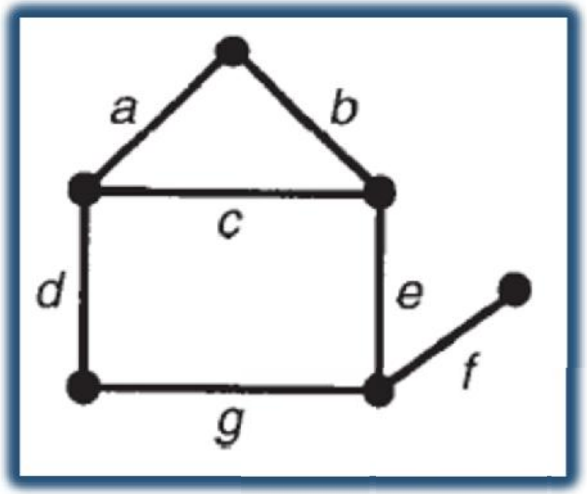
A cycle is thus a circuit with **some additional properties**.

- A vertex by itself is not a cycle, because a cycle must include at least one edge.
- If two cycles consist of the same vertices and edges, we regard them as the same cycle.

For example, a cycle with edges $e_1e_2e_3e_4$ is the same cycle as $e_2e_3e_4e_1$ and $e_3e_2e_1e_4$.

Paths and circuits

- **Note** that when we write down a cycle, we do not need to state a starting vertex.



- There are **three cycles**: $abc, cegd, abegd$.