НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ УНИВЕРСИТЕТ

Discrete Transparent Boundary Conditions

for the Equation of Rod Transverse Vibrations

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Transparent Boundary Conditions (TBCs) on the boundary $\vec{x} \in \partial V \subset \mathbb{R}^n$ for the equation

$$\partial_t u = Au + f$$

provide the same solution of the mixed problem as the solution of Cauchy problem with prolonged initial functions and forcing f to all space $\vec{x} \in \mathbb{R}^n$ by zero.

For $1D$ differential wave equation $\partial_t^2 u = c^2 \partial_x^2 u$ the TBCs are local: $\partial_t u = \pm c \partial_x u$ on the border $[-L/2, L/2] \subset \mathbb{R}$.

However, usually such TBCs are non-local and contain convolutions with respect to time t and all variables that are tangential to ∂V.

The equation of transverse vibrations of a rod (beam) with a circular cross section

$$\rho \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left[R^2 \rho \frac{\partial^3 u}{\partial x\, \partial t^2}\right] + \frac{\partial^2}{\partial x^2}\left[E R^2 \frac{\partial^2 u}{\partial x^2}\right] = f \qquad (1)$$

has many applications. Here

- ρ is the density of the rod,
- R is the radius of the cross section of the rod,
- E is the Young's modulus of the rod,
- $u \equiv u(t, x)$ is the transverse displacement of the rod,
- $x \in [-L/2, L/2] \subset \mathbb{R}$, L is the length of the rod,
- f describes external force (forcing).

Eq. (1) is not resolved with respect to the solution's higher (second) derivative with respect to time, and is supplemented by two initial conditions:

$$u(0, x) = U_0(x), \quad \partial_t u(0, x) = U_1(x),$$

Consider Eq. (1) with constant coefficients and no forcing f:

$$\rho\frac{\partial^2 u}{\partial t^2} - R^2\rho\frac{\partial^4 u}{\partial^2 t\,\partial^2 x} + ER^2\frac{\partial^4 u}{\partial^4 x} = 0, \tag{2}$$

Approximate it by the implicit finite-difference scheme of the Crank – Nicolson type (uniform grid with steps τ with respect to time t, and h with respect to spatial variable x):

$$\begin{gathered}\sigma\left(u_{2h}^{n+1} + u_{-2h}^{n+1} + u_{2h}^{n-1} + u_{-2h}^{n-1}\right) + \beta\left(u_{h}^{n+1} + u_{-h}^{n+1} + u_{h}^{n-1} + u_{-h}^{n-1}\right) + \\ +\alpha\left(u_0^{n+1} + u_0^{n-1}\right) + \gamma\left(u_{-h}^{n} + u_h^n\right) + \delta u_0^n = 0, \\ u(0, x_i) = u_0(x_i), \quad u(\tau, x_i) = u_\tau(x_i), \quad i = 0, \ldots, N.\end{gathered} \tag{3}$$

Here $\nu = ER^2\rho^{-1}\cdot\tau^2 h^{-4}$, $\mu = R^2\cdot h^{-2}$ and $\alpha = 1 + 3\nu + 2\mu$, $\beta = -2\nu - \mu$, $\gamma = 2\mu$, $\delta = -2 - 4\mu$, $\sigma = \nu/2$.

1. Let us apply the $\mathcal{Z}$-transform (discrete analogue of the Laplace transform) with respect to time to Eq. (3) and obtain a linear ordinary finite-difference equation with respect to the spatial variable m; the ordinary equation depending on the parameter $z \in \mathbb{C}$ (dual to discrete time n).
2. We construct for the homogeneous finite-difference 4-th order equation the fundamental set of solutions $\{Y_j(m)\}_{j=1}^{4}$, such that solutions $Y_1,\ Y_2$ decrease as $m \to +\infty$, and solutions $Y_3,\ Y_4$ decrease as $m \to -\infty$.
3. We construct the finite-difference boundary operators for the right end $x = L/2$ such that the conditions are fulfilled on the functions $Y_1,\ Y_2$ and for the left end $x = -L/2$ — on the functions $Y_3,\ Y_4$.

4. For the right segment's end decompose the obtained decreasing (as $m \to +\infty$) solutions into a Laurent series in a neighbourhood of $z = \infty$. For the left end — as $m \to -\infty$.
5. Apply the inverse $\mathcal{Z}$-transform $z \mapsto n$ to obtain the coefficients of the DTBCs.
6. Construct vectorial rational functions. The corresponding polynomials are symbols of the Approximate DTBCs (ADTBCs).

After $\mathcal{Z}$-transform of Eq. (3), we get characteristic homogeneous equation

$$\sigma\left(z^2+1\right)\left[\lambda+\lambda^{-1}\right]^2+\left(\beta\left(z^2+1\right)+\gamma z\right)\left[\lambda+\lambda^{-1}\right]+ \\ +\delta z+(\alpha-2\sigma)\left(z^2+1\right)=0. \tag{4}$$

Substitute auxiliary variable $\eta=\lambda+\lambda^{-1}$ in Eq. (4) and get

$$\sigma\left(z^2+1\right)\eta^2+\left(\beta\left(z^2+1\right)+\gamma z\right)\eta+\delta z+(\alpha-2\sigma)\left(z^2+1\right)=0. \tag{5}$$

Roots of Eq. (5) are

$$\eta_{1,2}(z)=\frac{-\beta\left(z^2+1\right)-\gamma z}{2\sigma\left(1+z\right)}\mp \\ \mp\frac{\sqrt[+]{\left(\beta\left(z^2+1\right)+\gamma z\right)^2-4\sigma\left(z^2+1\right)\left[\delta z+(\alpha-2\sigma)\left(z^2+1\right)\right]}}{2\sigma\left(1+z\right)}, \tag{6}$$

Construction of DTBCs

Series expansion

We develop functions (6) into the Laurent series at $z = \infty$:

$$\eta_{1,2}(z) = \frac{1}{\nu}\sum_{k=0}^{\infty}(-1)^k z^{-2k}\left[(\mu+2\nu)\left(1+z^{-2}\right) - 2\frac{\mu}{z}\mp\right.$$

$$\left.\mp\sqrt{\mu^2-2\nu}\left(1-\frac{1}{z}\right)\left(\frac{1}{z^2}-2\frac{\mu^2}{\mu^2-2\nu}\frac{1}{z}+1\right)\sum_{n=0}^{\infty}\mathrm{P}_n\left(\frac{\mu^2}{\mu^2-2\nu}\right)z^{-n}\right] \quad (7)$$

where P_n is a Legendre polynomial of degree n. Note that as $z \to \infty$

$$\eta_j(z) = \vartheta_j + r_j(z), \quad (8)$$

where $r_j(z) \to 0$ at $j = 1, 2$, and

$$\vartheta_j = \frac{1}{2\sigma}\left[\beta \mp \sqrt{\beta^2 - 4\sigma(\alpha - 2\sigma)}\right] = 2 + \frac{\mu}{\nu} \mp \frac{1}{\nu}\sqrt{\mu^2 - 2\nu}. \quad (9)$$

Let us resolve the relation $\eta = \lambda + \lambda^{-1}$ as a quadratic Eq.

$$\lambda^2 - \eta\lambda + 1 = 0. \tag{10}$$

For both auxiliary functions η_1, η_2 we obtain the following roots of characteristic Eq. (4):

$$\lambda_1 = \frac{\eta_1(z)}{2} - \sqrt{\frac{\eta_1^2(z)}{4} - 1}, \quad \lambda_2 = \frac{\eta_2(z)}{2} - \sqrt{\frac{\eta_2^2(z)}{4} - 1},$$
$$\lambda_3 = \frac{\eta_1(z)}{2} + \sqrt{\frac{\eta_1^2(z)}{4} - 1}, \quad \lambda_4 = \frac{\eta_2(z)}{2} + \sqrt{\frac{\eta_2^2(z)}{4} - 1}.$$

We obtain the Laurent series for the characteristic roots in a vicinity of the point $z = \infty$:

$$\lambda_{1,3}(z) = \frac{\eta_1(z)}{2} \mp \sqrt{\frac{\vartheta_1^2}{4} - 1} \cdot \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \, (2n)! \, r_1^n(z)}{(1-2n) \, n! \, 4^n \, (\theta_1 + 2)^n} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \, (2n)! \, r_1^n(z)}{(1-2n) \, n! \, 4^n \, (\theta_1 - 2)^n}, \tag{11}$$

$$\lambda_{2,4}(z) = \frac{\eta_2(z)}{2} \mp \sqrt{\frac{\vartheta_2^2}{4} - 1} \cdot \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \, (2n)! \, r_2^n(z)}{(1-2n) \, n! \, 4^n \, (\theta_2 + 2)^n} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \, (2n)! \, r_2^n(z)}{(1-2n) \, n! \, 4^n \, (\theta_2 - 2)^n}, \tag{12}$$

where $\eta_{1,2}$ are taken from (7), ϑ_j — from (9), $r_j(z) = \eta_j(z) - \vartheta_j$, $j = 1, 2$.

The following inequalities are fulfilled as $z \to \infty$:

$$|\lambda_1|,\ |\lambda_2| < 1 < |\lambda_3|,\ |\lambda_4|.$$

Therefore, as $m \to +\infty$ it is possible to derive decreasing λ_1^m, λ_2^m, and increasing λ_3^m, λ_4^m solutions of equation's (3) $\mathcal{Z}$-transform. They form the fundamental set of solutions.

As for differential Eq. (2), for correctness of mixed initial-boundary value problem for finite-difference Eq. (3) two boundary conditions at each edge of the rod are required. We start by construction of $\mathcal{Z}$-image of the boundary conditions for the left edge in the form

$$P_1(z^{-1})\,v(0) + Q_1(z^{-1})\,v(h) + R_1(z^{-1})\,v(2h) + S_1(z^{-1})\,v(3h) = 0,$$
$$P_2(z^{-1})\,v(0) + Q_2(z^{-1})\,v(h) + R_2(z^{-1})\,v(2h) + S_2(z^{-1})\,v(3h) = 0,$$

where v is $\mathcal{Z}$-image of u. The equations correspond to

$$\sum_{j=0}^{\infty} p_{kj}\,u_0^{n-j} + \sum_{j=0}^{\infty} q_{kj}\,u_1^{n-j} + \sum_{j=0}^{\infty} r_{kj}\,u_2^{n-j} + \sum_{j=0}^{\infty} s_{kj}\,u_3^{n-j} = 0,\ k = 1,\, 2, \tag{13}$$

where values p_{kj}, q_{kj}, q_{kj} and r_{kj} are the Laurent series coefficients before the term $1/z^j$ of the functions $P_k(z^{-1})$, $Q_k(z^{-1})$, $R_k(z^{-1})$, $S_k(z^{-1})$ correspondingly in a vicinity of the point $z = \infty \in \mathbb{C}$.

Two linearly independent boundary conditions will provide transparency property, iff for the increasing Cauchy problem solutions $\nu(m) = \lambda_3^m$ and $\nu(m) = \lambda_4^m$ the symbols of the boundary conditions $\langle P_k,\ Q_k,\ R_k,\ S_k\rangle$, $(k = 1,\ 2)$ fulfil the following equations:

$$\begin{aligned} P_k + Q_k\,\lambda_3 + R_k\,\lambda_3^2 + S_k\,\lambda_3^3 = 0, \\ P_k + Q_k\,\lambda_4 + R_k\,\lambda_4^2 + S_k\,\lambda_4^3 = 0. \end{aligned} \tag{14}$$

We relax the requirements to symbols of the operators of ICP boundary conditions, and exchange analytic functions in Syst. (14) by polynomials and exact equalities by asymptotic (as $z \to \infty$) equalities:

$$\begin{cases} P_k(z^{-1}) + Q_k(z^{-1})\,\lambda_3 + R_k(z^{-1})\,\lambda_3^2 + S_k(z^{-1})\,\lambda_3^3 = \mathbf{O}\left(z^{-K_k}\right), \\ P_k(z^{-1}) + Q_k(z^{-1})\,\lambda_4 + R_k(z^{-1})\,\lambda_4^2 + S_k(z^{-1})\,\lambda_4^3 = \mathbf{O}\left(z^{-K_k}\right). \end{cases} \tag{15}$$

If we choose normalisation condition at $k = 1$:

$$P_1(0) = p_{1,0} = 1,\ Q_1(0) = q_{1,0} = 0,$$

we obtain the first boundary condition in the form:

$$u_0^n + \sum_{j=1}^{\deg P_1} p_{1j}\, u_0^{n-j} + \sum_{j=1}^{\deg Q_1} q_{1j}\, u_1^{n-j} + \sum_{j=0}^{\deg R_1} r_{1j}\, u_2^{n-j} + \sum_{j=0}^{\deg S_1} s_{1j}\, u_3^{n-j} = 0. \tag{16}$$

Similarly, we choose normalisation condition at $k = 2$:

$$P_2(0) = p_{2,0} = 0,\ Q_2(0) = q_{2,0} = 1,$$

to obtain the preboundary value:

$$u_1^n + \sum_{j=1}^{\deg P_2} p_{2j}\, u_0^{n-j} + \sum_{j=1}^{\deg Q_2} q_{2j}\, u_1^{n-j} + \sum_{j=0}^{\deg R_2} r_{2j}\, u_2^{n-j} + \sum_{j=0}^{\deg S_2} s_{2j}\, u_3^{n-j} = 0. \tag{17}$$

The Hamiltonian (energy) of the rod is a sum of its kinetic K and potential P energies $\mathcal{H}[u] = \mathrm{K}[u] + \mathrm{P}[u]$:

$$\mathrm{K}[u] = \frac{1}{2}\int\limits_{-L/2}^{L/2} \rho \left[\left(\frac{\partial u}{\partial t}\right)^2 + R^2 \left(\frac{\partial^2 u}{\partial t\,\partial x}\right)^2 \right] \mathrm{d}x, \ \mathrm{P}[u] = \frac{1}{2}\int\limits_{-L/2}^{L/2} ER^2 \left(\frac{\partial^2 u}{\partial x^2}\right)^2 \mathrm{d}x.$$

Energy approximation:

$$\hat{\mathcal{H}}\left[u^{n+1/2}\right] = h\left[\frac{1}{2}\left(\vartheta_0^{n+1/2} + \vartheta_N^{n+1/2}\right) + \sum_{j=1}^{N-1} \vartheta_j^{n+1/2}\right], \tag{18}$$

where for $j = 1, \ldots, N-1$ we have

$$\vartheta_j^{n+1/2} = \rho\left(\frac{u_j^{n+1} - u_j^n}{\tau}\right)^2 + \rho R^2 \left(\frac{u_{j+1}^{n+1} - u_{j+1}^n - u_{j-1}^{n+1} + u_{j-1}^n}{2h\tau}\right)^2 +$$
$$+ ER^2 \left(\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1} + u_{j+1}^n - 2u_j^n + u_{j-1}^n}{2h^2}\right)^2.$$

To determine the stability regions on (h, τ) plane we introduce three stability criteria:

1. Energy criterion:

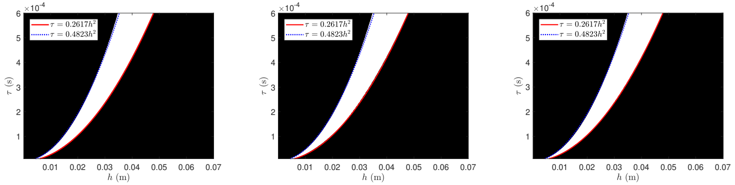
$$\|u^{n+1/2}\|_{\mathcal{H}} \equiv \sqrt{\hat{\mathcal{H}}\left[u^{n+1/2}\right]} \leq \|u^{1/2}\|_{\mathcal{H}},$$

2. C-norm criterion:

$$\|u^n\|_{\mathrm{C}} \equiv \max_{0 \leq j \leq N} |u_j^n| \leq \|u^0\|_{\mathrm{C}},$$

3. L^2 criterion:

$$\|u^n\|_{\mathrm{L}^2} \equiv \sqrt{h\left[\frac{1}{2}\left((u_0^n)^2 + (u_N^n)^2\right) + \sum_{j=1}^{N-1}(u_j^n)^2\right]} \leq \|u^0\|_{\mathrm{L}^2}.$$

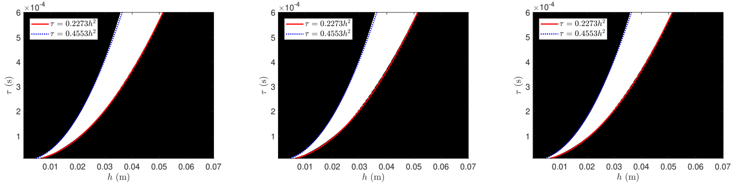


(a) $\mathcal{H}$ (b) C (c) L^2

Figure 1: The domain of stability on the (h, τ) plane for two ADTBCs. Set of polynomial degrees: $\langle 4, 4, 8, 8 \rangle$. Physical parameters of the rod ρ, E, R and L are the same as in Table 1.

Here and further we denote the symbol of ADTBCs obtained with polynomial degrees $\deg P_k = d_{1,k}$, $\deg Q_k = d_{2,k}$, $\deg R_k = d_{3,k}$ and $\deg S_k = d_{4,k}$ with $k = 1, 2$ as

$$\langle P_k, Q_k, R_k, S_k \rangle \equiv \langle d_{1,k}, d_{2,k}, d_{3,k}, d_{4,k} \rangle.$$



(a) $\mathcal{H}$ (b) C (c) L^2

Figure 2: The domain of stability on the (h, τ) plane for two ADTBCs. Set of polynomial degrees: $\langle 5, 3, 9, 7 \rangle$. Physical parameters of the rod ρ, E, R and L are the same as in Table 1.

For our experiments we choose rod's parameters (that are similar to steel) and steps:

$$
\begin{array}{r|l||c|l}
\rho & = 7860\,\mathrm{kg\,m^{-3}} & L & = 1\,\mathrm{m} \\
E & = 210 \cdot 10^9\,\mathrm{Pa} & T & = 0.3\,\mathrm{s} \\
R & = 10^{-3}\,\mathrm{m} & \nu & \approx 4.2748 \\
h & = 0.02\,\mathrm{m} & \mu & = 0.0025 \\
\tau & = 1.6 \cdot 10^{-4}\,\mathrm{s} & &
\end{array}
$$

Table 1: Parameters of numerical experiments

We set initial conditions for Eq. (2) as $u(0,x) = x\exp\left(-\frac{x^2}{0.02}\right)$ and $\frac{\partial u}{\partial t}(0,x) = 0$ for $x \in [-L/2,\ L/2]$.

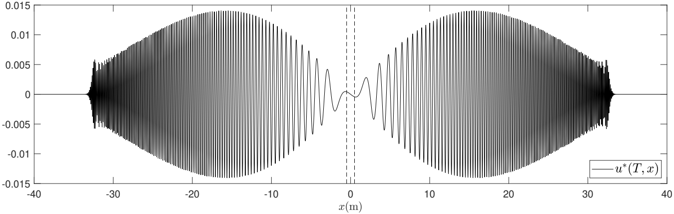


Figure 3: The reference solution u^* on the very extended segment $[-40L, 40L]$ at the final time moment $T = 0.3$. Two vertical dash lines indicate the borders of the considered segment $x \in [-L/2, L/2]$.

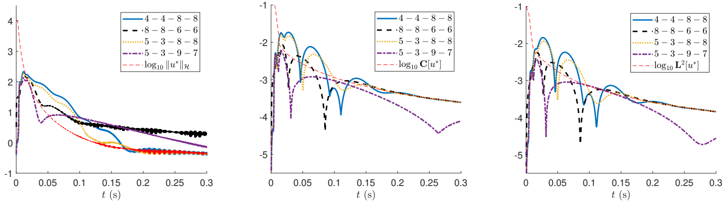
To evaluate the dynamics of error of Eq. (3) solution u under ADTBCs, we use

a) $\log_{10}\sqrt{\hat{\mathcal{H}}[u(t,x)-u^*(t,x)]}$,

b) $\log_{10}\left[\max_x |u(t,x)-u^*(t,x)|\right]$,

c) $\log_{10}\|u(t,x)-u^*(t,x)\|_2$.

The latter is approximated using trapezoidal method.
Results of numeric experiment with different ADTBCs sets are presented in Fig. 4.



(a) $\mathcal{H}$ (b) C (c) L^2

Figure 4: Common logarithm of (a) $\hat{\mathcal{H}}$, (b) C-norm, (c) L^2-norm of the difference between the reference solution u^* and solutions with ADTBCs.

In practice simple homogeneous boundary conditions (i.e., Dirichlet, Neumann) lead to partial or complete reflection of outgoing waves back into calculation area (sometimes with increased amplitude). Consider some "usual" homogeneous boundary conditions:

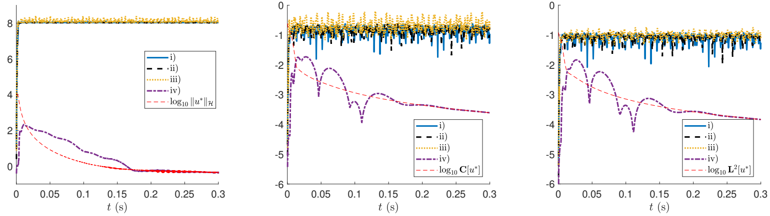
i) $u|_\Gamma = 0,\ \frac{\partial u}{\partial x}\Big|_\Gamma = 0 \implies u_0^n = u_1^n = 0,$

ii) $u|_\Gamma = 0,\ \frac{\partial^2 u}{\partial x^2}\Big|_\Gamma = 0 \implies u_0^n = 0,\ u_1^n = u_2^n/2,$

iii) $\frac{\partial^2 u}{\partial x^2}\Big|_\Gamma = 0,\ \frac{\partial^3 u}{\partial x^3}\Big|_\Gamma = 0 \implies u_0^n = 3u_2^n - 2u_3^n,\ u_1^n = 2u_2^n - u_3^n,$

iv) ADTBCs with polynomials degrees $\langle P_k, Q_k, R_k, S_k\rangle = \langle 4, 4, 8, 8\rangle,\ k = 1,\ 2.$

Fig. 5 shows the dynamics of solutions errors that are calculated using various boundary conditions:



(a) $\mathcal{H}$ (b) C (c) L^2

Figure 5: Common logarithm of (a) $\hat{\mathcal{H}}$, (b) C-norm, (c) L^2-norm of the difference between the reference solution u^* and solutions with “usual” boundary conditions.

Conclusions

1. The ADTBCs were constructed for the finite-difference Crank – Nicolson implicit approximation of the equation of rod transverse vibrations with a circular cross section.
2. Special vectorial version of the rational Hermite – Padé approximation provides economical and precise realisations of the ADTBCs.
3. Stability regions depend on the symbol of ADTBCs.
4. "Usual" homogeneous boundary conditions do not have transparency property.
5. The proposed algorithm could be used for different approximations of various evolutionary linear equations.

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3. Stability regions depend on the symbol of ADTBCs.
4. "Usual" homogeneous boundary conditions do not have transparency property.
5. The proposed algorithm could be used for different approximations of various evolutionary linear equations.
6. ADTBCs for a compact approximation are being studied.

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Thank you for your attention!

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