Dualization of a Monotone Boolean Function

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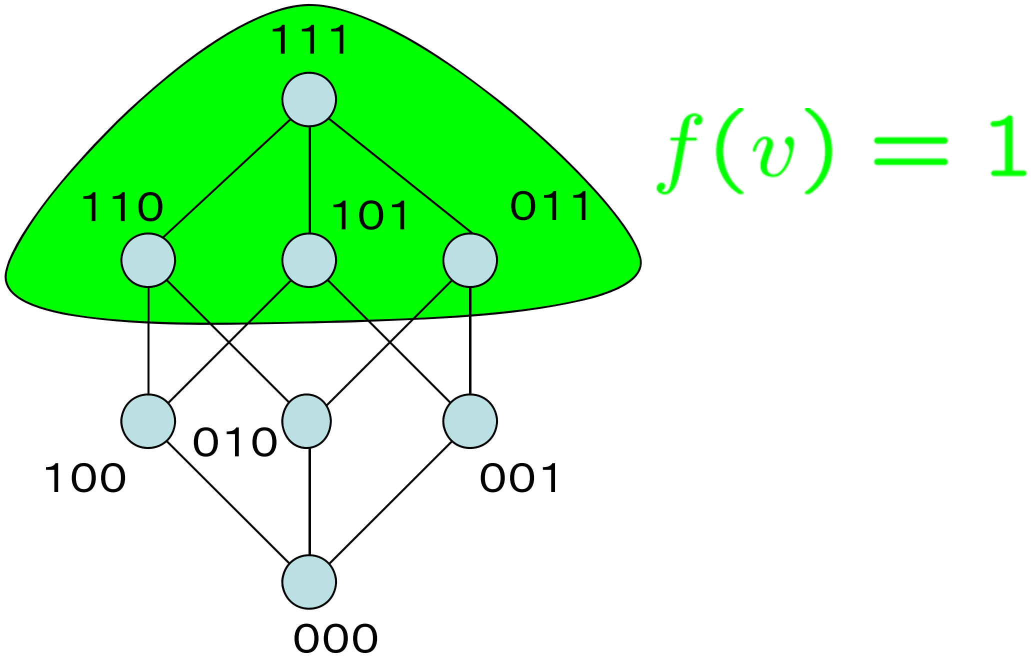
Definition

Boolean function $f : \{0,1\}^n \to \{0,1\}$

Monotone: $v \leq w \Longrightarrow f(v) \leq f(w)$



Definition

Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$

Monotone: $v \leq w \Longrightarrow f(v) \leq f(w)$

Conjunctive normal form (CNF) Prime CNF

$$\varphi = (x_1 \vee x_2)(x_2 \vee x_3)(x_3 \vee x_1)$$

~~$\overline{x}_i$~~

Disjunctive normal form (DNF)

$$\psi = x_1 x_2 \vee x_2 x_3 \vee x_3 x_1 \vee \cancel{x_1 x_2 x_3}$$

Prime DNF

Monotone Dualization

Input: A CNF φ of (a monotone function) f.

Output: Prime DNF ψ of f.

$$\text{Ex. } \varphi = (x_1 \vee x_2)(x_2 \vee x_3)(x_3 \vee x_4)$$

$$\equiv \xcancel{x_1 x_2 x_3} \vee x_1 x_2 x_4 \vee x_1 x_3 \vee \xcancel{x_1 x_3 x_4} \vee \cdots$$

$$\equiv x_1 x_3 \vee x_2 x_3 \vee x_2 x_4 = \psi$$

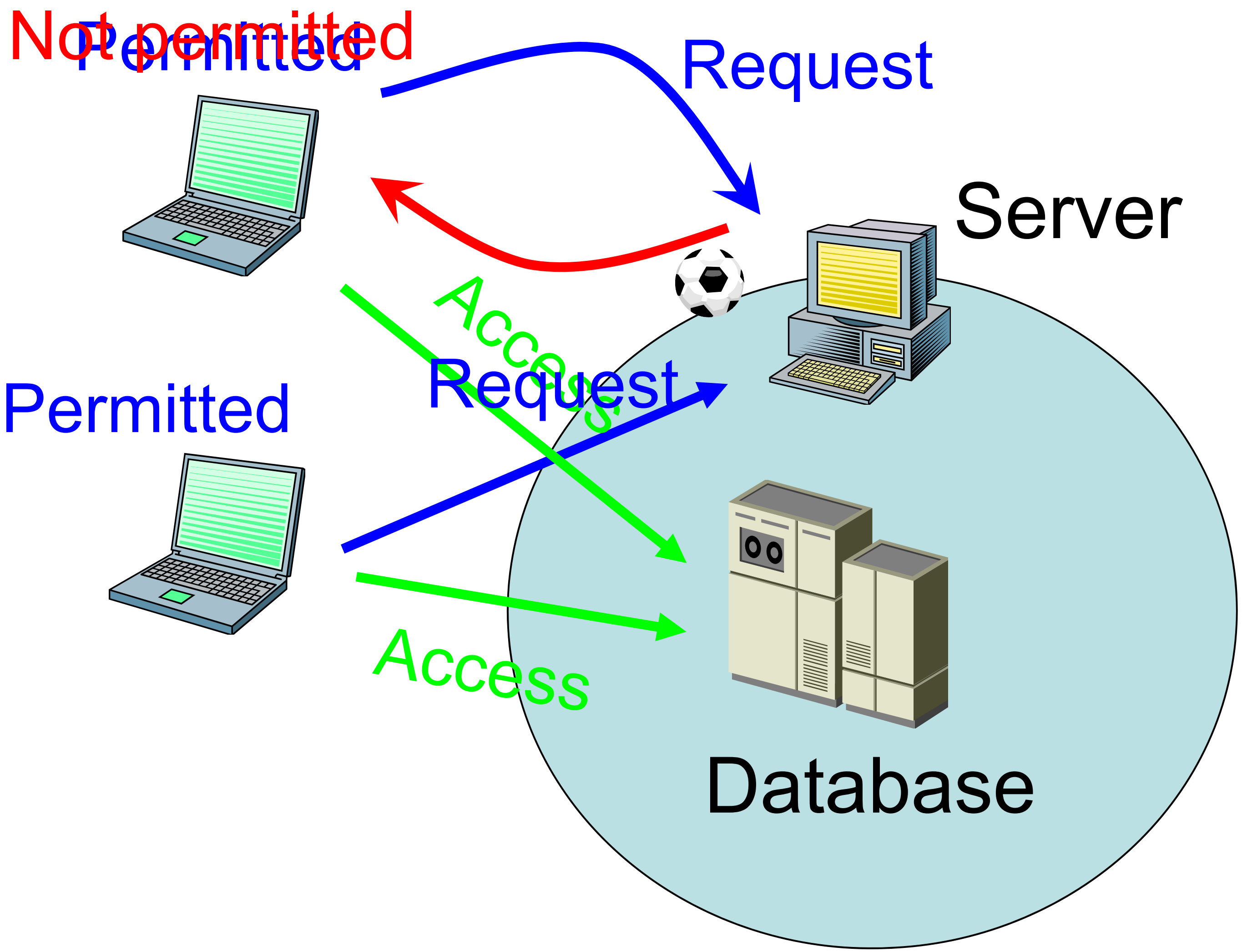
Applications

Artificial Intelligence, Database theory,
Distributed System, Mathematical Programming,
Game theory, Computational Learning,
Formal Concept Analysis, …………

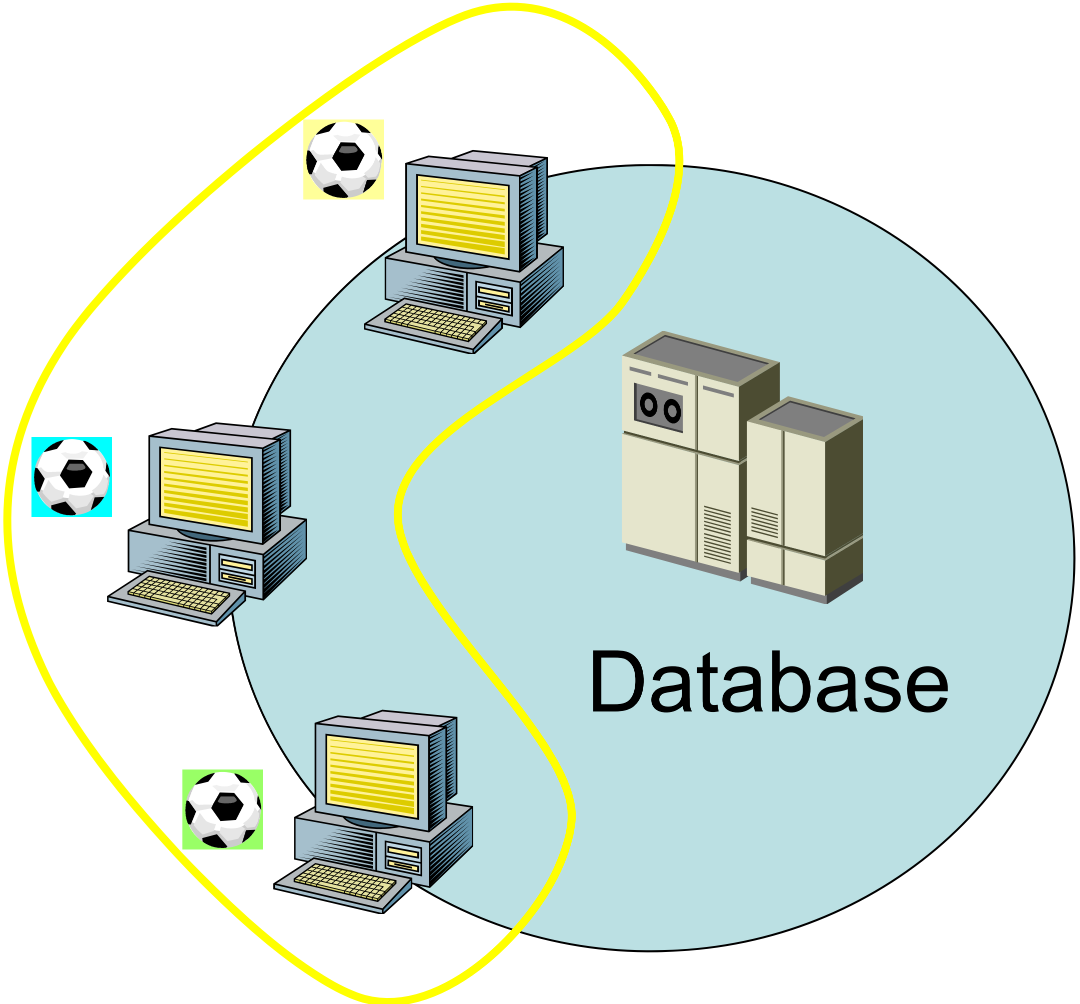
Many (Quasi-)Poly equivalent problems

[Eiter, Gottlob 91], [Bioch, Ibaraki 93],
[Domingo, Mishara, Pitt 99], [Eiter, Makino, 2003] ,
[Boros, Gurvich, Khachiyan, Makino 2000,2002 2004],
[Khachiyan, Boros, Elbassioni, Gurvich, Makino, 2001, 2003],
[Khachiyan, Boros, Elbassioni, Gurvich, 2003],
……………………………………………………………

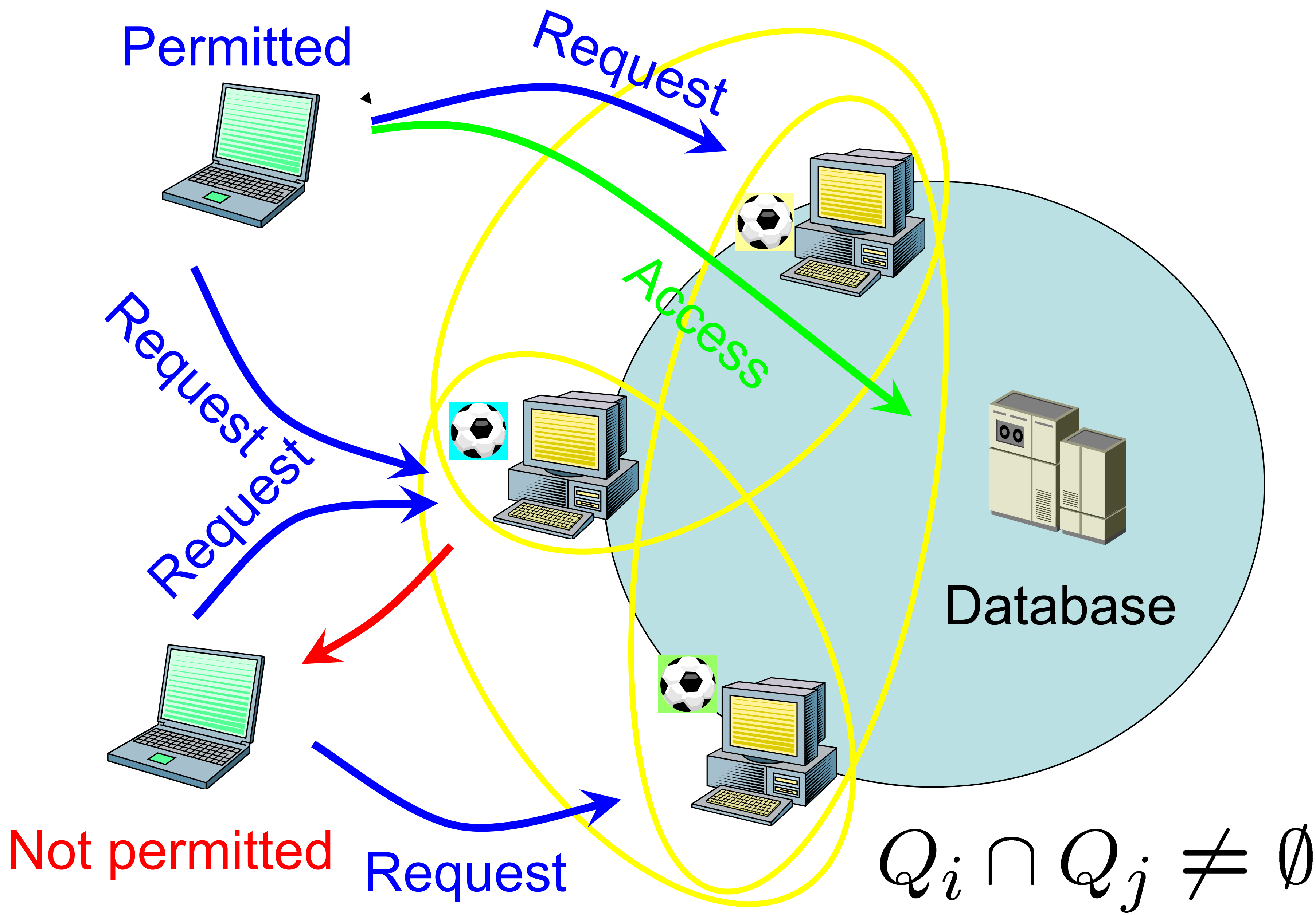
1. Coterie in distributed system
2. Relational database



Problem：fault-tolerance



Problem： fault-tolerance

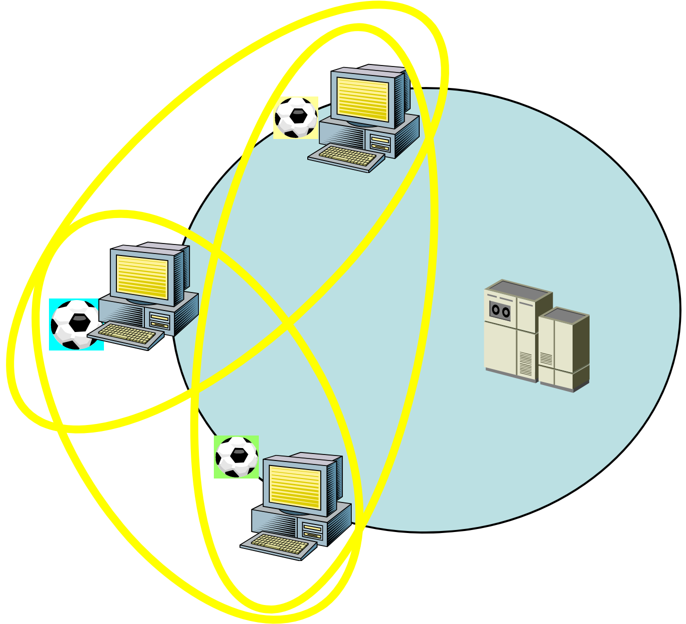

Permitted
Request
Access
Request
Request
Database
Not permitted
Request
$Q_i \cap Q_j \neq \emptyset$

Coterie

$$\mathcal{C} = \{Q_1, \ldots, Q_m\} \subseteq 2^V$$

$$\forall i \neq j\colon\ Q_i \cap Q_j \neq \emptyset$$

$$Q_i \not\subseteq Q_j$$



ND:

$$\nexists Q \subseteq V : \mathcal{C} \cup \{Q\} \setminus \{Q_i \in \mathcal{C} \mid Q_i \supset Q\}$$

is coterie

Deciding if coterie is ND:
poly. equivalent to monotone dualization

Functional dependency in relational database

Ex.

Teacher	Hour	Room
Ullman	Mon 10:30--12:00	120
Fagin	Mon 10:30--12:00	110
Fagin	Wed 13:30--15:00	120
Ullman	Wed 13:30--15:00	110
Ullman	Mon 10:30--12:00	100
Maier	Wed 13:30--15:00	120

FD

$\{T,H\} \Rightarrow R$

$\{H,R\} \Rightarrow T$

Database Design

Design by examples   Poly. equivalent to monotone dualization

Input: A set $\mathcal{F}$ of FDs

Output: Relation R that satisfies $\mathcal{F}$ only
(Armstrong Relation)

Input: A relation R

Output: A set of FDs that holds in R

Monotone Dualization

Input: A CNF φ of (a monotone function) f.

Output： Prime DNF ψ of f.

例: $\varphi = (x_1 \vee x_2)(x_2 \vee x_3)(x_3 \vee x_4)$

$$\equiv x_1 x_3 \vee x_2 x_3 \vee x_2 x_4 \;=\; \psi$$

Complexity？

Output size: Large

Input $\varphi = (x_1 \vee y_1)(x_2 \vee y_2) \cdots (x_k \vee y_k)$

$k = 3$

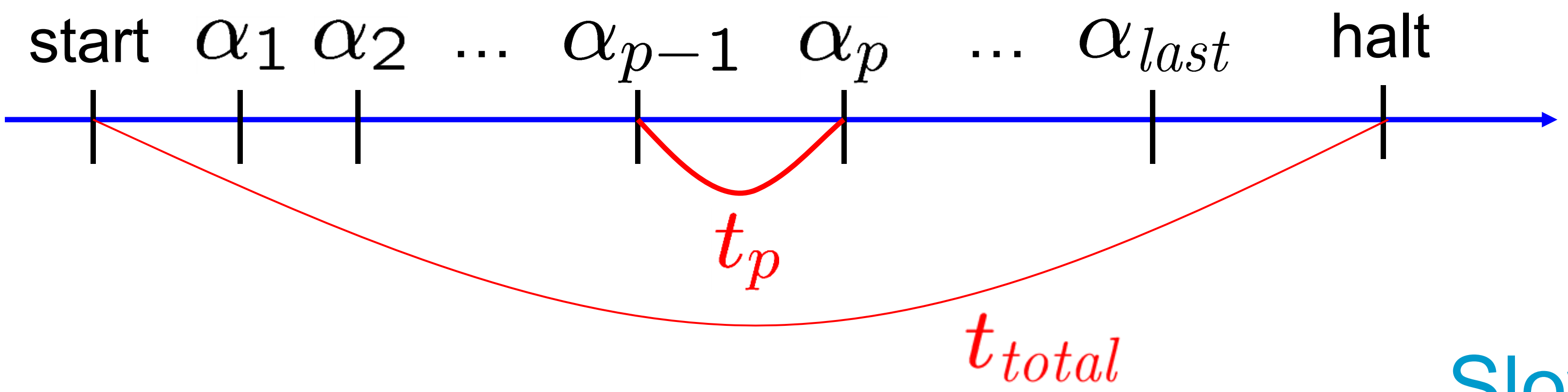
$$\varphi = (x_1 \vee y_1)(x_2 \vee y_2)(x_3 \vee y_3)$$

$$\psi = x_1x_2x_3 \vee x_1x_2y_3 \vee x_1y_2x_3 \vee x_1y_2y_3 \vee y_1x_2x_3 \vee y_1x_2y_3 \vee y_1y_2x_3 \vee y_1y_2y_3$$

Output: $\psi = \bigvee_{z_i \in \{x_i, y_i\}} (\wedge_{i=1}^{k} z_i)$ $\quad |\psi| = 2^k$

Monotone Dualization: Generation problem !!

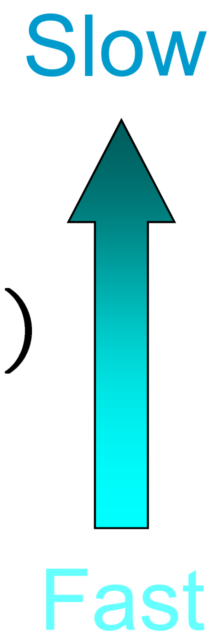
Complexity of generation problem



Output P: $t_{total} = poly(\text{input} + \text{output})$

Incremental P: $t_p = poly(\text{input} + \sum_{i=1}^{p-1} |\alpha_i|)$

P delay: $t_p = poly(\text{Input})$



Monotone Duality

Input：monotone CNF φ & monotone DNF ψ.

Question：$\varphi \equiv \psi$？

[Bioch, Ibaraki 93]

Monotone Dualization: Output P

$\Updownarrow$

Monotone Dualization：Incremental P

$\Updownarrow$

Monotone Duality：P

Polynomial？ OPEN

Bioch, Boros, Crama, Domingo, Eiter, Elbassioni, Fredman, Gaur, Gogic,Gottlob, Gunopulos, Gurvich, Hammer, Ibaraki, Johnson, Kameda, Kavvadias, Khachiyan, Khardon, Kogan, Krishnamurti, Lawler, Lenstra, Lovasz, Mannila, Mishra, Papadimitriou, Pitt, Rinnoy Kan, Sideri, Stavropoulos, Tamaki, Toinonen, Uno, Yannakakis, ... (until 2002), Kuznetsov, ……..

Polynomial？ OPEN

茨木俊秀．単調論理関数の同定問題とその複雑さ,
離散構造とアルゴリズム III,室田一雄（編）近代科学社,
pp. 1--33, 1994. Toshihide Ibaraki

Johnson. Open and closed problems in NP-completeness.
Lecture given at the International School of Mathematics
``G. Stampacchia'': Summer School ``NP-Completeness:
The First 20 Years'', Erice, Italy, June 20-27, 1991.

Lovasz. Combinatorial optimization: Some problems
and trends, DIMACS Technical Report 92-53, 1992.

Papadimitriou. NP-completeness: A retrospective,
In: Proc. 24th International Colloquium on Automata,
Languages and Programming (ICALP), pp.2--6,
Springer LNCS 1256, 1997.

Mannila. Local and Global Methods in Data Mining:
Basic Techniques and Open Problems In: Proc. 29th
ICALP, pp.57--68, Springer LNCS 2380, 2002.

Eiter, Gottlob. Hypergraph Transversal Computation
and Related Problems in Logic and AI, In: Proc. European
Conference on Logics in Artificial Intelligence (JELIA),
pp. 549-564, Springer LNCS 2224, 2002

Best known

[Fredman, Khachiyan, 96]

Quasi-Polynomial

$N^{0(\log N / \log\log N)}$ time,

where $N = |\varphi| + |\psi|$

[Gaur, Krishnamurti, 99], [Tamaki, 00],

[Elbassioni 06], [Boros, Makino 09]

Monotone Duality

Input： monotone CNF φ & monotone DNF ψ.

Question： $\varphi \equiv \psi$?

[Bioch, Ibaraki 93]

Monotone Dualization: Output P — Output Quasi-P

⇕

Monotone Dualization： Incremental P — Incremental Quasi-P

⇕

Monotone Duality： P — Quasi-P

Non-Monotone Duality

Input: monotone CNF φ & monotone DNF ψ.

Question: $\varphi \not\equiv \psi$?

Non-Monotone Duality in NP

Evidence v : $\varphi(v) \neq \psi(v)$

$\mathrm{NP} \not\subseteq \mathrm{DTIME}(N^{\log N})$ is believed

Non-Monotone Duality $\neq$ NP−complete

Guessed Bits

[Eiter, Gottlob, Makino, 02]

$o(\log^2 N)$ guessed bits

Non-Monotone Duality $\subseteq \beta_2\mathrm{P}$

$\mathrm{NP} \not\subseteq \beta_2\mathrm{P}$ is believed

Non-Monotone Duality $\neq$ NP−complete

Probabilistically？

[Shmulevich, Korshunov, Astola, 01]
Almost all monotone CNFs is dualizable in polynomial delay.

$\mathcal{M}(n)$: Class of monotne CNFs of n variables

$\mathcal{M}_4(n)$: Class of 4-tight CNFs of n variables

$$\lim_{n\to\infty} \frac{|\mathcal{M}_4(n)|}{|\mathcal{M}(n)|} = 1.$$

[Makino, Ibaraki, 97] k-tight CNF: P delay

Current Research

1. Polynomially solvable subclasses

2. Generalization of quasi-polynomial time algorithm

Limited CNF

1. k-clause CNF (#clauses $\leqq k$)

$$\varphi = (x_1 \vee x_2)x_4(x_3 \vee x_1)$$ 3-clause CNF

2. k-CNF (Each clause contains $\leqq k$ literals)

$$\varphi = (x_1 \vee x_2)x_4(x_3 \vee x_1)$$ 2-CNF

2 1 2

3. Read-k CNF (Each variable appears $\leqq k$ times)

$$\varphi = (x_1 \vee x_2)x_4(x_3 \vee x_1)$$ Read-2 CNF

x_1 2 x_2 1 x_3 1 x_4 1

⋮

1. k-clause CNF（#clauses $\leqq k$）

$\varphi = (x_1 \vee x_2)x_4(x_3 \vee x_1)$ 3-clause CNF

k : constant

Expanded DNF has $\leqq n^k$ terms （Input） P

$k = \log n$

[Makino 03] Incremental P

2. k-CNF （Each clause contains ≦ k literals）

$$\varphi = (x_1 \vee x_2) x_4 (x_3 \vee x_1) \quad \text{2-CNF}$$

2 1 2

$k = 2$ ⟺ Generating maximal cliques in graph

[Tsukiyama et al. 77], [Chiba, Nishizeki, 85], [Johnson, Yannakakis, Papadimitriou, 80], [Makino, Uno, 04]

$O(M(n)) = O(n^{2.376})$ Delay

k : constant (≥ 3) P delay ?

[Boros et al., 98], [Eiter, Gottlob, 95], [Eiter, Gottlob, Makino, 03] Incremental P

$k = \log n$???

3. Read-k CNF（Each variable appears $\leqq k$ times）

$$\varphi = (x_1 \vee x_2)x_4(x_3 \vee x_1)$$ Read-2 CNF

x_1 2 x_2 1 x_3 1 x_4 1

k : constant

[Domingo, Mishra, Pitt, 99] P delay

$k = \log n$

[Eiter, Gottlob, Makino, 03] Incremental P

P delay？

Decomposition Schemes

$$\varphi = \overset{c_1}{(x_1 \vee x_2)}\overset{c_2}{(x_2 \vee x_3 \vee x_4)}\overset{c_3}{(x_1 \vee x_3)} \cdots \underset{c_m}{(x_4 \vee x_5)}$$

φ_i: CNF having the first i clauses in φ

Ex. $\varphi_1 = (x_1 \vee x_2)$

$\varphi_2 = (x_1 \vee x_2)(x_2 \vee x_3 \vee x_4)$

$\varphi_3 = (x_1 \vee x_2)(x_2 \vee x_3 \vee x_4)(x_1 \vee x_3)$

ψ_i: Prime DNF $(\equiv \varphi_i)$ Objective ψ_m

$$\psi_1 = \varphi_1$$

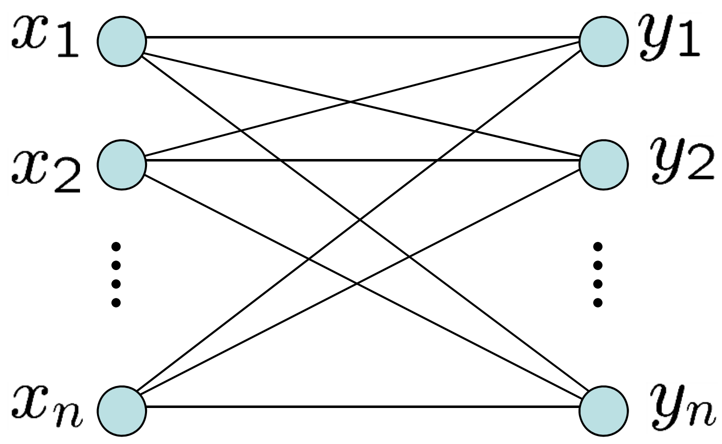
$$\psi_i \equiv \psi_{i-1} \wedge c_i, \quad i = 2, \cdots, m$$

$$\varphi = \overset{c_1}{(x_1 \vee x_2)}\overset{c_2}{(x_2 \vee x_3 \vee x_4)}\overset{c_3}{(x_1 \vee x_3)} \cdots \underset{c_m}{(x_4 \vee x_5)}$$

$$\psi_1 = \varphi_1$$
$$\psi_i \equiv \psi_{i-1} \wedge c_i, \quad i = 2, \cdots, m$$

$$\psi_1 = \varphi_1 = x_1 \vee x_2$$
$$\psi_2 (\equiv (x_1 \vee x_2)(x_2 \vee x_3 \vee x_4))$$
$$= x_1 x_3 \vee x_1 x_4 \vee x_2$$
$$\psi_3 (\equiv (x_1 x_3 \vee x_1 x_4 \vee x_2)(x_1 \vee x_3))$$
$$= x_1 x_3 \vee x_1 x_4 \vee x_1 x_2 \vee x_2 x_3$$

Efficient algorithm？



$$\varphi = \bigwedge_{i,j \in \{1,\ldots,n\}} (x_i \vee y_j)$$

$$\psi = x_1 x_2 \ldots x_n \vee y_1 y_2 \ldots y_n$$

$$\varphi_n = (x_1 \vee y_1)(x_2 \vee y_2) \cdots (x_n \vee y_n)$$

$$\psi_n = \bigvee_{z_i \in \{x_i, y_i\}} (\wedge_{i=1}^{n} z_i) \qquad |\psi_n| = 2^n$$

Good ordering？ No！ [Takata 02]

$$\varphi = (x_1 \vee x_2)(x_1 \vee x_3)(x_2 \vee x_3 \vee x_4)(x_1 \vee x_4)$$

$$\varphi_i = \bigwedge_{c \in \varphi:\, c \subseteq \{x_1, \ldots, x_i\}} c$$

$$\varphi_0 = \varphi_1 = 1$$

$$\varphi_2 = (x_1 \vee x_2)$$

$$\varphi_3 = (x_1 \vee x_2)(x_1 \vee x_3)$$

$$\varphi_4 = \varphi$$

$$\Delta_3 = (x_1 \vee x_3)$$

ψ_i: Prime DNF $(\equiv \varphi_i)$ Objective ψ_m

$$\psi_0 = 1$$

$$\psi_i \equiv \psi_{i-1} \wedge \Delta_i, \quad i = 1, \cdots, n$$

$$\Delta_i = \varphi_i \setminus \varphi_{i-1}$$

$$\varphi = (x_1 \vee x_2)(x_1 \vee x_3)(x_2 \vee x_3 \vee x_4)(x_1 \vee x_4)$$

$$\varphi_i = \bigwedge_{c \in \varphi:\, c \subseteq \{x_1, \ldots, x_i\}} c$$

ψ_i: Prime DNF $(\equiv \varphi_i)$ Objective ψ_m

$\Delta_i = \varphi_i \setminus \varphi_{i-1}$

$$\psi_0 = 1$$

$$\psi_i \equiv \psi_{i-1} \wedge \Delta_i, \quad i = 1, \cdots, n$$

$|\psi_i| \leq |\psi|$ for $i = 1, \ldots, n$

Decomposition Scheme II

[Lawler, Lenstra, Rinnoy Kan, 80]

[Eiter, Gottlob, Makino, 03]

Decomposition Scheme II

+

[Johnson, Yannakakis, Papadimitriou, 88]

2 - CNF (maximal cliques in graph)

[Eiter, Gottlob, Makino, 03]

degenerate CNF, read-k CNF, acyclic CNF,
CNF with bounded treewidth, k-CNF, …

unified, Efficient, Widely applicable

Current Research

1. Polynomially solvable subclasses
2. Generalization of quasi-polynomial time algorithm

Monotone Dualization

$\Updownarrow$

$$Ax \geq 1,\ x \geq 0 \text{ where } A \in \{0,1\}^{m\times n}$$

Computing the Hilbert basis for the ideal
(Generating all minimal integral solutions)

Ex. $\varphi = (x_1 \vee x_2)(x_2 \vee x_3)(x_3 \vee x_4)$

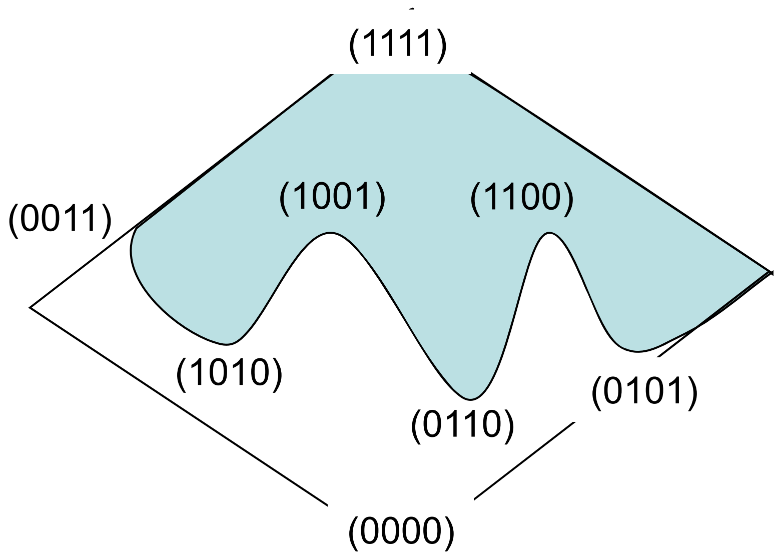
$$\equiv x_1x_3 \vee x_2x_3 \vee x_2x_4 = \psi$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Ex. $\varphi = (x_1 \vee x_2)(x_2 \vee x_3)(x_3 \vee x_4)$

$$\equiv x_1x_3 \vee x_2x_3 \vee x_2x_4 \;=\; \psi$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \;\; x_1, x_2, x_3, x_4 \geq 0$$



Generalization

Monotone Dualization

$\Updownarrow$

$Ax \geq 1,\ x \geq 0$ where $A \in \{0,1\}^{m\times n}$

Computing the Hilbert basis for the ideal
(Generating all minimal integral solutions)

$Ax \geq b,\ 0 \leq x \leq u$ where

$A \in \mathbb{R}_+^{m\times n},\ b \in \mathbb{R}^m,\ u \in (\mathbb{R} \cup \{+\infty\})^n$

Generating all minimal integral solutions

Conjecture [Lawler, Lenstra, Rinnoy Kan, 80]

All minimal integral solutions of monotone linear system cannot be generated in Output-P, unless P=NP

The corresponding decision problem is NP-complete

Th [Boros, Elbassioni, Gurvich, Khachiyan, Makino, 01]

All minimal integral solutions of monotone linear system cannot be generated in incremental Quasi-P.

$N^{Polylog N}$

Th [Makino, Ibaraki, 94]

All maximal integral infeasible solutions of monotone linear system cannot be generated in Output-P, unless P=NP

Monotone separable inequalities

$$\sum_{i=1}^{n} f_{ij}(x_i) \geq t_j, \quad j = 1, \ldots, m,$$

where, $f_{ij} : \mathbb{R} \to \mathbb{R}$ monotone & P-computable

$f_{ij}(x) \geq f_{ij}(y)$ if $x \geq y$

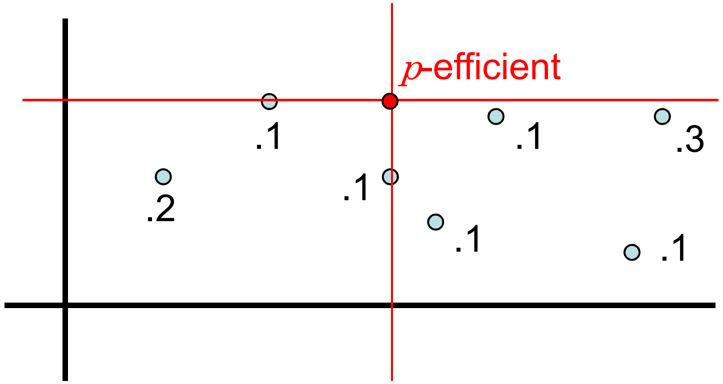
Th [Boros, Elbassioni, Gurvich, Khachiyan, Makino, 03]

All minimal integral solutions of monotone separable inequalities can be generated in incremental Quasi-P.

Efficient points in discrete probabilistic distribution

ξ : random variable with finite support $S \subseteq \mathbb{Z}^n$

$$\sum_{q \in \mathcal{S}} \Pr[\xi = q] = 1, \ \Pr[\xi = q] > 0 \text{ for } q \in \mathcal{S}$$



$\circ \in S$

$p = 0.45$

$p \in (0,1)$に対して

p-efficient $x \in \mathbb{Z}^n$: minimal point s.t. $\Pr[\xi \le x] > p$

p-inefficient $x \in \mathbb{Z}^n$: maximal point s.t. $\Pr[\xi \le x] \le p$

Generating efficient and inefficient points

Application to Stochastic programming

Th [Boros, Elbassioni, Gurvich, Khachiyan, Makino, 03]

All inefficient points can be generated in incremental Quasi-P time.

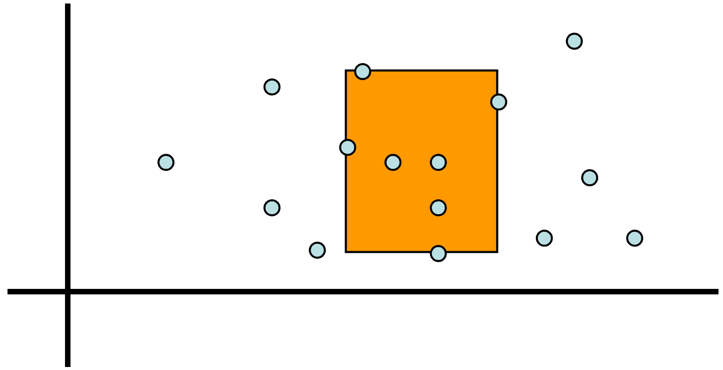
Th [Boros, Elbassioni, Gurvich, Khachiyan, Makino, 03]

All efficient points cannot be generated in Output P time, unless P=NP.

maximal k-box (k-hyperrectangle)

$\mathcal{S}$: Set of point in $\mathbb{R}^n$

整数 $k (\leq |\mathcal{S}|)$



$\circ \in S$

$k = 3$

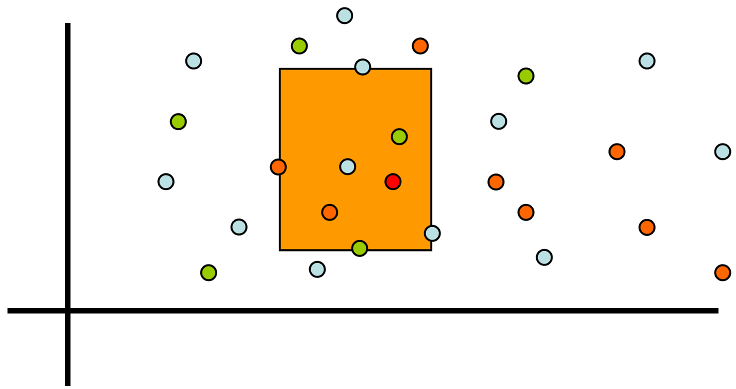
k-box

$\prod_{i=1}^{n} [l_i, u_i] \subseteq \mathbb{R}^n$: Interior contains at mot k point

Generalization of maximal k-box

$$\mathcal{S} \subseteq \mathbb{R}^n,\ C : \mathcal{S} \mapsto \{1, \ldots, r\},\ w : \mathcal{S} \mapsto \mathbb{R}_+,$$

$$t \in \mathbb{R}_+^r,\ \delta \geq 0$$



$$\circ \in S$$

$$w \equiv 1$$

$$t(\,\bullet\,) = 2$$

$$t(\,\circ\,) = 1$$

$$t(\,\bullet\,) = 1$$

$$\text{diameter}\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \leq \delta$$

Generating maximal k-boxes

Applications to machine learning, computational geometry, and data mining

Th [Boros, Elbassioni, Gurvich, Khachiyan, Makino, 03]

All maximal k-boxes can be generated in incremental Quasi-P time.

Dual-bounded generation problems

Boros, Elbassioni, Gurvich, Khachiyan, Makino, 01,02,03,04

1. Reduce to generating minimal feasible points in **good** monotone system in good partial ordered set

$$\{x \in \mathbb{Z}^n \mid \ell \leq x \leq u\}$$

2. Generate both minimal feasible and maximal infeasible points

Extend Fredman-Khachiyan Quasi-P algorithm

Dual-bounded generation problems

Needed

Not needed

2. Generate both minimal feasible and maximal infeasible points

Extend Fredman-Khachiyan Quasi-P algorithm

$\mathcal{F}$: set of minimal feasible points

$\mathcal{F}^d$: set of maximal infeasible points

In general $|\mathcal{F}^d| \gg |\mathcal{F}|$

In general, $|\mathcal{F}^d| \gg |\mathcal{F}|$

Dual-bounded $|\mathcal{F}^d| \leq quasi\text{-}poly(\text{input}, |\mathcal{F}|)$

Dual-bounded ⇨ Output Qusai-P

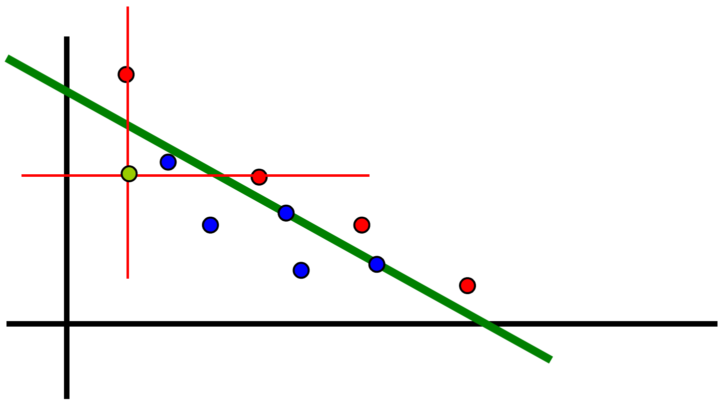
Uniform dual-bounded $\forall \mathcal{H} \subseteq \mathcal{F}$

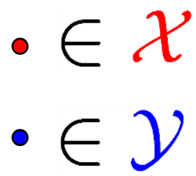
$$|\mathcal{F}^d \cap \mathcal{H}^d| \leq quasi\text{-}poly(\text{input}, |\mathcal{H}|)$$

Uniform dual-bounded ⇨ Incremental Quasi-P

Intersection Inequality

$\mathcal{X}, \mathcal{Y} \subseteq \mathbb{R}^n$: Finite set





(P1) $\exists w(x) \, (= \sum_{i=1}^{n} w_i x_i) = t$:

$$w(x) > t \geq w(y) \quad (\forall x \in \mathcal{X}, \ \forall y \in \mathcal{Y}).$$

(P2) $\forall x \, (\neq) \, x' \in \mathcal{X}, \ \exists y \in \mathcal{Y}$: $y \geq x \wedge x'$.

$$\mathcal{Y} \neq \emptyset \implies |\mathcal{X}| \leq n|\mathcal{Y}|$$

Generalization of intersection inequalities

$\mathcal{P}_i \ (i = 1, \ldots, n)$: poset s.t. $\forall x, y \in \mathcal{P}_i$, $\exists x \wedge y \in \mathcal{P}_i$

$x^- = \{y \in \mathcal{P}_i \mid y \preceq x\}$

(P1) $\exists w : \cup_{i=1}^n \mathcal{P}_i \mapsto \mathbb{R}_+$, $t \in \mathbb{R}_+$:

$$\sum_{i=1}^n w(x_i^-) > t \geq \sum_{i=1}^n w(y_i^-) \ \forall x \in \mathcal{X}, y \in \mathcal{Y}.$$

(P2) $\forall x' \neq x'' \in \mathcal{X}$, $\exists y \in \mathcal{Y}$: $y \succeq x' \wedge x''$.

$q_i(x) = |\{z \in \mathcal{P}_i : z \not\preceq x, z$ has a child $z' \preceq x\}|$

$\mathcal{Y} \neq \emptyset \implies |\mathcal{X}| \leq \sum_{y \in \mathcal{Y}} \sum_i q_i(y_i)$

Conclusion

1. Monotone dualization (def. & applications)

2. Complexity of monotone dualization
   （time, guessed bits, probabilistically）
3. P-solvable classes
4. Generalization of montone dualization
   （Quasi-P solvable）

Fredman-Khachiyan アルゴリズム

$$\varphi \equiv \psi\ ?$$

もしYESならば,

例: $\varphi = (x_1 \vee x_2)(x_2 \vee x_3)(x_3 \vee x_4)$

$$\equiv x_1 x_3 \vee x_2 x_3 \vee x_2 x_4 \quad = \psi$$

$$\mathcal{F} = \{\{1,2\},\{2,3\},\{3,4\}\}$$

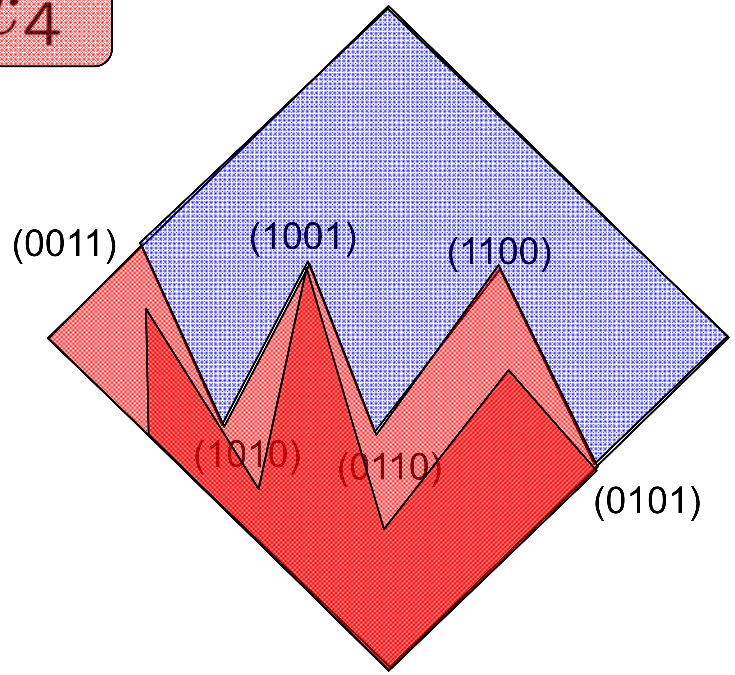
$$\mathcal{G} = \{\{1,3\},\{2,3\},\{2,4\}\}$$

性質1 $\forall F \in \mathcal{F}, \forall G \in \mathcal{G} : F \cap G \neq \emptyset$

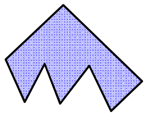
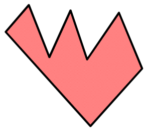
例: $\varphi = (x_1 \vee x_2)(x_2 \vee x_3)(x_3 \vee x_4)$

$\equiv x_1x_3 \vee x_2x_3 \vee x_2x_4 = \psi$

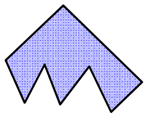
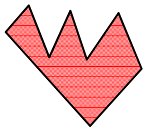
$\overline{\varphi} \equiv \overline{x}_1\overline{x}_2 \vee \overline{x}_2\overline{x}_3 \vee \overline{x}_3\overline{x}_4$



もし $\varphi \equiv \psi$ ならば,

 $+$  $= \{0,1\}^n$

もし 性質1 $\forall F \in \mathcal{F}, \forall G \in \mathcal{G} : F \cap G \neq \emptyset$ ならば,

 $\cap$  $\neq \emptyset$

例: $\varphi = (x_1 \vee x_2)(x_2 \vee x_3)(x_3 \vee x_4)$

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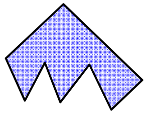
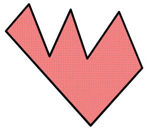
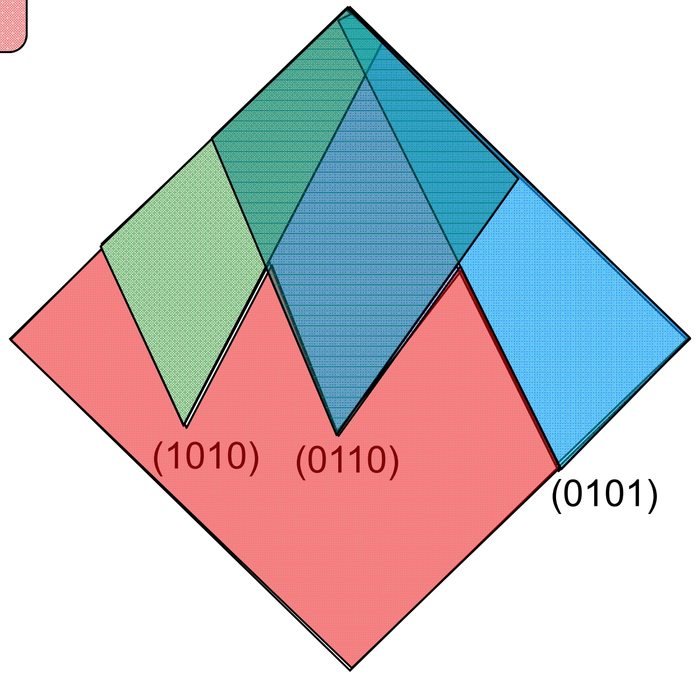
 $+$  $= \{0,1\}^n$?



$$\sum_{G \in \mathcal{G}} 2^{n-|G|} + \sum_{F \in \mathcal{F}} 2^{n-|G|} \geq 2^n \ ?$$

NOならば,

$v : \varphi(v) \neq \psi(v)$が多項式時間で求められる

例: $\varphi = (x_1 \vee x_2)(x_2 \vee x_3)(x_3 \vee x_4)$

$$\equiv x_1 x_3 \vee x_2 x_3 \vee x_2 x_4 = \psi$$

$$\overline{\varphi} \equiv \overline{x}_1\overline{x}_2 \vee \overline{x}_2\overline{x}_3 \vee \overline{x}_3\overline{x}_4$$

YESならば, $\displaystyle\sum_{G \in \mathcal{G}} 2^{n-|G|} + \sum_{F \in \mathcal{F}} 2^{n-|F|} \geq 2^n.$

$\ell = \min\{|H| \mid H \in \mathcal{G} \cup \mathcal{F}\}$ とすると

$$(|\mathcal{G}| + |\mathcal{F}|)2^{n-\ell} \geq 2^n \iff (|\mathcal{G}| + |\mathcal{F}|) \geq 2^\ell$$

$$\iff \ell \leq \log(|\mathcal{G}| + |\mathcal{F}|)$$

例: $\varphi = (x_1 \vee x_2)(x_2 \vee x_3)(x_3 \vee x_4)$

$$\equiv x_1x_3 \vee x_2x_3 \vee x_2x_4 \quad = \psi$$

$$\overline{\varphi} \equiv \overline{x}_1\overline{x}_2 \vee \overline{x}_2\overline{x}_3 \vee \overline{x}_3\overline{x}_4$$

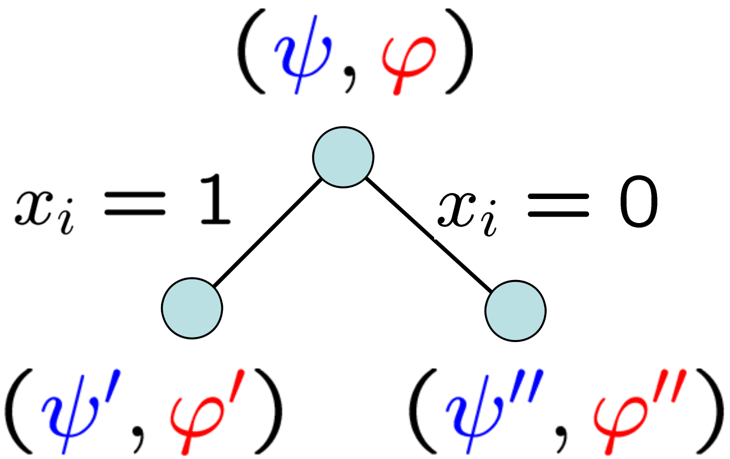
YESならば, $\sum_{G \in \mathcal{G}} 2^{n-|G|} + \sum_{F \in \mathcal{F}} 2^{n-|F|} \geq 2^n.$

$$\min\{|H| \mid H \in \mathcal{G} \cup \mathcal{F}\} \leq \log(|\mathcal{G}| + |\mathcal{F}|)$$

性質1 $\forall F \in \mathcal{F}, \forall G \in \mathcal{G} : F \cap G \neq \emptyset$ より

$\exists i \in H^*(\in \mathcal{G})$:

$$|\{F \in \mathcal{F} \mid i \in F\}| \geq |\mathcal{F}| / \log(|\mathcal{G}| + |\mathcal{F}|)$$



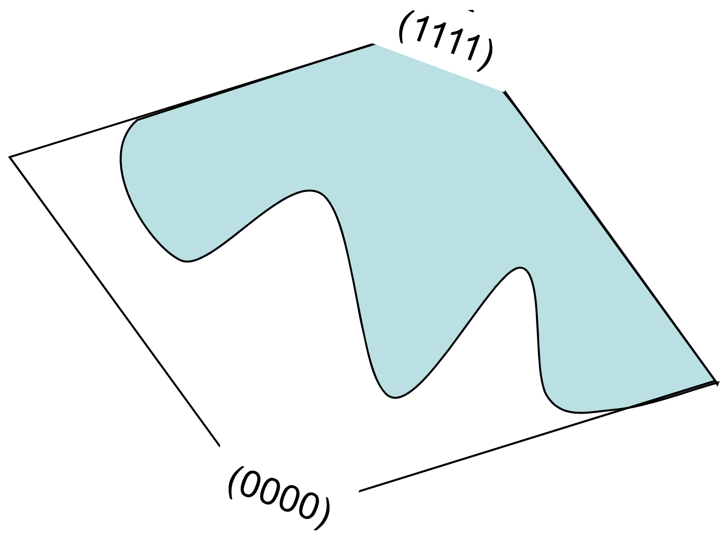
$$|\psi'| \leq |\psi|$$

$$|\varphi'| \leq (1 - 1/\log(|\psi| + |\varphi|))|\varphi|$$

$$|\psi''| \leq |\psi| - 1$$

$$|\varphi''| \leq |\varphi|$$

$N^{O(\log^2 N)}$ 時間　ただし, $N = |\varphi| + |\psi|$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