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Complex turbulent exchange coefficient in Akerblom-Ekman model

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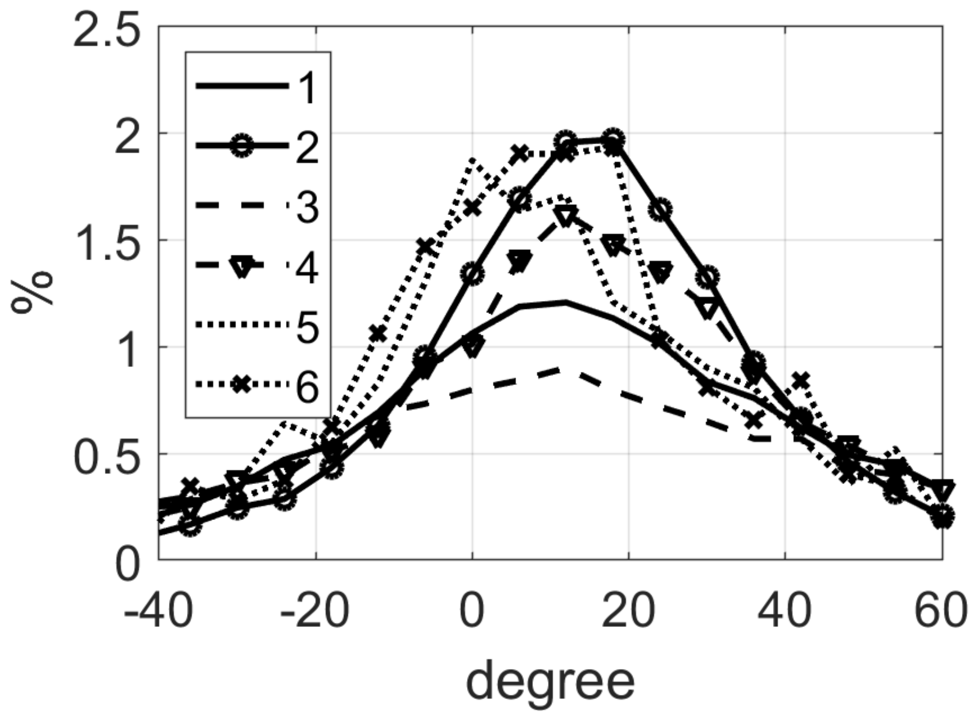
The traditional **Akerblom – Ekman model** describes the dynamics of wind speed in the boundary layer (BL) of the atmosphere or an ocean on a rotating planet:

$$\begin{cases} \dfrac{d}{dz}\left[k(z)\dfrac{du}{dz}\right] = -l\left(v - v_g\right), \\ \dfrac{d}{dz}\left[k(z)\dfrac{dv}{dz}\right] = l\left(u - u_g\right), \end{cases} \qquad (1)$$

where $u(z), v(z)$ are the required horizontal wind components, u_g, v_g are the geostrophic wind on the BL upper boundary. The vertical variable $z \in [0, H_{max}]$ is the height above the Earth's surface, H_{max} is the thickness of the boundary layer, $l = \sin\varphi \times 1.45842 \times 10^{-4} / s$ is the Coriolis parameter, φ is the geographic latitude, $k(z) > 0$ is the coefficient of the turbulent exchange. **The system (1) is singular iff $k(z_0) = 0$ for some $z_0 \in (0, H_{max})$.**

If $k(z) = const$ then the wind rotation angle in BL is equal to 45°.

Observed wind rotation angle – histogram



Conditional probability distributions of the wind rotation angle for various latitude zones and various subsamples: 1, 3, 5 – to the South from the 50° N, 2, 4, 6 – to the North from the 50° N; 1, 2 – full subsample, 3, 4 – deep subsample, 5,6 – stable subsample.

The typical rotation degree is $\sim 15^{\circ}$ – **3 times less then theoretical.**

Eq. (1) is invariant with respect to the group of rotations around the vertical axis **SO(2)**. The group SO(2) commutes only with skew-symmetric operators. Therefore, we will also consider the more general system:

$$\begin{cases} \frac{d}{dz}\left[\gamma(\mathrm{z})\sin(\varphi)\frac{dv}{dz}+k(\mathrm{z})\frac{du}{dz}\right] = -l\left(v-v_g\right), \\ \frac{d}{dz}\left[k(\mathrm{z})\frac{dv}{dz}-\gamma(\mathrm{z})\sin(\varphi)\frac{du}{dz}\right] = l\left(u-u_g\right), \end{cases} \tag{2}$$

where the second coefficient of turbulent exchange $\gamma(z)$ plays the role of the regularizator for (1): when the first coefficient $k(z_0)=0$, (2) does not become degenerate.

The cofactor $\sin(\varphi)$ was added into Eq. (2) to adjust the results of our numerical experiments with data from South hemisphere.

The complex form

Rewrite (2) in a complex form: $w = u + \mathrm{i}v$, $w_g = u_g + \mathrm{i}v_g$, and $\kappa = k - \mathrm{i}\gamma\sin(\varphi)$:

$$\frac{d}{dz}\left[\kappa(\mathrm{z})\frac{dw}{dz}\right] = \mathrm{i}l\left(w - w_g\right). \qquad (3)$$

For $\kappa(z) = const$ the wind rotation angle is equal $\arg\sqrt{\frac{\mathrm{i}}{\kappa}} = \frac{1}{2}\,\mathrm{atan}\,\frac{k}{\gamma\sin\varphi}$.

Therefore the observable wind rotation angle 10-20^{O} corresponds to the values of the ratio

$$\gamma\sin\varphi/k \approx 1.2 - 2.7$$

The quadratic programing problem (QPP)

To reduce the order of differentiation, we integrate Eq. 2 with respect to z:

$$\kappa(z)\frac{dw}{dz} = -\psi + c \tag{4}$$

where $c \in \mathbb{C}$ are constant of integration, a function $\psi(z)$ satisfy the following equations:

$$\frac{d\psi}{dz} = \mathrm{i}l\left(w_g - w\right), \quad \int_0^H \psi(z)\,dz = 0.$$

We will search $\kappa(z)$ as a solution of QPP. This QPP minimizes the mean relative residual of (4) over N vertical profiles:

$$L\left(\kappa(z), c_j\right) = \frac{1}{N}\sum_{j=1}^{N}\frac{1}{W_j}\int_0^{H_j}\left|\kappa(z)\frac{dw_j}{dz} + \psi\left(z\right) - c_j\right|^2 dz \rightarrow \min_{\kappa(z),\left\{c_j\right\}}, \tag{5}$$

where $W_j = \int_0^{H_j}\left|\psi\left(z\right)\right|^2 dz$ for the normalization. With this normalization $\min_{c_j} L\left(0, c_j\right) = 1$.

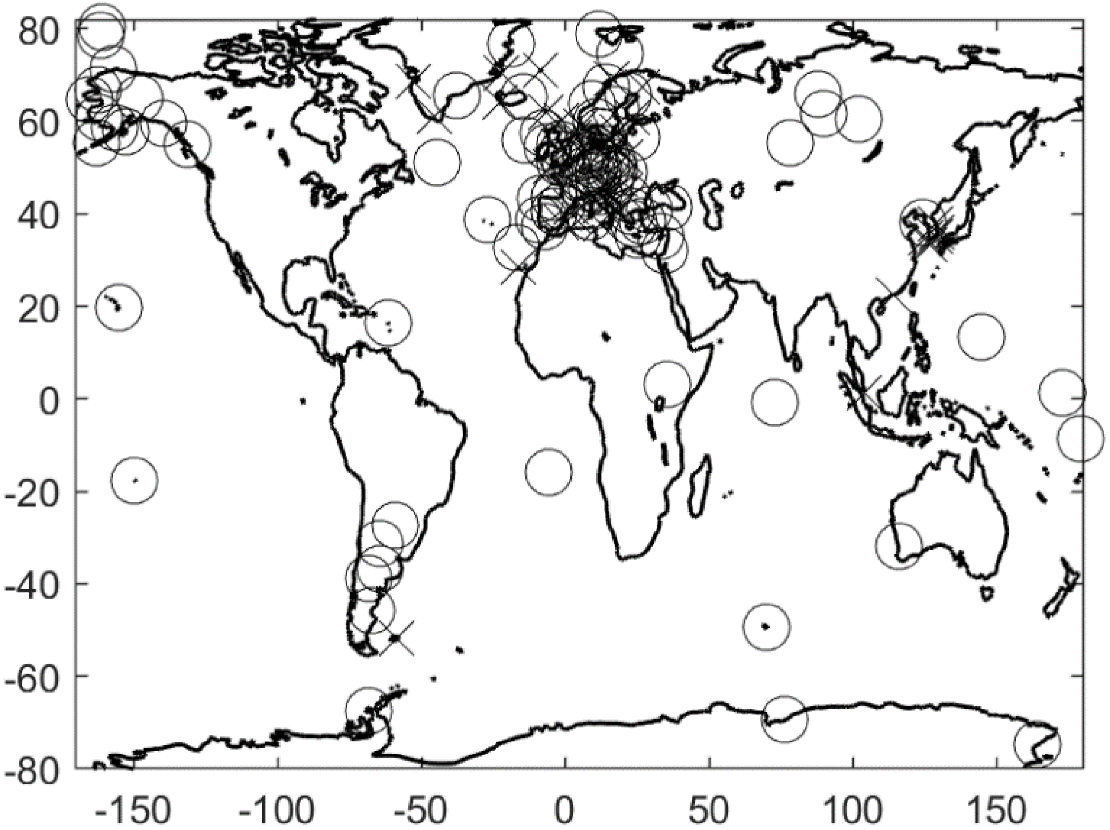
Let $0 < \Lambda < 1$ be the minimum of the functional L. The value $100\%\left(1-\Lambda\right)$ is interpreted as the average coefficient of determination. It is presented in Table 2.

Dataset

We use the dataset from **26142** profiles, which satisfy the following conditions:

1. The measurement unit for wind speed is $0.1 m/s$.
2. The mean vertical resolution is good (more than 25 points in the layer 0-1000 m).
3. **The boundary layer thickness $H_j > 100m$.**
4. The variability of the wind in the boundary layer is greater than $2.5 m/s$.
5. The absolute value of difference between the altitude of the aerological station and the altitude of the lowest level of the BUFR profile is no more $5m$.

Geographical location of the aerological stations



Geographical location of 111 stations, from which the radiosonde data were assimilated. Crosses mark **28 “intensive” stations**, with a large number (more than 400) of the profiles

Boundary layer's thickness

We use a standard definition of the boundary layer's thickness H_j as the minimal positive root of the following equation:

$$\Theta_j\left(H_j\right) = \Theta_{V,j}(0), \qquad (5)$$

where Θ is a potential temperature and Θ_V is a potential virtual temperature

The dataset of BUFR profiles during the period from Apr. 4, 2018 to Nov. 29, 2019

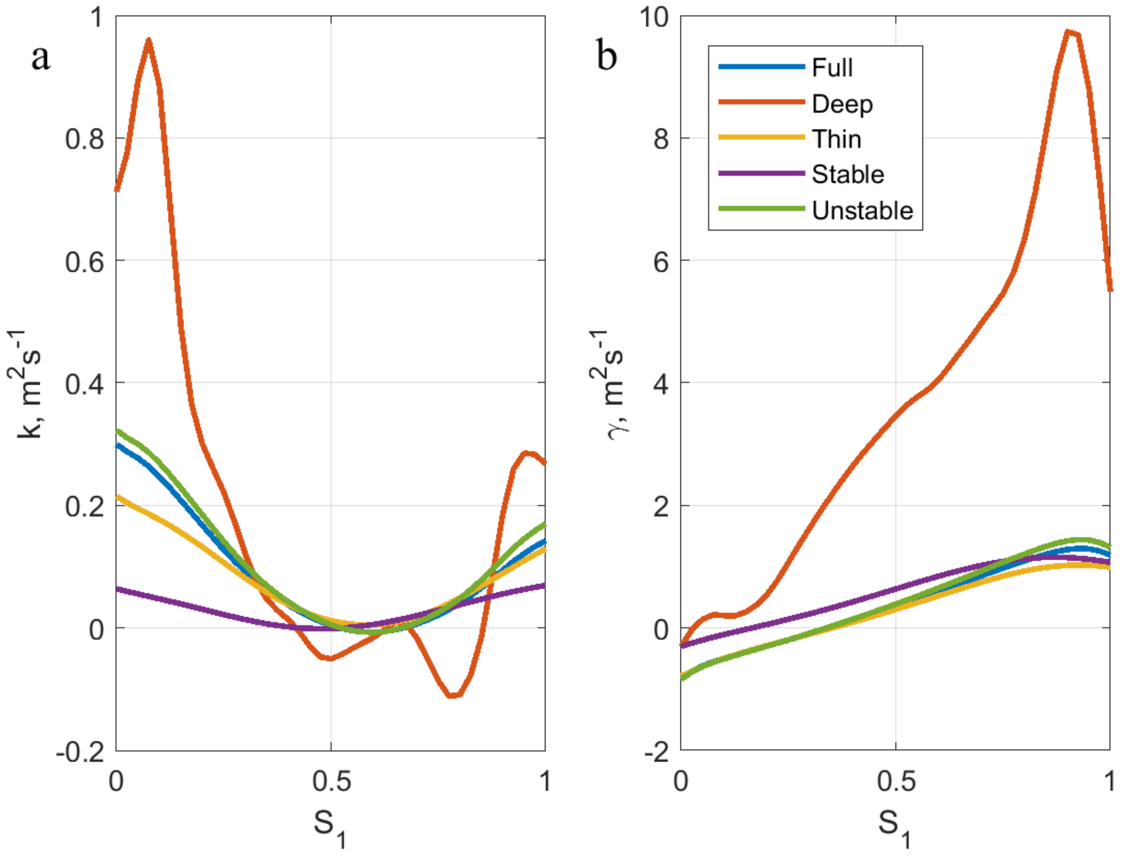
Subsample name	Addition condition	Profiles	Boundary layer thickness H_j
Full	None	26142	671±516m
Deep	$H_j > 1000m$	8462	1592±454m
Thin	$H_j < 500m$	12051	270±115m
Stable	$Ri(z) > 0.3$ for any $z \in \left[0; H_j\right]$	2622	201±105m
Unstable	$Ri(z_0) < 0.2$ for some $z_0 \in \left[0; H_j\right]$	22584	742±517m

The Richardson number

The Richardson number Ri is the dimensionless function of the height *z:*

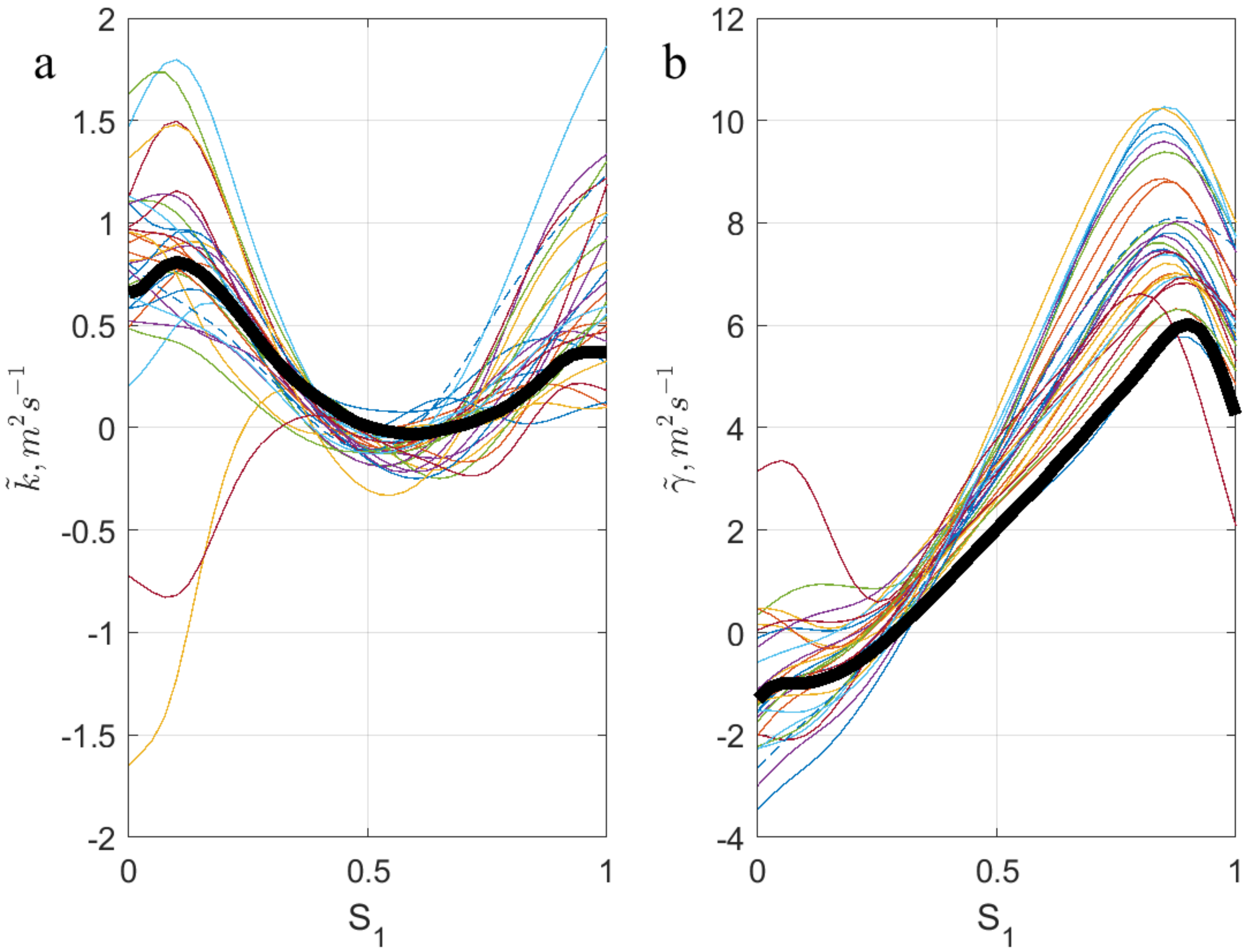
$$Ri(z) = \frac{g}{\Theta} \frac{\dfrac{\partial \Theta}{\partial z}}{\left(\dfrac{\partial u}{\partial z}\right)^2 + \left(\dfrac{\partial v}{\partial z}\right)^2}. \tag{6}$$

The values $Ri > Ri_c = 0.25$ correspond to stable stratifications, $Ri < Ri_c$ correspond to unstable ones, and $Ri < 0$ correspond to strictly unstable stratifications of an atmospheric column (a temperature inversion layer exists in the column).



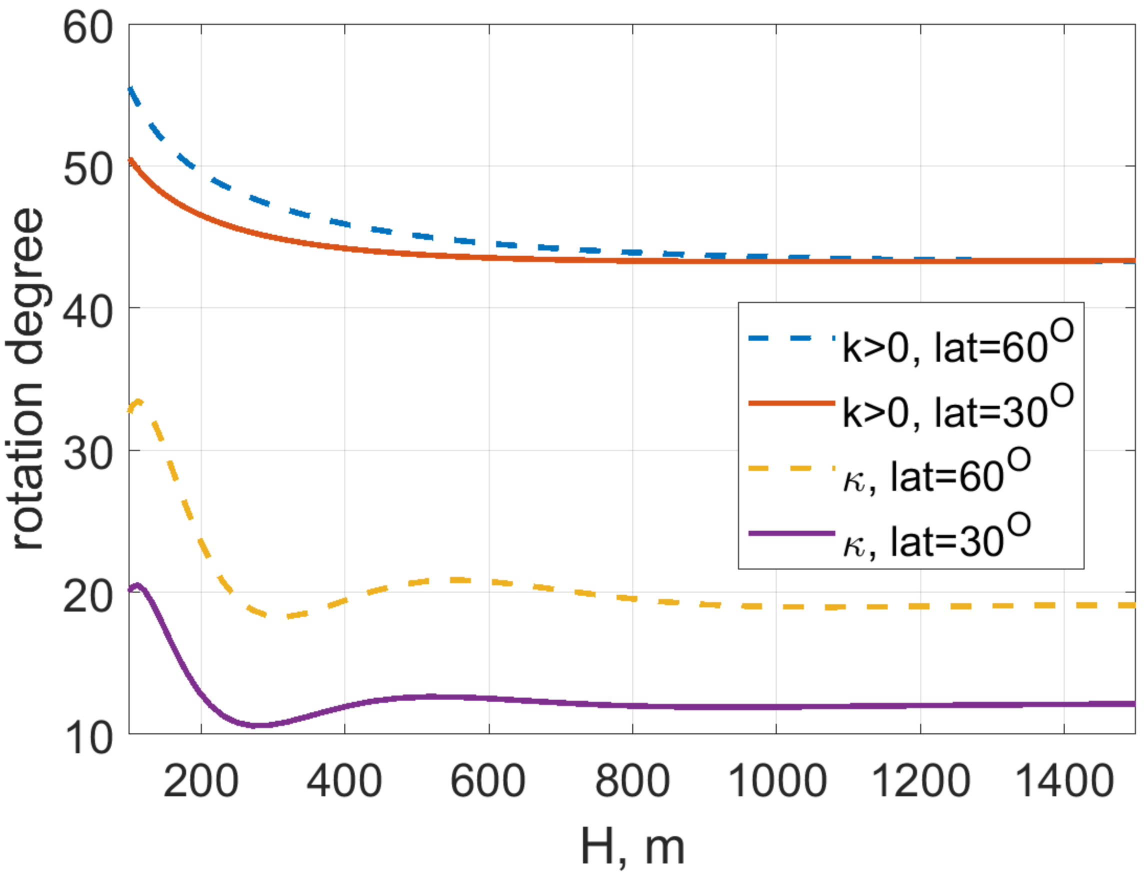
The optimal (for different subsamples) coefficient of turbulent exchange κ depending on the relative height $S_1 = z / H$: a) the real part $k(S_1)$; b) the imaginary part $\gamma(S_1)$. We consider also normalized parameters: $\kappa = \frac{\tilde{\kappa} \cdot H_j}{1000m}$

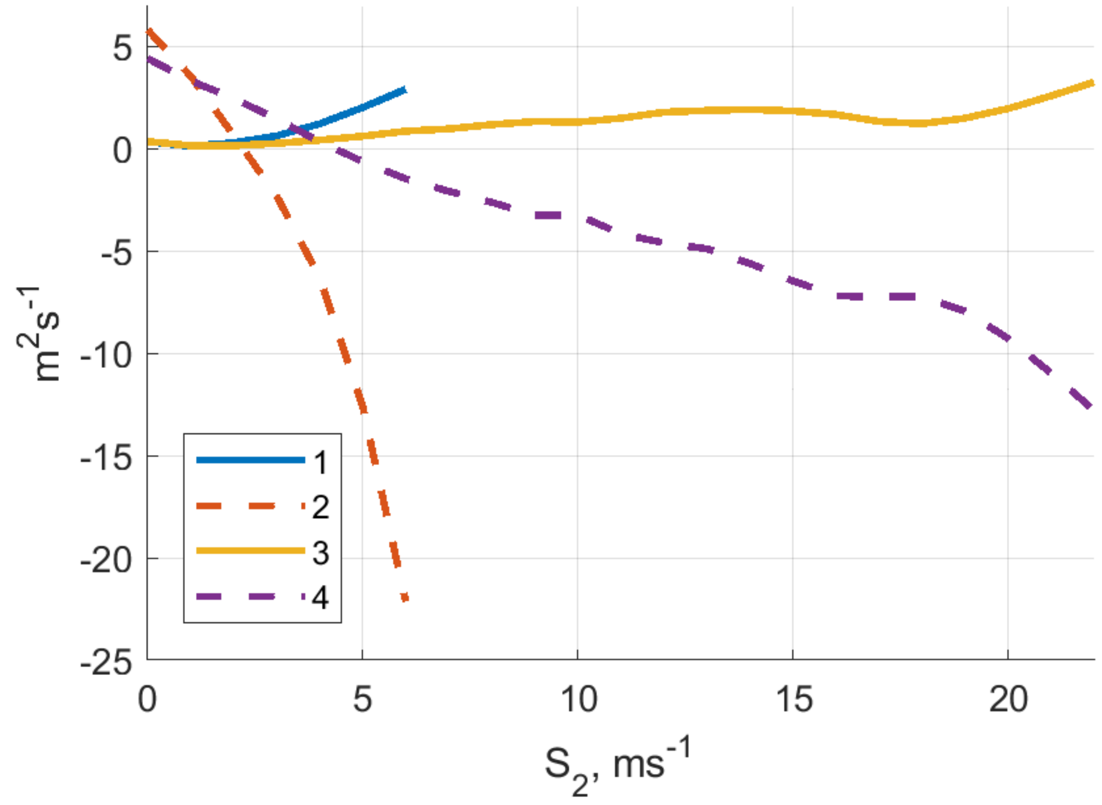
a) the real part $\tilde{k}(S_1)$, b) the imaginary part $\tilde{\gamma}(S_1)$.



The optimal normalized coefficients of turbulent exchange $\tilde{\kappa}(S_1)$ depending on the relative height $S_1 = z / H_j$, were determined separately for 28 “intensive” stations.

The wind rotation angle for the optimal $\kappa = H\tilde{\kappa}(z/H)$

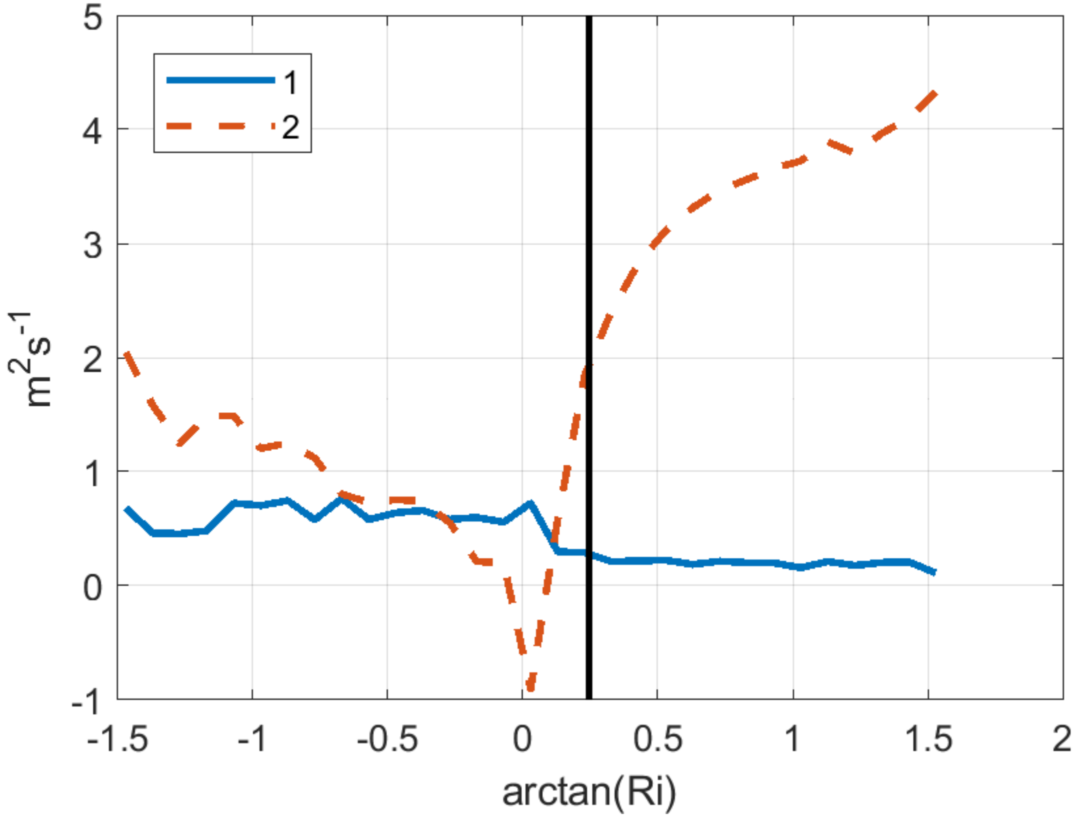




Optimal coefficient of turbulent exchange $\tilde{\kappa}$ (full subsample): the real part $\tilde{k}\left(S_2\right)$ (graphs 1, 3); the imaginary part $\tilde{\gamma}\left(S_2\right)$ (graphs 2, 4) depending on the wind shear module $S_2 = shear(z) = \sqrt{\left(u(z)-u_g\right)^2 + \left(v(z)-v_g\right)^2}$ $(m \cdot s^{-1})$:

for stable subsample (graphs 1, 2),

for unstable subsample (graphs 3,4)



Optimal coefficients of turbulent exchange $\tilde{\kappa}$ (Full subsample): the real part $\tilde{k}(S_3)$ (graph 1); the imaginary part $\tilde{\gamma}(S_3)$ (graph 2) depending on the $S_3 = \arctan(Ri)$, where Ri – Richardson number. Black vertical line shown the critical Richardson number $Ri = Ri_c = 0.25$. For $Ri > 0.5$ we have the ratio $\gamma / k > 20$

Optimized coefficients		κ	$k>0,\gamma=0$	$\tilde{\kappa}$	$\tilde{k}>0,\gamma=0$	Ratio
Subsample	Atmospheric parameter(s) S	The mean coefficient of determination $100\%(1-\Lambda)$				$\frac{1-\Lambda(\tilde{\kappa}\in\mathbb{C})}{1-\Lambda(\tilde{\kappa}\in\mathbb{R})}$
Full	Relative height S_1	38,5%	11,7%	**48,3%**	13,8%	3,5
Deep		34,6%	7,7%	**35,0%**	8,0%	4,5
Thin		62,3%	17,4%	**72,6%**	20,9%	3,3
Stable		65,9%	10,2%	**77,5%**	11,6%	**7,1**
Unstable		38,9%	12,3%	**48,2%**	14,0%	3,5
Full	Wind shear modulus S_2	37,6%	12,2%	46,3%	15,1%	3,1
Deep		29,2%	8,3%	29,8%	8,6%	3,4
Thin		59,9%	18,2%	67,7%	22,4%	3,0
Stable		59,7%	11,6%	67,8%	13,5%	5,5
Unstable		39,1%	12,6%	46,7%	15,3%	3,0
Full	Richardson number Ri	24,4%	11,4%	34,1%	12,6%	2,7
Deep		29,2%	7,4%	29,9%	7,5%	4,1
Thin		36,0%	16,8%	43,0%	19,8%	2,1
Stable		33,0%	9,7%	39,4%	10,3%	4,0
Unstable		28,6%	11,8%	35,3%	12,8%	2,7

Full	Relative height S_1 and wind shear modulus S_2	41,2%	12,7%	**53,4%**	15,4%	3,4
Deep		35,2%	8,7%	**35,7%**	9,1%	4,0
Thin		65,2%	18,9%	**76,7%**	22,9%	3,3
Stable		66,3%	11,7%	**78,1%**	13,5%	6,3
Unstable		42,0%	13,1%	**53,4%**	15,7%	3,4
Full	Relative height S_1 and Richardson number Ri	39,5%	12,4%	50,3%	13,8%	3,6
Deep		34,7%	7,9%	35,1%	8,1%	4,5
Thin		63,6%	18,4%	74,7%	21,4%	3,4
Stable		65,9%	10,2%	77,5%	11,6%	**7,1**
Unstable		40,6%	12,9%	50,1%	14,1%	3,6

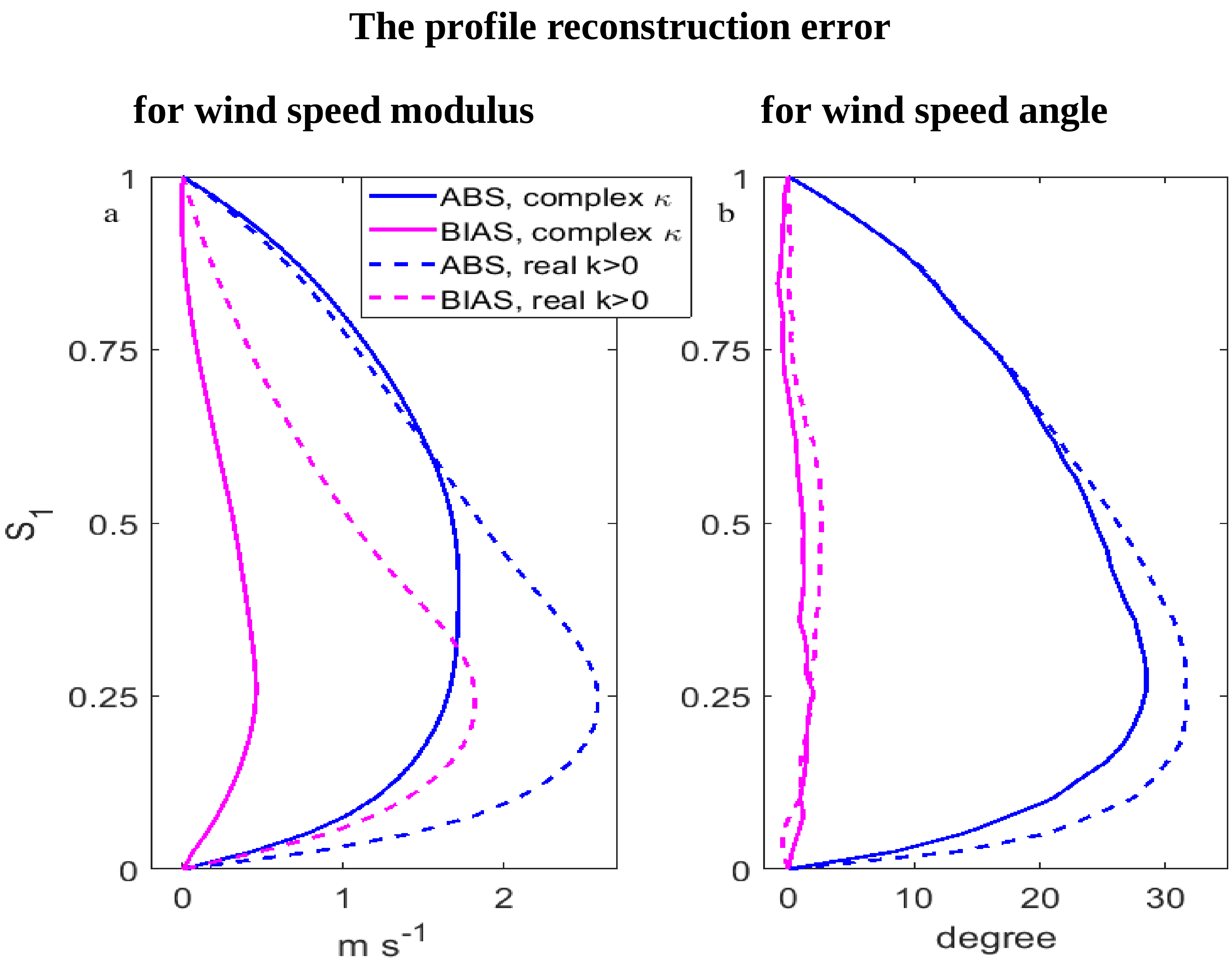
Comparison of the BUFR profiles and model's solutions

Let us represent the coefficient of the turbulent exchange κ in the form $\kappa = H\tilde{\kappa}\left(z/H\right)$. Then we can find the solution $\hat{w}_j\left(z,\kappa,w_0\right)$ of Eq. 2 with the Dirichlet boundary conditions $w\left(H\right)=w_g$, $w\left(0\right)=w_0$ and estimate the mean error of the profile reconstruction:

$$ABS_{speed}\left(S_1,\kappa,w_0\right)=\frac{1}{N}\sum_{j=1}^{N}\left\|\hat{w}_j\left(S_1H_j,\kappa,w_0\right)\right|-\left|w_j\left(S_1\right)\right\|,$$

$$ABS_{direction}\left(S_1,\kappa,w_0\right)=\frac{1}{\tilde{N}}\sum_{j=1}^{\tilde{N}}\left|\arg\hat{w}_j\left(S_1H_j,\kappa,w_0\right)-\arg w_j\left(S_1\right)\right|,$$

Here we exclude from the formulas for $ABS_{direction}$ the terms with small velocities $\left|\hat{w}_j\right|$ or $\left|w_j\right|\leq 2m/s$, when the determination of the wind's direction is not clear. The limit of the sums in these formulas is smaller: $\tilde{N}\approx 0.69N$.



The profile reconstruction error
for wind speed modulus
for wind speed angle
a
b
ABS, complex κ
BIAS, complex κ
ABS, real k>0
BIAS, real k>0
S1
1
0.75
0.5
0.25
0
0
1
2
m s-1
0
10
20
30
degree

Conclusion

1. The original theory of Akerblom – Ekman, predicted 45° wind rotation in the boundary layer. We observed the rotation angle is an average of three times smaller.
2. We include the coefficient γ in the model, the consistence with BUFR data increase up to 7 times for stable stratification and up to 3.5 for unstable. The coefficient γ can be interpreted as a coefficient in the imaginary part of coefficient κ;
3. We compare the universal coefficient κ, both on unique parameter: relative height $S_1 = z / H$, or on the wind shear S_2, or on the Richardson number Ri. The relative height is preferable
4. The wind speed bias for model with complex κ is 4 times less then for the model with real $k > 0$.

http://method.meteorf.ru/ansambl/ansambl.html

Thank you for attention

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