# Two dimensional boundary value problem for the 2-nd order elliptic equation with discontinuous coefficient

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Elliptic linear differential equations such as Poisson (1) and Helmgoltz (2) equations describe stationary solutions (e.g. for diffusion, heat conductivity, and for distribution of the electrostatic potential)

$$-div(\vartheta(\vec{x})grad(u)) = f(\vec{x}), \quad \vec{x} \in G, \tag{1}$$

$$-div(\vartheta(\vec{x})grad(u)) + \rho(\vec{x})u = f(\vec{x}). \tag{2}$$

In many physical and technical cases the media is not homogeneous and its properties (described by coefficients $\vartheta(\vec{x})$ and $\rho(\vec{x})$) are discontinuous.

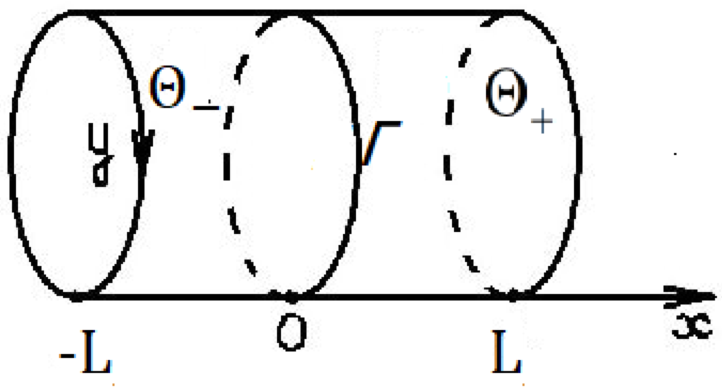


***Fig. 1.*** Model: domain G – surface of the cylinder, jump-line Γ – central circle at $x = 0$, $\vartheta$ - piecewise continuous coefficient: $\vartheta_-$ in the left part of the cylinder, $\vartheta_+$ - in the right. Here $x \in [-L, L],\ y \in [0, 2\pi)$.

On the edges of the cylinder G the Dirichlet boundary conditions are posed, on the jump-line $\Gamma$ the Kirchhoff conditions are fulfilled:

$$[u] = 0, \qquad (3a) \qquad\qquad [\vartheta \partial_n u] = 0. \qquad (3b)$$

Here [ ] is an amplitude of the jump on line Γ, $\partial_n$ is the normal derivative on $\Gamma$. We assume that $L = \pi$, in other cases we should renormalize our domain.

## Compact approximation:

The cylinder $G$ is covered by a uniform grid with $N$ knots on the circle and $N$+1 knots on its generatrix. Due to the chosen size the grid's diameter in coordinates <*x, y*> is equal to $h = \dfrac{2\pi}{N}$. Let us define for every point with the too dimensional index $\vec{j}$ of the grid a pair of difference operators $A_{\vec{j}}$ and $P_{\vec{j}}$, which approximate the differential problem (2 - 3) and are applied to the functions $u$ and $f$, respectively, and a pair of stencils for these operators (grid points where operators have non-zero coefficients).

The operators $A_{\vec{j}}$ and $P_{\vec{j}}$ should be exact on a set of test functions: $(u_k, f_k)$, where $k$=1,…, $K$, $f_k = L[u_k]$, $L$ is the differential operator in the left-hand side of Eq. (1). Therefore $\forall k \ A_{\vec{j}} u_k = P_{\vec{j}} f_k$. Coefficients of the operators $A_{\vec{j}}$ and $P_{\vec{j}}$ for any grid index $i$ are found by solving a "local SLAE" of order $K$+1. These coefficients form $|\vec{j}|$-th lines of the "global SLAE" $\mathbf{A}\vec{U} = \mathbf{P}\vec{F}$, where $\vec{U}$ is a finite-difference solution on the grid, $N \times (N+1)$ □ $K$. At points at the edge of $G$ boundary conditions are approximated. $A_{\vec{j}}$ and $P_{\vec{j}}$ have non-zero coefficients only at $K$ points of their stencils, all the other elements of line $|\vec{j}|$ in the global SLAE are zeros. As $K$ □ $N$, matrixes of the global SLAE are rather sparse.

The points of the grid are divided in four groups due to approximation method

Type I – points inside $G$ far from $\Gamma$

Type II – points, which templates intersect with $\Gamma$

Type III – points on line $\Gamma$

Type IV – points on the edges of $G$

**Points of type I**

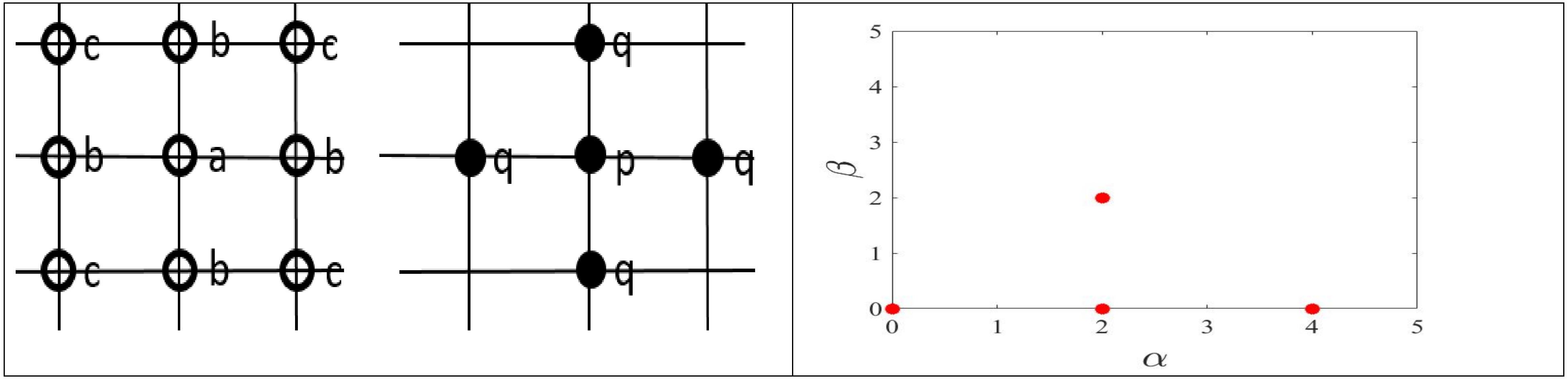


***Fig. 2.*** *In the left part the scheme's stencils are presented and a Newton's diagram in the right part for monomials* $x^{\alpha}y^{\beta}$

Let us use monomial $x^{\alpha} y^{\beta}$ as test functions. Due to the symmetry of the stencils with respect to vertical and horizontal axis, the monomials powers $\alpha, \beta$ are even. As the grid's diameters with respect to x and y are equal, the equations for test functions $x^{\alpha} y^{\beta}$ and $x^{\beta} y^{\alpha}$ will also be the same. That is why to achieve 4-th accuracy order we need to take only the following test functions: $1, x^2, x^4, x^2 y^2$.

The coefficients will be as follows: $a = 1, b = -0.2, c = -0.05, p = 0.2h^2, q = 0.025h^2$.

## Points of type II

For points near to line $\Gamma$ one should take the same stencils as for point of type I (view Fig.2). At the same time one should take in account that the right-hand side $f$ is not defined on $\Gamma$, but there exist left and right-hand limits: $f_-$ and $f_+$, respectively. Therefore when constructing the global SLAE, we assume for point to the left from $\Gamma$ the right-hand side is equal to $f_-$, and for points to the right from $\Gamma$ the right-hand side is equal to $f_+$

# Points of type III

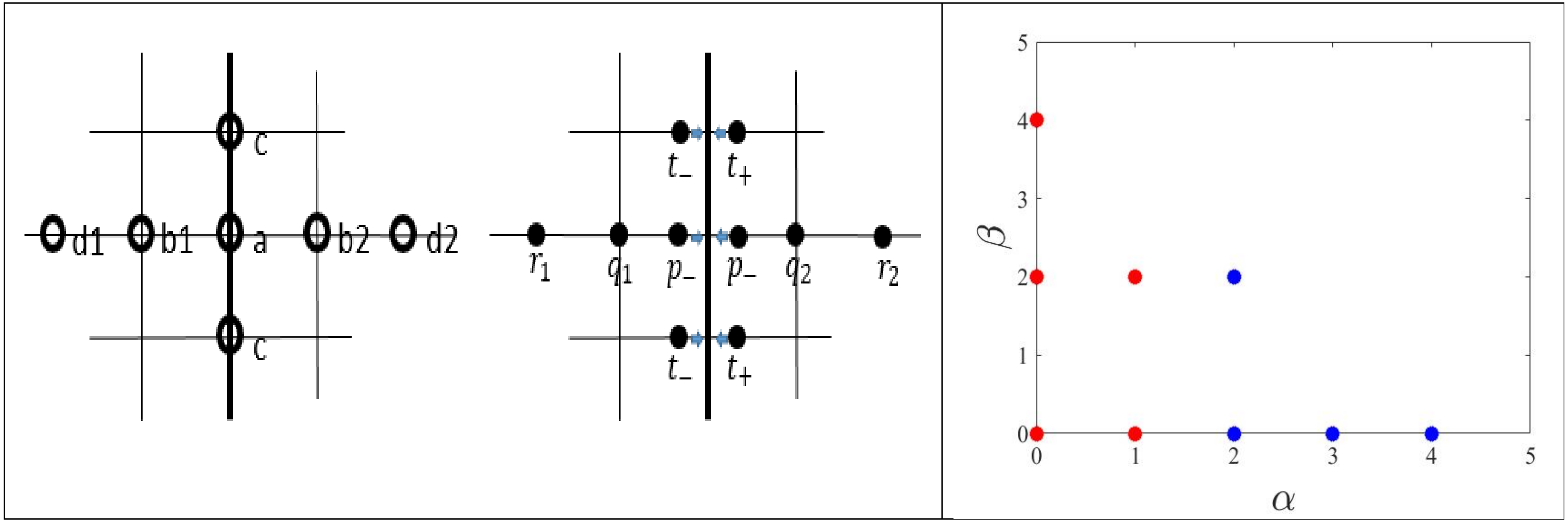


***Fig.3.*** *Stencils and a Newton's diagram for points of type III. Blue points mark two test function: one with sign(x) and one without*

We assume that the solution of the differential problem $u$ is a piecewise analytic function, which has two different Teylor series on the right and on the left side from line $\Gamma$:

$$u(x,y)=\sum a_{ij}x^i y^j, x \le 0$$

$$u(x,y)=\sum b_{ij}x^i y^j, x \ge 0$$

From the Kirchhoff conditions (3a,b) we get: $a_{0j} = b_{0j}$, $\vartheta_- a_{0j} = \vartheta_+ b_{0j}$. We take the following test functions: $1, \dfrac{x}{\vartheta}, x^2, sign(x)x^2, x^3, sign(x)x^3, x^4, sign(x)x^4, y^2, \dfrac{y^2 x}{\vartheta}, x^2 y^2, sign(x)x^2 y^2, y^4$. Here we also use the symmetry of the stencils with respect to horizontal axis.

The right-hand side on line $\Gamma$ is a two valued function ($f_+$ и $f_-$), that is why the stencil for $f$ has two coefficients on the jump-line $\Gamma$ $t_-$ & $t_+, p_-$ & $p_+$). Coefficints $t_-, p_-$ refere to the left-hand limit ($f_-$) and $t_+, p_+$ to the right-hand limit ($f_+$).

After solving the "small SLAE" we obtain the following coefficients for the compact approximation:

$$a = 1,\ c = -\frac{1}{5},\ b_1 = -\frac{2\vartheta_+}{\vartheta_+ + \vartheta_-},\ b_2 = -\frac{2\vartheta_+}{\vartheta_+ + \vartheta_-},\ d_1 = \frac{\vartheta_-}{15(\vartheta_+ + \vartheta_-)},\ d_2 = \frac{\vartheta_+}{15(\vartheta_+ + \vartheta_-)},$$

$$r_1 = r_2 = -\frac{h^2}{36(\vartheta_+ + \vartheta_-)}, q_1 = q_2 = -\frac{7h^2}{90(\vartheta_+ + \vartheta_-)},\ t_- = t_+ = \frac{h^2}{60(\vartheta_+ + \vartheta_-)},\ p_- = p_+ = \frac{7h^2}{60(\vartheta_+ + \vartheta_-)}.$$

**At points of type IV – Dirichlet boundary condition**

## Constructions of the “global SLAE”

After calculating the coefficients of difference operators at each point the “global” matrices ***A*** and ***P*** are constructed. ***A*** is a square matrix, it’s size is $M \times M$ (M – number of grid’s knots). Matrix ***P*** counts points on *Γ* two times, as ($f$) is a two-valued function, therefore the size of ***P*** is $M \times (M + N)$.

To solve the SLAE one needs to invert matrix ***A***, that is why it is important to provide good conditionality of ***A***. Local operators $A_i$ are exact on the constant test function, therefore $\forall i \ \sum_j a_{ij} = 0$. If $a_{ii} > 0$, and all other weight are negative, then:

$$\sum_{j \neq i} | a_{ij} | = a_{ii} \ . \tag{4}$$

Zero lies on the edge of Gershgorin’s circles, which contain the spectrum of ***A***. At boundary points of cylinder G the diagonal of ***A*** dominates. Therefore, one can hope that 0 won’t be included in ***A***’s spectrum, so the matrix ***A*** is invertible.

## Tests, proving the scheme's order.

The scheme's order can be evaluated the following way. We consider a smooth function $\tilde{u}$ and construct a new function $u(x,y)=g(x,y)\tilde{u}(x,y)$, where $g$ – is a piecewise linear function by $x$: $g(x,y)=1+a(y)(x+|x|)$. Function a(y) is defined so that $u$ fulfills the Kirchhoff conditions (3 a, b). Therefore $a(y)=\partial_x\tilde{u}(0,y)\dfrac{\vartheta_- - \vartheta_+}{2\vartheta_+\tilde{u}(0,y)}$. Then we calculate $f$ as $f=L[u]$, so $u$ is the exact solution of problem (2-3) with the right hand-side $f$ and suitable non=homogeneous Dirichlet boundary conditions.

The norm of the error (E) is evaluated as follows: $E=\left\|u_{appr}-u\right\|_C$. Here $u_{appr}$ is a solution that is calculated by the difference scheme. Below the graphs of errors depending on $N$ are presented for three schemes: classic, compact, and compact with Richardson extrapolation. Experiments proving the scheme's accuracy order were made for huge $\kappa=10000$.

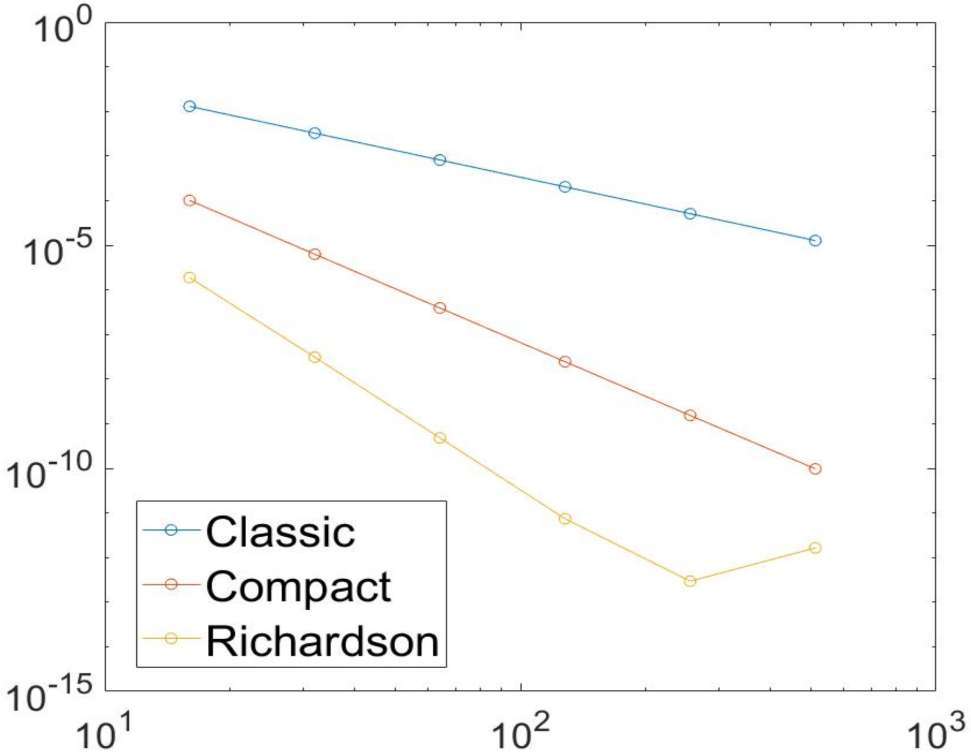


***Fig.4.*** *The schemes' error depending on the number N in loglog coordinates. The exact solution:*

$$u = \begin{cases} \vartheta_+ x\sin(y), & x \le 0 \\ \vartheta_- x\sin(y), & x \ge 0 \end{cases}.$$

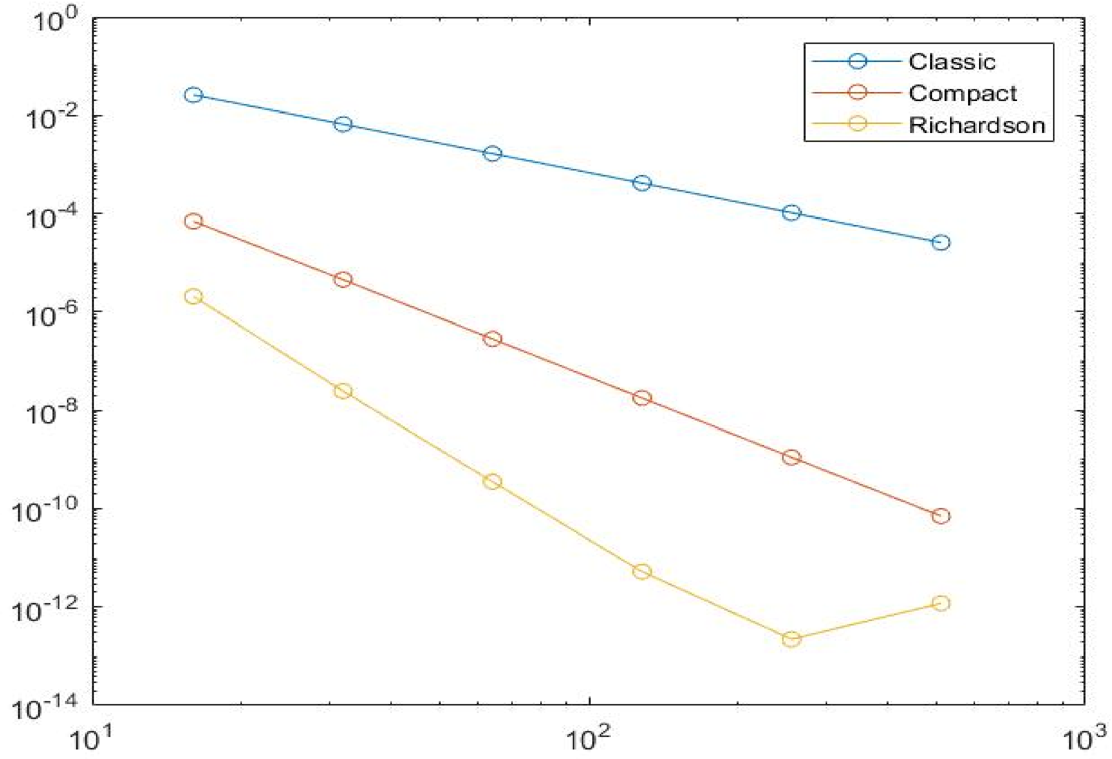


***Fig.5.*** *Same results for the exact solution:*

$$\tilde{u} = x^6 \sin(y).$$

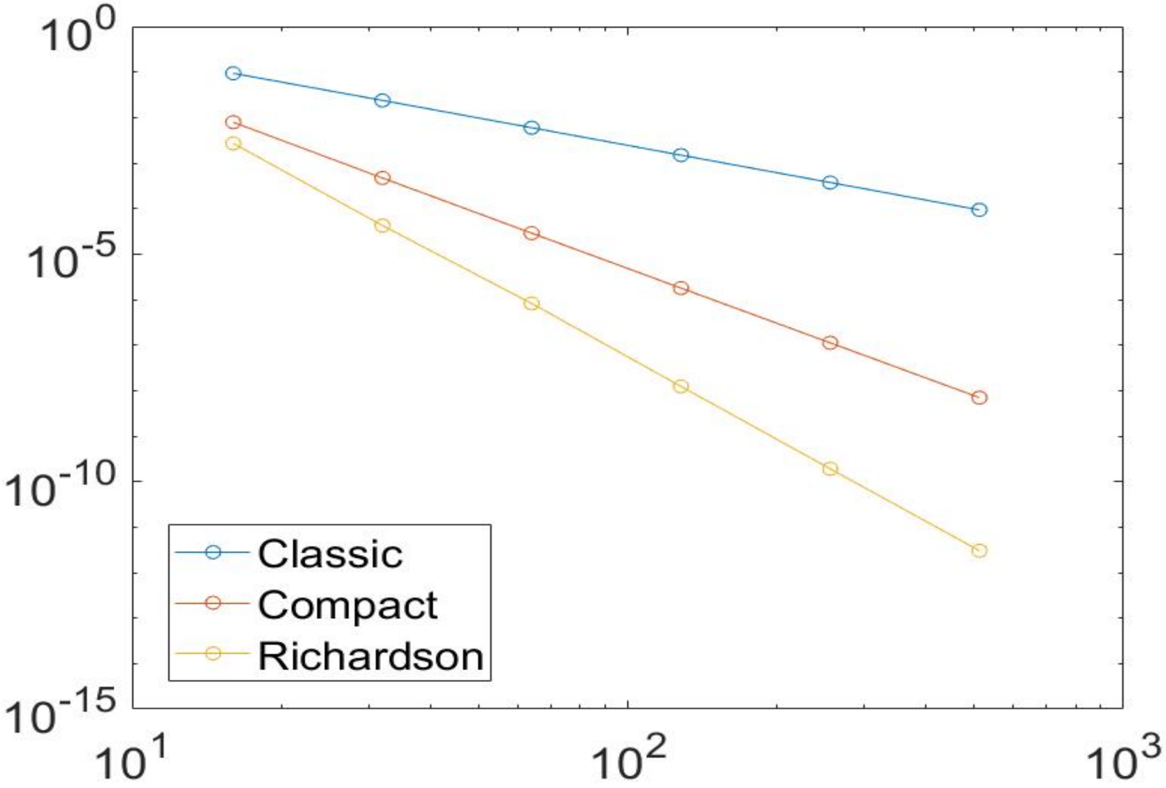


***Fig.6.*** *Same results for the exact solution:*

$\tilde{u} = \sin(y + x) + 2$.

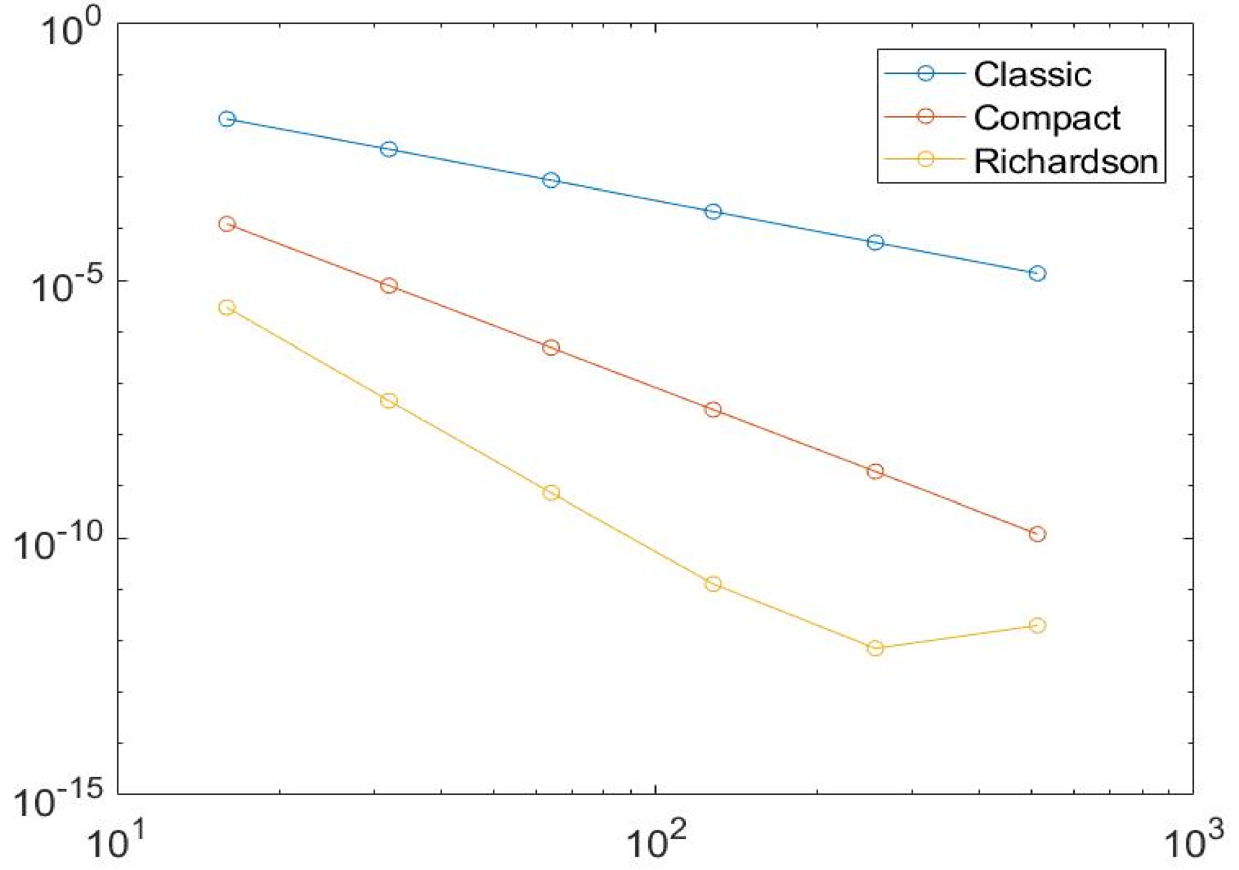


***Fig.7.*** *Same results for the exact solution:*

$\tilde{u} = \sin(x)$

## A sample solution of the Dirichlet problem.

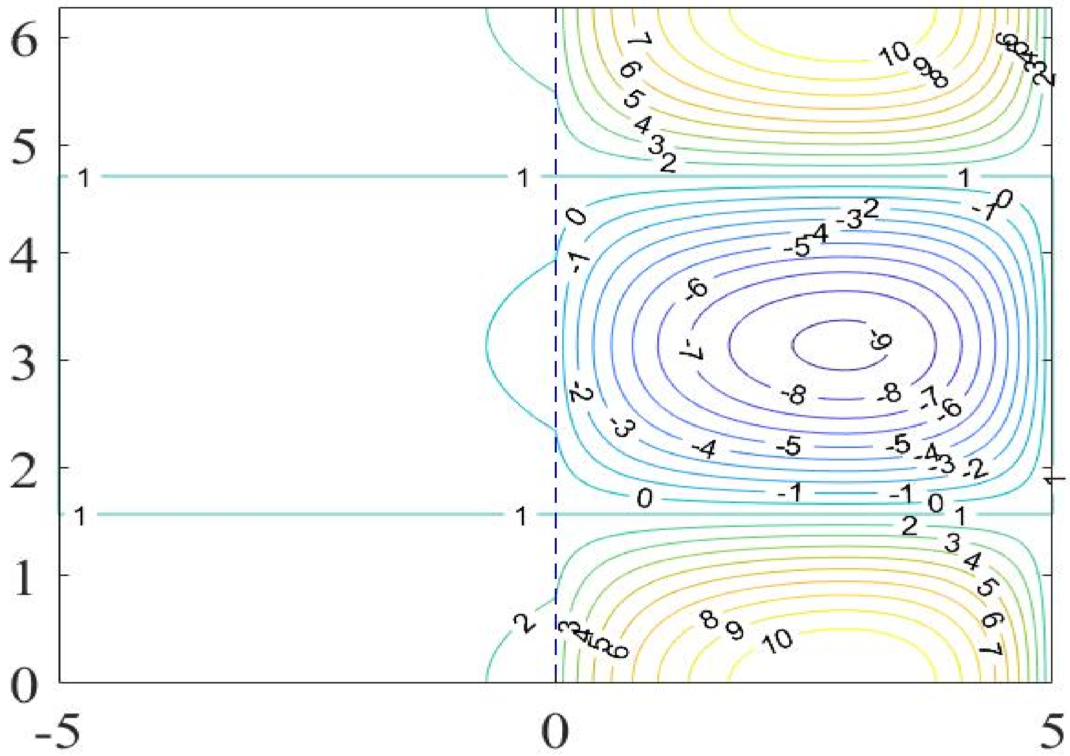

***Fig.8.*** *Isolines of a sample solution at* $\kappa = 10$ *, right-hand side* $f = \cos(y)$ *and boundary conditions* $U(L, y) = U(-L, y) \equiv 1$ *This picture was drawn for N = 100.*

## Experiments for homogeneous media

In this case the jump-line does not exists and all grid point have either type I or IV. The stencils for inner points of G are shown on Fig.2.

For an exact solution u one can take any function from $C^4$. Below the results of numerical experiments are presented.

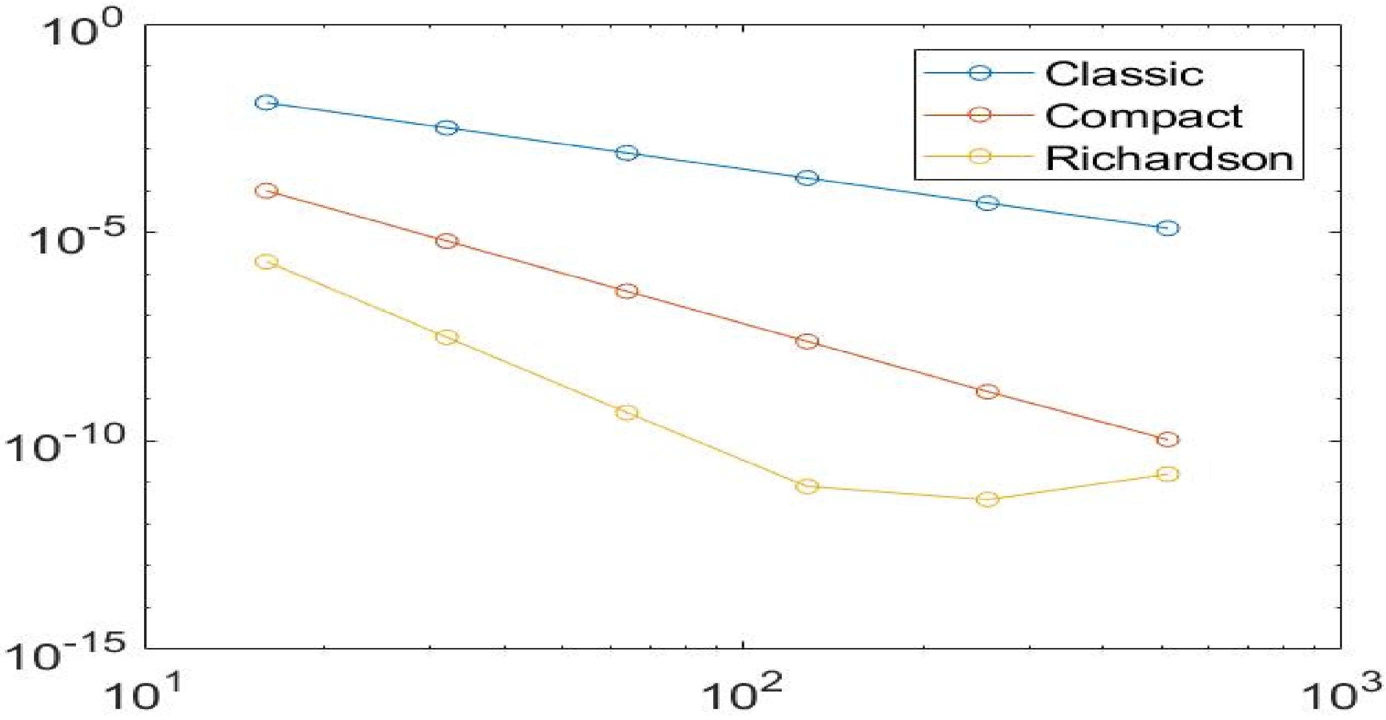


***Fig.9.*** *Schemes' error depending on N in loglog coordinates. The exact solution:* $\tilde{u} = \sin(x)$

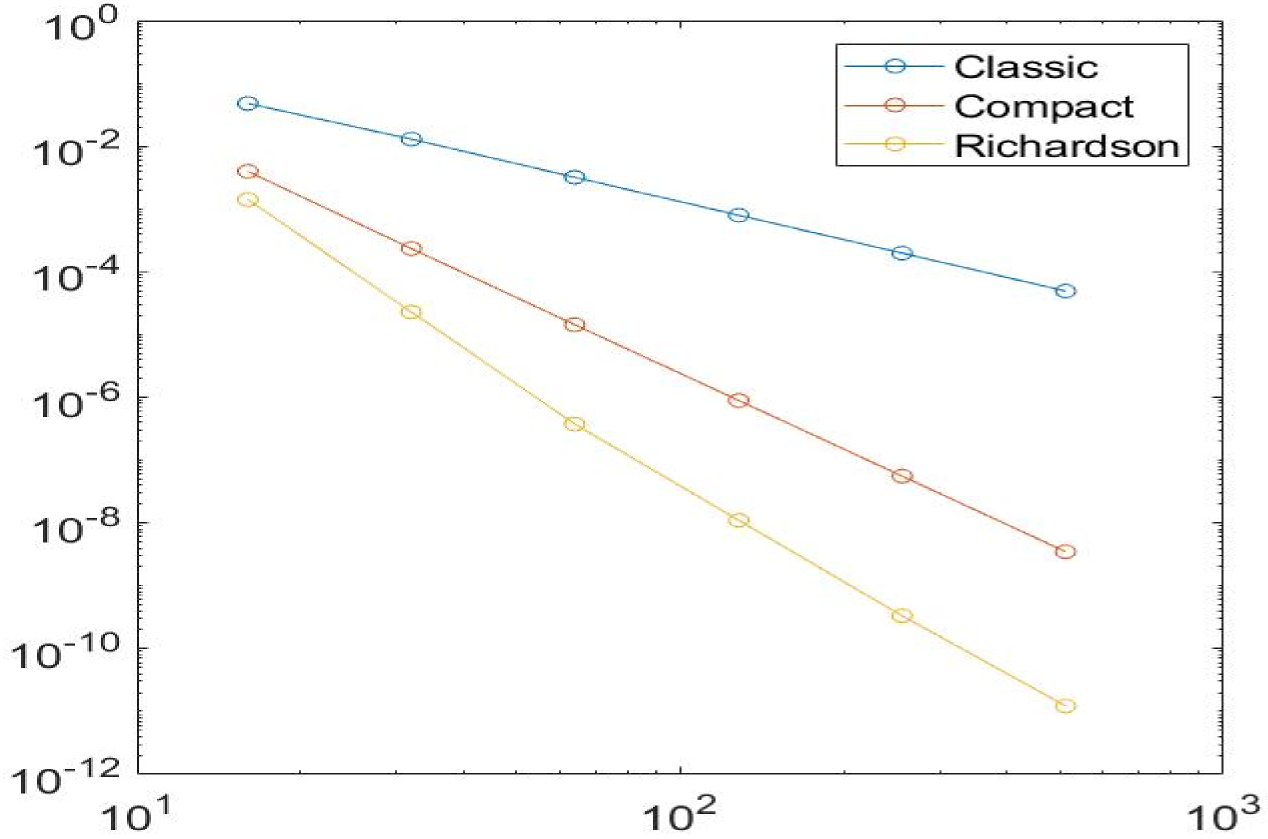


***Fig.10.*** *Same results for the exact solution:* $\tilde{u} = \sin(x + y)$

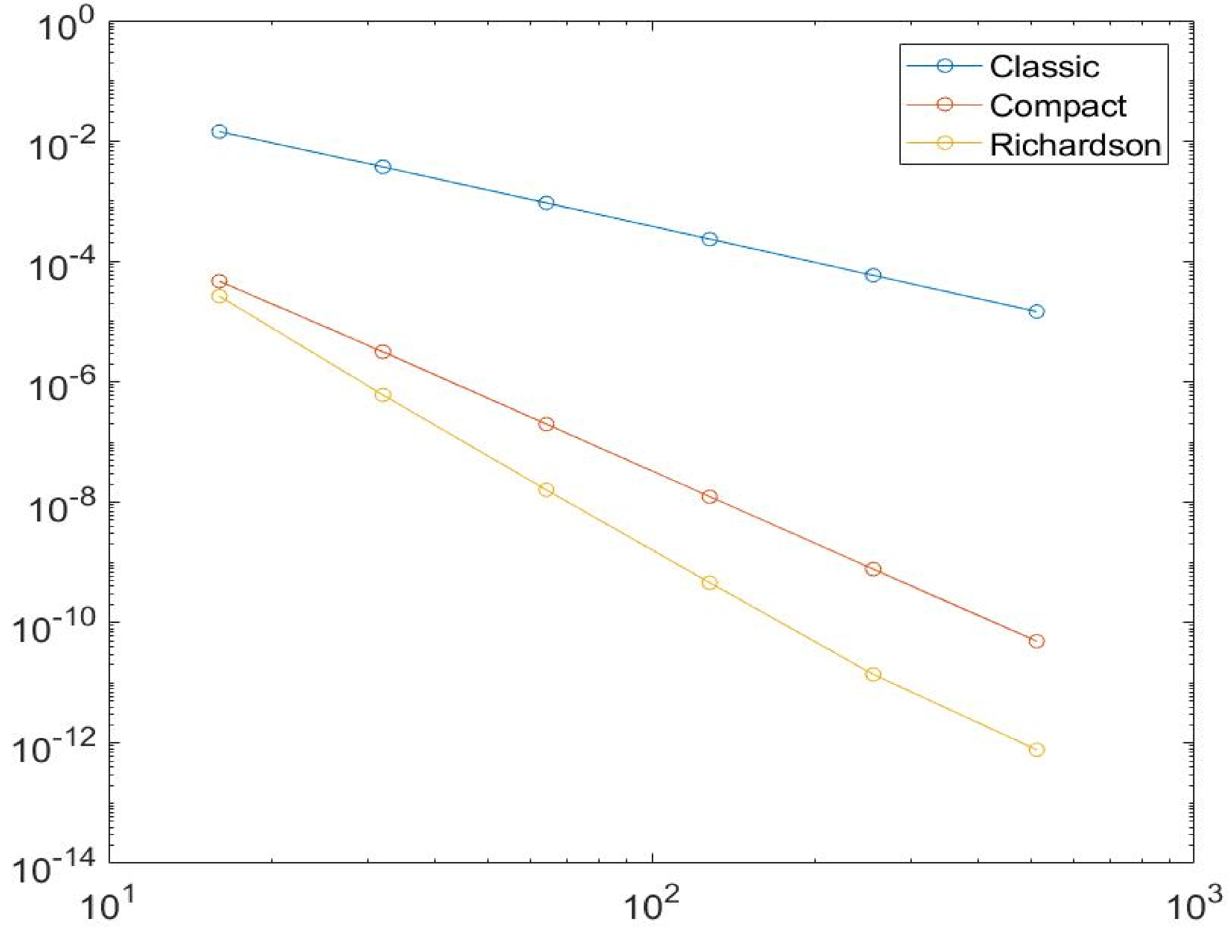


***Fig.11.*** *Same results for the exact solution:*
$\tilde{u} = x^6 \sin(x)$

**Classic divergent scheme (in comparison with the compact one)**

The idea of this scheme is in difference approximation of the derivatives as follows:

$$(u_x{}')_{i-1/2,j} \approx \frac{u_{ij} - u_{i-1,j}}{h}, \qquad (u_y{}')_{i,j-1/2} \approx \frac{u_{ij} - u_{i,j-1}}{h}.$$

Therefore, we get the following approximation of the Laplace operator:

$$(\bar{L}[u])_{ij} \approx h^{-2}\left\{\left[\vartheta\left(i+\frac{1}{2},j\right)\left(u_{(i+1),j} - u_{i,j}\right) - \vartheta\left(i-\frac{1}{2},j\right)\left(u_{i,j} - u_{(i-1),j}\right)\right] + \right.$$
$$\left. + \left[\vartheta\left(i,j+\frac{1}{2}\right)\left(u_{i,(j+1)} - u_{i,j}\right) - \vartheta\left(i,j-\frac{1}{2}\right)\left(u_{i,j} - u_{i,(j-1)}\right)\right]\right\}$$

At points of type **I** and **II** equation (1) is approximated: $\vartheta L[u]_{ij} = f_{ij}$.

At points of type III one can approximate the solution with two quadratic polynomial from each side of $\Gamma$ and write the Kirchhoff condition (3), which will give the following relation between the polynomials' coefficients:

$$\vartheta_- u_{i-2\,j} - 4\vartheta_- u_{i-1\,j} + 3(\vartheta_- + \vartheta_+)u_{ij} - 4\vartheta_+ u_{i+1\,j} + \vartheta_+ u_{i+2\,j} = 0. \qquad (5)$$

## Richardson extrapolation

Our algorithm that solves the differential problem depends on the grid's step $h \sim \frac{1}{N}$, and the following asymptotic is fulfilled:

$$u_h(\vec{x}) = u(\vec{x}) + C(\vec{x})h^{\nu} + o(h^{\nu}) \tag{6}$$

Therefore, we obtain for the step $2h$:

$$u_{2h}(\vec{x}) = u(\vec{x}) + C(\vec{x})2^{\nu} h^{\nu} + o(h^{\nu}) \tag{7}$$

We obtain from estimations (6-7): $C(\vec{x}) = \dfrac{u_h(\vec{x}) - u_{2h}(\vec{x})}{(1-2^{\nu})h^{\nu}}$, $u(\vec{x}) = u_{2h}(\vec{x}) - C(x)(2h)^{\nu} + o(h^{\nu})$.

We compare two solutions $u_{2h}(\vec{x})$ и $u_h(\vec{x})$ only on the coarsest grid with step $2h$.

The compact scheme's accuracy order is equal to 4, therefore one should take $\nu = 4$.

## Helmgoltz equation

Compact approximation of Helmgoltz Eq. (2) can be reduced to compact approximation of the Poisson equation by using the following substitution: $g = f - \rho(x, y)u$. We construct the global SLAE for $u$ and $g$: $Au = Pg \Leftrightarrow (A + \bar{\rho}P)u = Pf \Leftrightarrow Bu = Pf$, where the matrix $B = A + P\bar{\rho}$, $\bar{\rho}$ - the diagonal matrix with grid values of the coefficient $\rho$ in Eq. (2. If $\rho$ is positive, the spectrum of the Helmgoltz operator will also be positive and matrix $B$ will be well conditioned. Otherwise we can't guarantee good conditionality of matrix $B$.

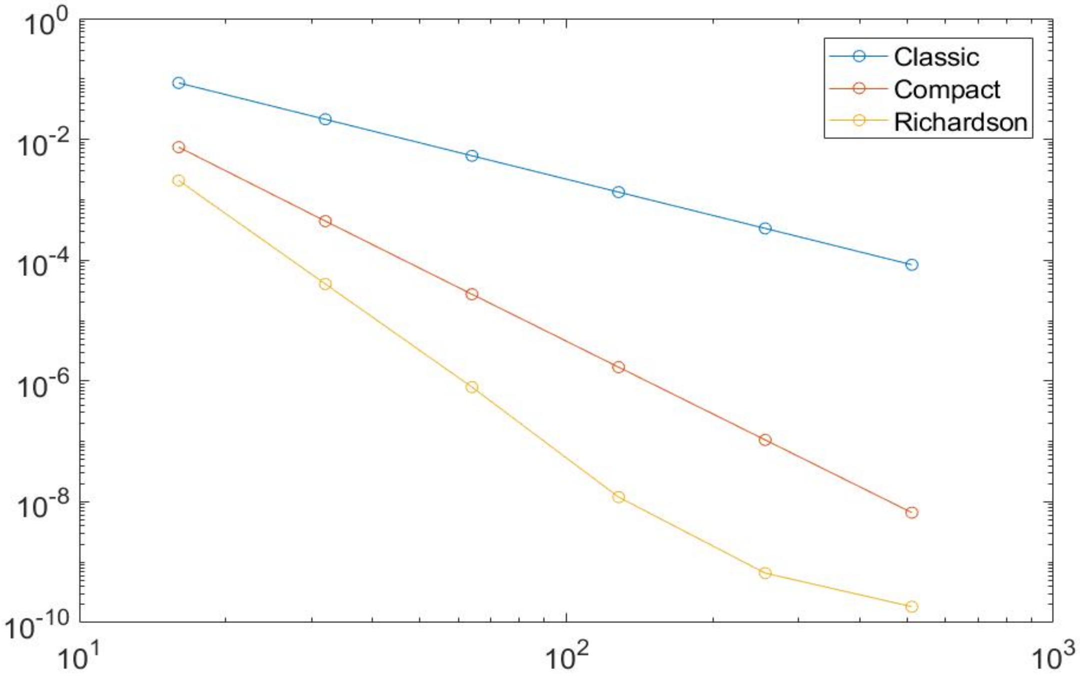


***Fig.12.*** *Schemes' error depending on $N$ in loglog coordinates for Helmgoltz equation.*
*Here* $\tilde{u} = x^6 \sin(y)$.
*The coefficient*

$$\rho(x, y) = \exp\left(-\frac{1}{y(y - 2\pi)}\right)$$

*is positive.*

**Case of discontinuous coefficient $\rho$ in Helmgoltz equation.**

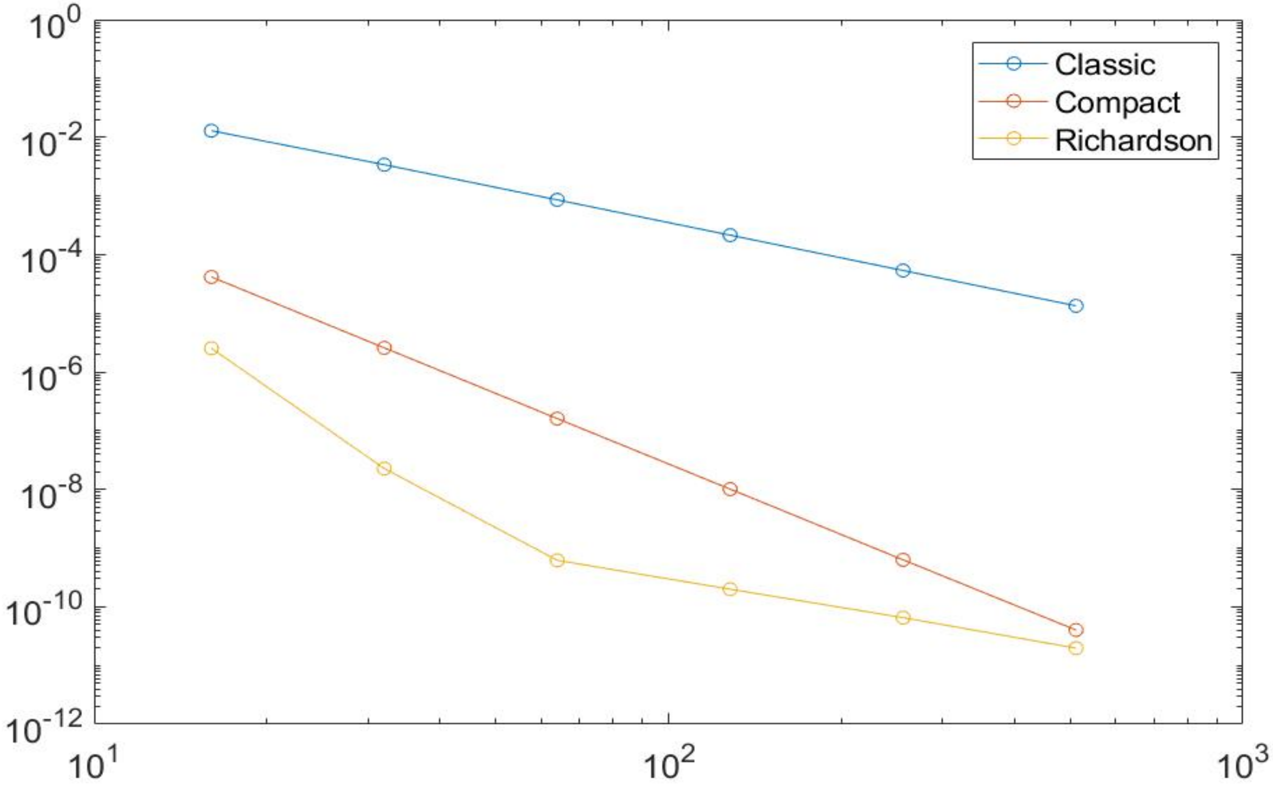


***Fig.13.*** *Same results for*

$$\tilde{u} = \sin(y + x) + 2$$

*and coefficient:*

$$\rho(x, y) = \begin{cases} 1, & x \le 0 \\ 10, & x \ge 0 \end{cases}.$$

## The Multigrid method

The idea of the multigrid method is that one needs to consistently apply several embedded into each other grids with resolutions $N_0$, $2N_0$, $4N_0$,...., $2^{k-1}N_0$, respectively. This method is effective as it allows to attenuate amplitudes of the problem's eigen functions rather fast for a wide diapason of wavenumbers. It happens so because each grid has its own diapason of fast attenuating eigen functions and the multigrid technic allows to combine them.

Transmission from a coarser grid to a finer: bilinear interpolation

Transmission from a finer grid to a coarser: simple restriction

We apply the grids in the following order:

Iterations start on the coarsest grid, which are followed by a series of refinements and smoothing relaxation iterations after each refinement. After that we start series of coarsening with smoothing iteration after each coarsening. This process is called a V-cycle. In this study we assume that the resolution of the coarsest grid $N_0 = 16$.

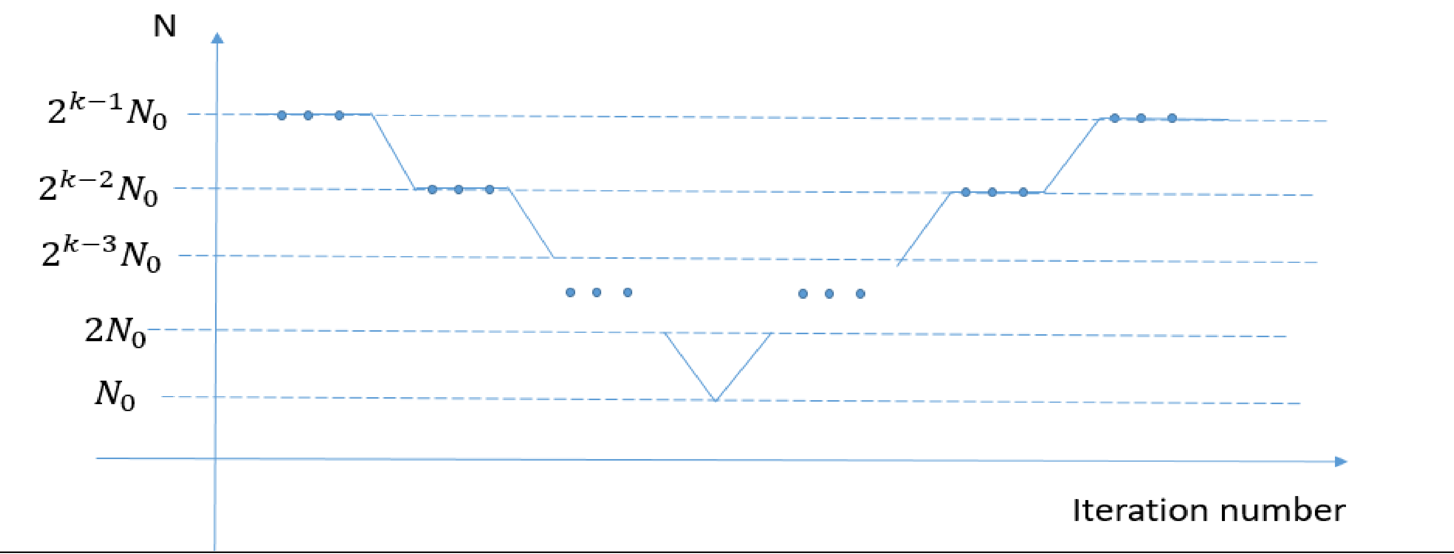


**Fig.14.** *Structure of a V-cycle.*

The efficiency of the multigrid method is described by two parameters: NN (normalized norm of the residual) and CC (computational cost – number of arithmetical operations +, -, /, *)

We provide below the results of experiments: NN and CC depending on the resolution of the finest grid.

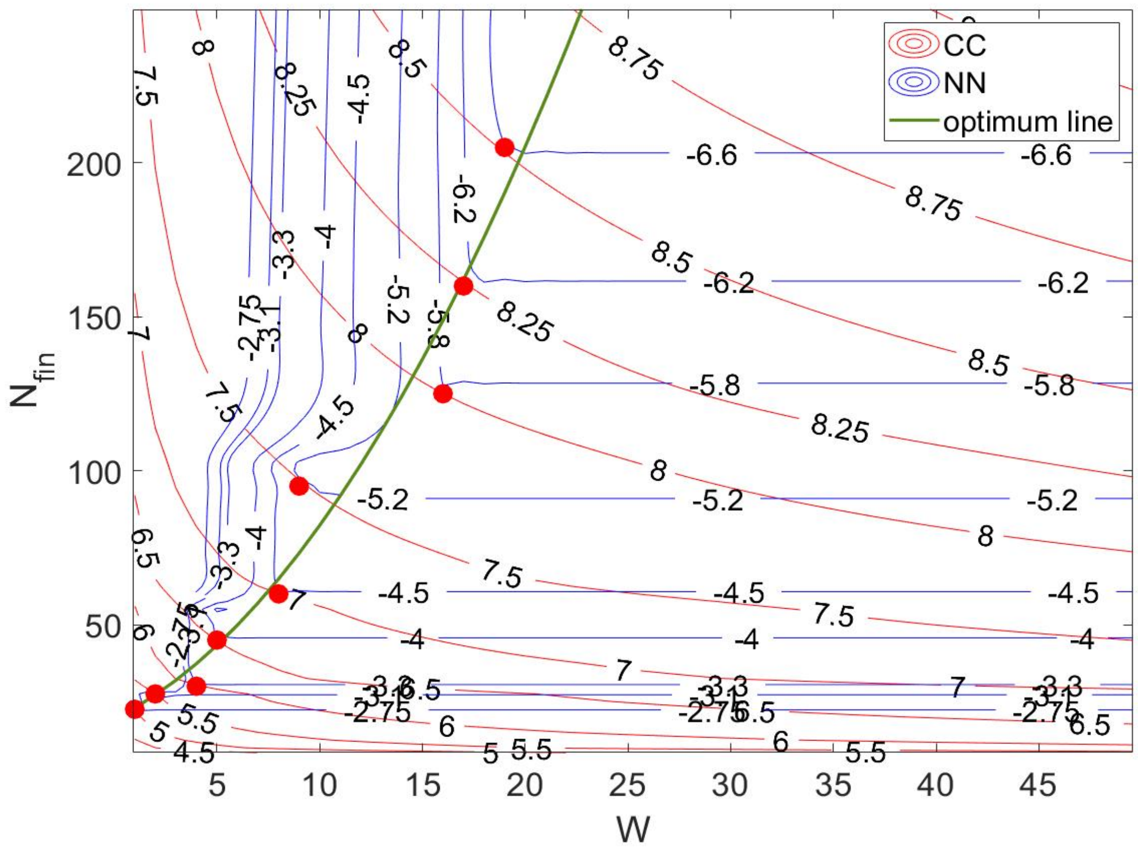


**Fig.15.** *Isolines of CC and NN for the compact scheme depending on the number of V-cycles (W) and the resolution of the finnest grid ( $N_{fin}$ ). The green line shows the optimal realtinon between W and $N_{fin}$.*
*Here* $u = \sin(x + y)$

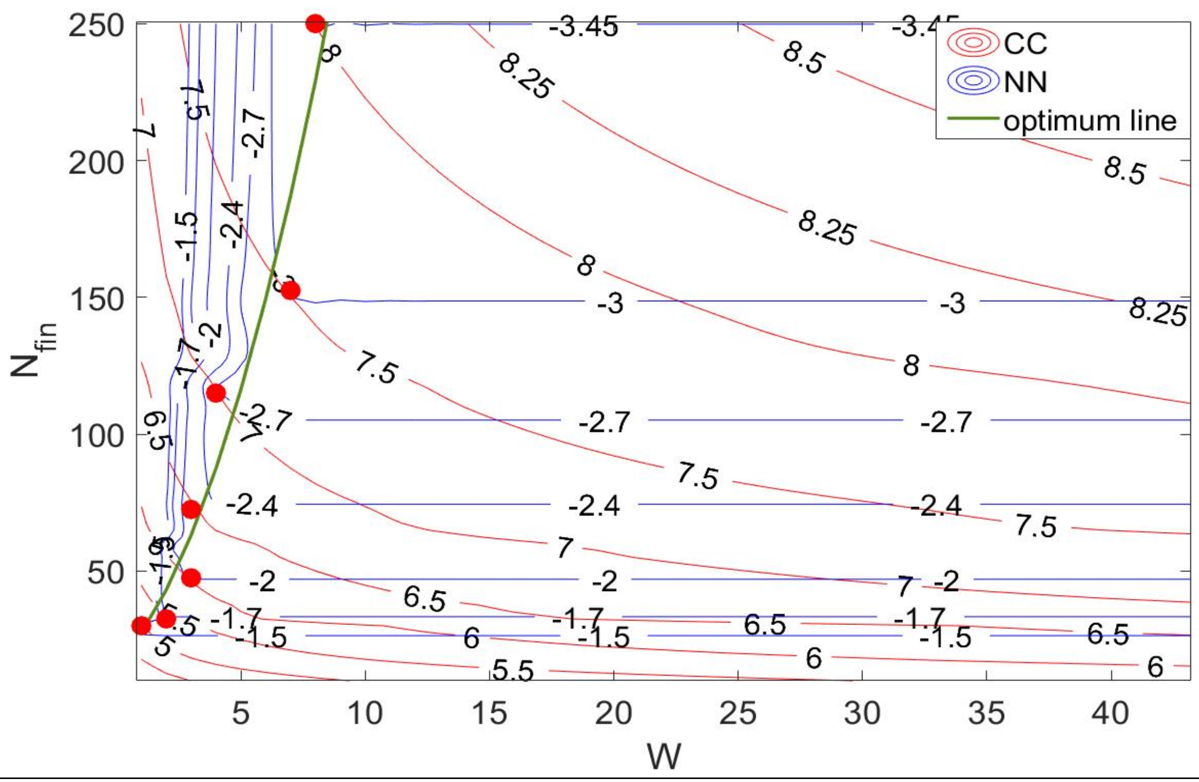


**Fig.16.** *Same results for the classic scheme.*

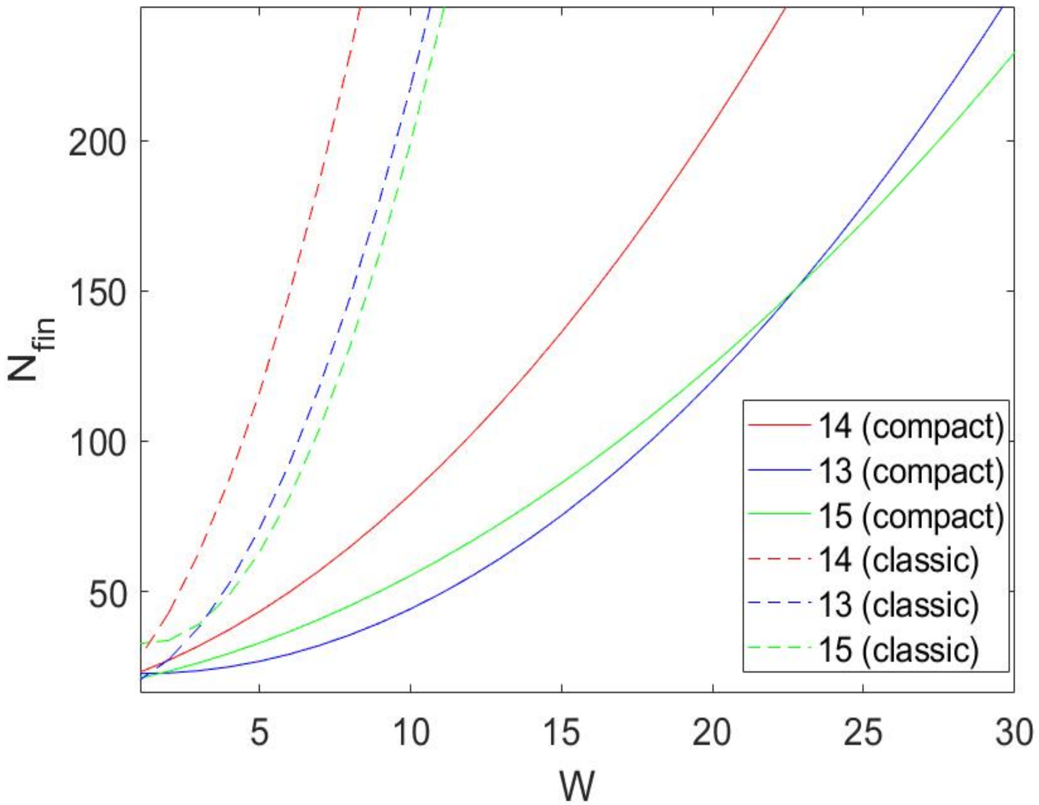


**Fig.17.** *Optimal curves for both schemes for various exact solutions:*

13) $\tilde{u} = \sin(x + y)$
*14)* $\tilde{u} = \sin(x)$
15) $\tilde{u} = x^6 \sin(y)$.


## Acknowledgements

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