Post's Functional Completeness Theorem

Definition: P_2 – is the set of **all** boolean functions. The system of boolean functions $\{f_1, f_2, \dots, f_i, \dots\}$ from P_2 is **functionally complete** if any boolean function can be constructed from them.

Examples:

- P_2 – is functionally complete
- $\{\overline{x}, x_1 \& x_2, x_1 \vee x_2\}$ - is functionally complete
- $\{0, 1\}$ – is not complete

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Theorem 1. Given two systems of functions from P_2:

$$F = \{f_1, f_2, \dots\} \qquad \textbf{(I)}$$

$$G = \{g_1, g_2, \dots\} \qquad \textbf{(II)}$$

If the system **(I)** is complete and every function from **(I)** can be constructed by functions from **(II)**, then system **(II)** is complete.

Proof: Let h – any function from P_2. As **(I)** is complete, then

$$h = C[f_1, f_2, \dots]$$

as every function from **(I)** can be constructed by functions from **(II):**

$$f_1 = C_1[g_1, g_2, \dots]$$
$$f_2 = C_2[g_1, g_2, \dots]$$

.... then

$$C[f_1, f_2, \dots] = C[C_1[g_1, g_2, \dots], C_2[g_1, g_2, \dots], \dots] = ([g_1, g_2, \dots]$$

- Or $$h = ([g_1, g_2, \dots]$$

q.e.d.

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Example: System $B = \{0, 1, x_1 \cdot x_2, x_1 + x_2\}$ - is complete.

Theorem 2 (Zhegalkin). Every function from P_2 can be presented as polynomial **mod 2**. It follows from the last example.

Definition: **T** – subset from P_2. Set of all boolean functions constructed from **T** – is called **closed class** of **T** and denotes as **[T]**.

Closed class features:

- $[T] \supseteq T$
- $[[T]] = [T]$
- If $T_1 \subseteq T_2$, then $[T_1] \subseteq [T_2]$
- $[T_1 \cup T_2] \supseteq [T_1] \cup [T_2]$

Another definition of functional completeness: **T** is functional complete, if $[T] = P_2$.

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Important closed classes of functions

- T_0 – class of all 0-preserving functions, such as $f(0, \dots, 0) = 0$.
- T_1 – class of all 1-preserving functions, such as $f(1, \dots, 1) = 1$.
- **S** – class of self-dual functions, such as

$$f(x_1, \dots, x_n) = \neg f(\neg x_1, \dots, \neg x_n)$$

- **M** – class of monotonic functions, such as :

$$\{x_1, \dots, x_n\} \leq \{y_1, \dots, y_n\}, \text{ if } x_i \leq y_i$$

if $\{x_1, \dots, x_n\} \leq \{y_1, \dots, y_n\}$

then $f(x_1, \dots, x_n) \leq f(y_1, \dots, y_n)$

- **L** – class of linear functions, which can be presented as:

$$f(x_1, \dots, x_n) = a_0 + a_1 \cdot x_1 + \dots + a_n \cdot x_n;\ a_i \text{ Č } \{0, 1\}$$

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Lemma 1

If function $f(x_1, \dots, x_n) \notin S$ (class of self-dual functions), then from this function, substituting functions x and $\neg x$, we can get not self-dual function of one variable, i.e. constant.

Proof:

As $f \notin S$, then there is a set $(a_1, \dots, a_n)$, such as $f(\neg a_1, \dots, \neg a_n) = f(a_1, \dots, a_n)$.

Consider functions $\varphi_i(x) = x^{a_i} \quad (i = 1, \dots, n)$.

Let $\varphi(x) = f(\varphi_1(x), \dots, \varphi_n(x))$.

Then we have

$$\varphi(0) = f\big(\varphi_1(0), \dots, \varphi_n(0)\big) = f(0^{a_1}, \dots, 0^{a_n}) = f(\overline{a_1}, \dots, \overline{a}_n) =$$
$$= f(a_1, \dots, a_n) = f(1^{a_1}, \dots 1^{a_n}) = f\big(\varphi_1(1), \dots, \varphi_n(1)\big) = \varphi(1).$$

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Lemma 2

If $f(x_1, \dots, x_n) \notin M$ (class of monotonic functions), then it is possible to get function $\neg x$ from function f, substituting constants **0** and **1** and function x.

Proof:

Sets $A = (a_1, \dots, a_i, \dots, a_n)$ and $B = (a_1, \dots, \neg a_i, \dots, a_n)$ are called neighboring by coordinate i.

Let us prove, that there is a pair of sets **A** and **B**, such as $f(A) > f(B)$.

- Since $f \notin M$, then there are sets A_1 and B_1, such as

$A_1 \leq B_1$ and $f(A_1) > f(B_1)$. If sets A_1 and B_1 are neighboring, then lemma is proven.

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If sets are not neighboring, then sets are different in t coordinates $(t > 1)$. Then between sets A_1 and B_1 we can insert $(t-1)$ sets, so that neighboring sets will be different in one coordinate.

Then at least for one pair of neighboring sets (denote them A and B), will be $f(A) > f(B)$.

Let A and B be neighboring by coordinate i.

Consider function $\varphi(x) = f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n)$:

$$\varphi(0) = f(a_1, \dots, a_{i-1}, 0, a_{i+1}, \dots, a_n) = f(A) > f(B) =$$
$$= f(a_1, \dots, a_{i-1}, 1, a_{i+1}, \dots, a_n) = F(1)$$

Then: $\varphi(0) = 1, \varphi(1) = 0, i.e. \varphi(x) = \neg x$.

Lemma is proven.

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Lemma 3

If $\boldsymbol{f(x_1, \dots, x_n)} \notin \boldsymbol{L}$ (class of linear functions), then, substituting constants **0** and **1** and functions $\boldsymbol{x}$ and $\neg \boldsymbol{x}$, and may be applying $\neg$ to $\boldsymbol{f}$, we can get function $\boldsymbol{x_1 \,\&\, x_2}$ from $\boldsymbol{f}$.

Proof:

Consider Zhigalkin's polynomial for f:

$$\boldsymbol{f(x1, \dots, xn) = \sum_{(i_1,\dots,i_s)}{}' a_{i_1 \dots i_s} x_{i_1} \dots x_{i_s}}$$

Since polynomial is non-linear, then it has a member with at least two factors. Denote them $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$. Then

$$\boldsymbol{\sum_{(i_1,\dots,i_s)}{}' a_{i_1 \dots i_s} x_{i_1} \dots x_{i_s} =}$$

$$\boldsymbol{= x_1\, x_2 f_1(x_3, \dots, x_n) + x_1 f_2(x_3, \dots, x_n) + x_2 f_3(x_3, \dots, x_n) + f_4(x_3, \dots, x_n)}$$

where, due to uniqueness of polynomial $\boldsymbol{f1(x_3, \dots, x_n) \neq 0.}$

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Let $\alpha_3, \dots, \alpha_n$ be such $f_1(\alpha_3, \dots, \alpha_n) = 1$.

Then

$$\varphi(x_1, x_2) = f(x_1, x_2, a_3, \dots, a_n) = x_1 x_2 + a x_1 + + \beta x_2 + \gamma,$$

where a, β, γ – constants 0 or 1.

Consider function $\psi(x_1, x_2) = \varphi(x_1 + \beta, x_2 + \alpha) + \alpha\beta + \gamma$

$$\varphi(x_1 + \beta, x_2 + \alpha) + \alpha\beta + \gamma = (x_1 + \beta)(x_2 + \alpha) + \alpha(x_1 + \beta) + \beta(x_2 + \alpha) + \gamma + \alpha\beta + \gamma = x_1 x_2$$

Then $\psi(x_1, x_2) = x_1 \& x_2$

Lemma is proven.

Post's Functional Completeness Theorem

Theorem (Post's Functional Completeness Theorem)

A system $\boldsymbol{F}$ of boolean functions is functionally complete if and only if for each of the **five defined classes** $\boldsymbol{T_0}, \boldsymbol{T_1}, \boldsymbol{S}, \boldsymbol{M}, \boldsymbol{L}$, there is a member of $\boldsymbol{F}$ which does not belong to that class.

Proof:

Necessity: Let $\boldsymbol{F}$ is functional complete, i.e. $[\boldsymbol{F}] = \boldsymbol{P_2}$.

Let us denote any of five classes as $\boldsymbol{R}$ and assume that $\boldsymbol{F}$Ċ$\boldsymbol{R}$.

Then, $\boldsymbol{P_2} = [\boldsymbol{F}]$Ċ$[\boldsymbol{R}] = \boldsymbol{R}$.

So, $\boldsymbol{R} = \boldsymbol{P_2}$, that is contradiction.

Necessity is proved.

Post's Functional Completeness Theorem

Sufficiency:

Let us assume, that F does not belong to any of five classes.

Then we can select subsystem [illegible] $= \{f_i, f_j, f_k, f_l, f_n\}$, which consists of no more than five functions, and also does not belong to any of five classes. These functions depend on the variables $x_1, \dots, x_n$.

I. Getting constants **0** and **1** from functions f_i, f_j, f_k.

Consider function $f_i \notin T_0$ (0-preserving).

Post's Functional Completeness Theorem

Two cases are possible:

1. $f_i(1, \dots, 1) = 1$, then $f_i(x, \dots, x) = 1$,

because $f_i(0, \dots, 0) = 1$ and $f_i(1, \dots, 1) = 1$

second constant **(0)** can be got by f_j:

$$f_j(1, \dots, 1) = 1$$

2. $f_i(1, \dots, 1) = 0$, then $f_i(x, \dots, x) = \neg x$,

because $f_i(0, \dots, 0) = 1$ and $f_i(1, \dots, 1) = 0$

Consider function $f_k \notin S$. As we have $\neg x$, then we can get constant according to Lemma 1.

Since we have $\neg x$, then we can get second constant.

So for both cases we can get constants **0** and **1**.

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II. Getting function $\neg x$ from constants **0, 1** and function f_m .

It can be done according to Lemma 2.

III. Getting function $x_1 \& x_2$ from constants **0, 1** and functions $\neg x$ and f_i.

It can be done according to Lemma 3.

So we have constructed functions $\neg x$ and $x_1 \& x_2$ by formulas under and therefore under F.

Corollary 1

For every closed class of functions T from P_2, if T P_2, then T is included at least in one of constructed classes.

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Definition

Class R of functions from P_2 is called **precomplete class**, if R is not complete and for every function f $(f \in P2, f \notin R)$, class $R \cup \{f\}$ – is complete.

Corollary 2

In the Boolean algebra there are only five precomplete classes:

$$T_0, T_1, S, M, L.$$

Post's Functional Completeness Theorem provides not only completeness criterion.

It allows (with Disjunctive normal form and Conjunctive normal form) to find formula for every boolean function by complete system functions.