Tropical Combinatorial Nullstellensatz and Fewnomials Testing

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Max-plus Semiring

Max-plus semiring (tropical semiring):

$$(K, \oplus, \odot),$$

where $K = \mathbb{R}$ or $K = \mathbb{Q}$ and

$$x \oplus y = \max\{x, y\},$$

$$x \odot y = x + y$$

Tropical Polynomials

Monomials:

$$M = c \odot x_1^{\odot i_1} \odot \ldots \odot x_n^{\odot i_n} = c + i_1 x_1 + \ldots + i_n x_n,$$

where $c \in \mathbb{K}$ and $i_1, \ldots, i_n \in \mathbb{Z}_+$

Notation: $\vec{x}^I = x_1^{\odot i_1} \odot \ldots \odot x_n^{\odot i_n}$

Polynomials:

$$f = \bigoplus_i M_i = \max_i M_i$$

Degree:
$\deg M = i_1 + \ldots + i_n,$
$\deg f = \max_i \deg(M_i)$

Roots

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A point $\vec{a} \in \mathbb{K}^n$ is a root of the polynomial f if the maximum $\max_i\{M_i(\vec{a})\}$ is either attained on at least two different monomials M_i or is infinite

A tropical polynomial $p(\vec{x})$ is a convex piece-wise linear function

The roots of p are non-smoothness points of this function

Example 1

$$f = 1 \oplus 2 \odot x \oplus 0 \odot x^{\odot 2} = \max(1, x + 2, 2x)$$

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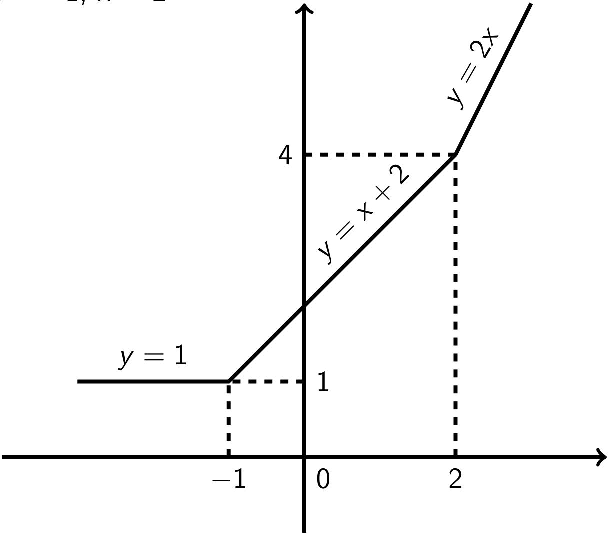

Example 2

$$f = 2 \oplus 0 \odot x \oplus 1 \odot y = \max(2, x, y + 1)$$

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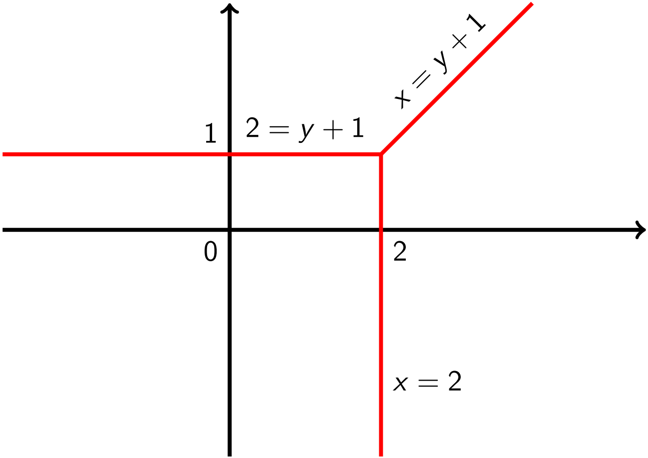

Motivation

- Algebraic geometry. Example: Mikhalkin's theorem on the enumeration of plane complex algebraic curves
- Mathematical physics
- Combinatorial optimization, scheduling problems
- Complexity theory: solvability problem for the systems of tropical linear polynomials is equivalent to mean payoff games

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Tropical analogs of classical objects are

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Tropical analogs of classical objects are

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- simple enough to be computationally accessible

What is Known?

Linear polynomials:
Analogs of the rank of matricies
Analog of matrix determinant
Analog of Gauss triangular form

Complexity of solvability problem: polynomially equivalent to mean payoff games (is in $\mathrm{NP} \cap \mathrm{coNP}$, not known to be in P)

General polynomials:
Radical of the tropical ideal studied
Analog of Nullstellensatz

Complexity of solvability problem: NP-complete

This Talk

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3. What is the size of the minimal set of points $R \subseteq \mathbb{K}^n$ such that any non-trivial polynomial with at most k monomials has a non-root in one of the points of R?

Combinatorial Nulstellensatz

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Denote $[k] = \{0, 1 \ldots, k\}$

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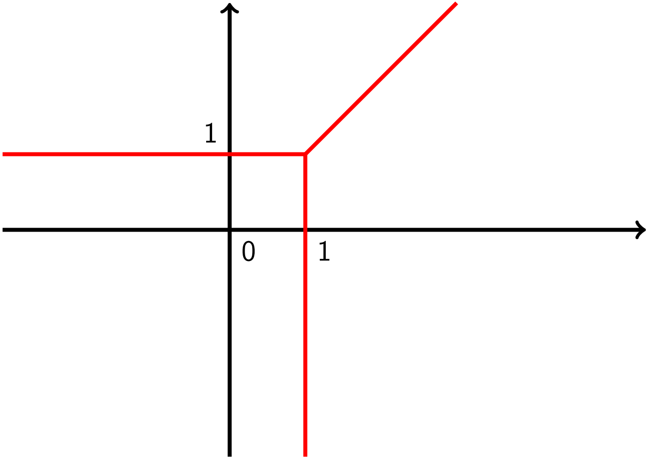
Can be extended to any $R = S = \mathrm{Supp}(p)$. Open in the classical setting!

Example, $d = 1$

$f = 1 \oplus 0 \odot x \oplus 0 \odot y = \max(1, x, y)$.

Roots:



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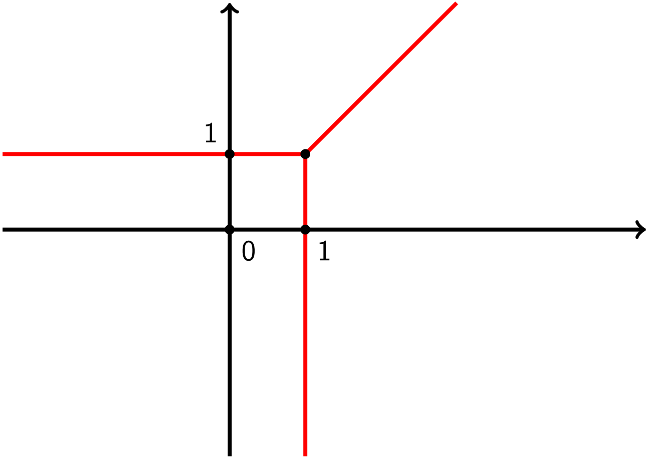

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Combinatorial Nulstellensatz

Theorem (Classical Combinatorial Nullstellensatz)

If p is of total degree at most nd and a monomial $x_1^d x_2^d \ldots x_n^d$ is in p, then p has a non-root in $[d]^n$

Thus it might be that $|\text{Supp}(p)| > |R|$ and p still must have a non-root in R

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If $|S| > |R|$, then there is a tropical polynomial p with $\mathrm{Supp}(p) = S$ and roots in all points of R

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Proof strategy: Look at the polynomial with varying coefficients, analyze as a tropical linear system, use known results for tropical linear systems

Schwartz-Zippel Lemma

Question 2 Given finite sets $R \subseteq \mathbb{R}^n$ and $S \subseteq \mathbb{Z}_+^n$, how many roots can a tropical polynomial p with $\mathrm{Supp}(p) \subseteq S$ have in the set R?

Classical case:

Theorem (Classical Schwartz-Zippel Lemma)

Let $R \subseteq \mathbb{R}$ be of size k and p be a non-zero polynomial of degree d. Then p has roots in at most dk^{n-1} points in R^n.

Tropical Schwartz-Zippel Lemma

Theorem

- *Let $R \subseteq \mathbb{R}$ be of size k and p be a non-zero tropical polynomial of degree d. Then p has roots in at most*

$$k^n - (k-d)^n \approx ndk^{n-1}$$

points in R^n

- *Exactly the same statement is true for the polynomials with individual degree of each variable at most d*
- *The bound is optimal*

For $d = 1$ this is Isolation Lemma

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Proof Idea.
Use Tropical Combinatorial Nullstellensatz □

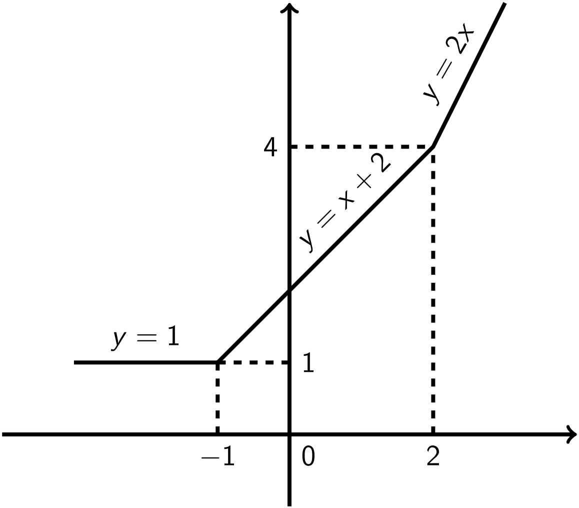
Universal Testing Set

Question 3 What is the size of the minimal set of points $R \subseteq \mathbb{K}^n$ such that any non-trivial polynomial with at most k monomials has a non-root in one of the points of R?

Classical case: $r = k$ (Grigoriev, Karpinski, Singer, Ben-Or, Tiwari, Kaltofen, Yagati)

Example, $n = 1$

$f = 1 \oplus 1 \odot x \oplus 0 \odot x^{\odot 2} = \max(1, x + 1, 2x)$



$k + 1$ monomials are needed for k roots, so $r = k$

Tropical Universal Testing Set, $\mathbb{K} = \mathbb{R}$

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For polynomials over $\mathbb{R}$ the minimal size r of the universal testing set for tropical polynomials with at most k monomials is equal to k

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Proof Idea.

Universal set: pick a set R of points whose coordinates are linearly independent over $\mathbb{Q}$

Let p vanish on R. Consider a graph: vertices are monomials, edges connect monomials that both have maximums on one of the roots in R

Show that the graph can have no cycles

□

Tropical Universal Testing Set, $\mathbb{K} = \mathbb{Q}$

Question 3 What is the size r of the minimal set of points $R \subseteq \mathbb{K}^n$ such that any non-trivial polynomial with at most k monomials has a non-root in one of the points of R?

Theorem

For the size of the minimal universal testing set over $\mathbb{Q}$ the following inequalities hold:

$$\frac{(k-1)(n+1)}{2} + 1 \leq r \leq k(n+1) + 1.$$

Proof Idea.

Upper bound: Count the dimension of semialgebraic set of sets of roots of tropical polynomials

Lower bound: Given set of points R construct polynomial with roots in all points of R inductively □

Tropical Universal Testing Set, $\mathbb{K} = \mathbb{Q}$

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Theorem

For $n = 2$ we have

$$r = 2k - 1$$

Proof Idea.

A universal set: vertices of a convex polygon □

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- Tropical Universal Testing Set
  Completely different for $\mathbb{R}$ and $\mathbb{Q}$
  Gap between lower and upper bound for $\mathbb{Q}$