Monotonicity of implicit schemes

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Abstract

Monotone finite-difference schemes have significant advantages in integrating partial differential equations of evolution. For explicit difference schemes, the monotonicity property is easy to check. Here, the monotonicity of implicit linear schemes is estimated using the theory of residues. Examples of linear implicit schemes, monotonic and non-monotonic are considered.

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1. Introduction

A finite-difference explicit scheme, approximating an evolutionary partial differential equation

$$u_j^{n+1} = H_j\left(\vec{u}^n\right), \tag{1}$$

(n is the step number in time, j is the step number in a spatial variable; $\vec{u}^n = \left\{u_j^n\right\}_{j=1}^N$ or $\left\{u_j^n\right\}_{j=-\infty}^{\infty}$) is called monotonic if for all j the functions H_j monotonically increase with respect to all their arguments.

The monotonicity of the finite-difference scheme for evolutionary equations guarantees it a number of advantages related to 1) the maximum principle for their solutions and 2) the possible convergence of the difference solution to a weak solution of the approximated differential problem, in the case when a quasi-linear partial differential equation of the first order (a conservation law) is approximated. A shock may exist in its solution for smooth initial data (or is present from the very beginning). For a more detailed discussion of these issues, see, for example, [1], [2], [4], [6], [7], [8].

Explicit linear one-layer scheme $\vec{u}^{n+1} = L\left[\vec{u}^n\right]$ on a three-point stencil: $u_j^{n+1} = au_{j-1}^n + bu_j^n + cu_{j-1}^n$ is monotonic iff all its three coefficients are non-negative. Similarly, monotonicity is defined on a stencil that is wider with respect to x. An implicit

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scheme is equivalent to an explicit one, but on an infinite stencil. If we consider a problem that is periodic with respect to x, then all points are included in the implicit scheme stencil.

The following important property is fulfilled for monotonic schemes. Let two grid functions satisfy the inequality $v_j \leq u_j,\ j \in \mathbb{Z}$. Then $[Lv]_j \leq [Lu]_j$ for $j \in \mathbb{Z}$.

Let us consider an implicit one-layer time scheme on a three-point stencil (the step in the space is denoted by h, in time τ) approximating a linear equation with constant coefficients:

$$a_1 u_{j-1}^{n+1} + b_1 u_j^{n+1} + c_1 u_{j+1}^{n+1} = a_0 u_{j-1}^n + b_0 u_j^n + c_0 u_{j+1}^n. \tag{2}$$

We can formally apply the Fourier transform $F_{x\to\xi}$ with respect to a spatial variable to this equality. Let's denote $\omega = \xi h$. Then the transition in the solution to the next step in time, in Fourier images is written as a multiplication by a rational function (the symbol of the transition operator):

$$\sigma(\omega) = \frac{a_0 \exp(-i\omega) + b_0 + c_0 \exp(i\omega)}{a_1 \exp(-i\omega) + b_1 + c_1 \exp(i\omega)}. \tag{3}$$

Thus, to check the monotonicity of the scheme, it is necessary to decompose into a Fourier series a rational function of an exponent with real coefficients $\sigma(\omega)$. All its Fourier coefficients must be non-negative. The purpose of this work is to obtain the corresponding conditions for 6 coefficients: a_0, b_0, c_0, a_1 b_1, c_1.

Remark. We do not consider all rational functions, but only with a denominator other than zero at $\omega \in \mathbb{R}$. Otherwise, the L^2-norm of the function $\sigma(\omega)$ is infinite. In terms of difference schemes, this means that it is required to inverse the matrix at each time step, which is either irreversible or poorly conditioned. Such implicit schemes cannot be used.

Remark. Another (equivalent) formulation of the question about the monotonicity of an implicit scheme on a three-point stencil is possible: under which conditions on the coefficients a_0, b_0, c_0, a_1 b_1, c_1 of matrices

$$A = \begin{pmatrix} b_1 & c_1 & 0 & \dots & 0 & a_1 \\ a_1 & b_1 & c_1 & 0 & \dots & 0 \\ 0 & a_1 & b_1 & c_1 & \dots & 0 \\ . & . & . & . & . & . \\ 0 & \dots & 0 & a_1 & b_1 & c_1 \\ c_1 & 0 & \dots & 0 & a_1 & b_1 \end{pmatrix} \quad B = \begin{pmatrix} b_0 & c_0 & 0 & \dots & 0 & a_0 \\ a_0 & b_0 & c_0 & 0 & \dots & 0 \\ 0 & a_0 & b_0 & c_0 & \dots & 0 \\ . & . & . & . & . & . \\ 0 & \dots & 0 & a_0 & b_0 & c_0 \\ c_0 & 0 & \dots & 0 & a_0 & b_0 \end{pmatrix}$$

all elements of the matrix $C = A^{-1}B$ are non-negative? We assume here that the boundary conditions are periodic.

2. Fourier coefficients estimation

If the denominator of the symbol $\sigma(\omega)$ has real roots, the implicit scheme (2) is degenerate. The real part of the denominator is equal to $b_1 + (c_1 + a_1)\cos(\omega)$ and

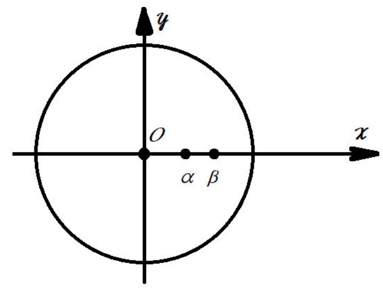

Figure 1: Case 1. Both roots of the denominator $\alpha \neq \beta$ are real and modulo less than 1.

its imaginary part is equal to $(c_1 - a_1)\sin(\omega)$. If $c_1 = a_1$, then the non-degeneracy condition is: $|b_1|> 2|a_1|$. If $c_1 \neq a_1$ then the denominator vanishes at $\omega = \pi n$ iff $b_1 = \pm(a_1 + c_1)$.

We will exclude these degeneracy cases from further consideration.

Remark. The absence of the degeneracy is a necessary, but not sufficient condition for the stability of the corresponding finite-difference scheme.

Let us calculate the Fourier coefficients:

$$C_k = \frac{1}{2\pi}\int_0^{2\pi} \frac{a_0 \exp(-i\omega) + b_0 + c_0 \exp(i\omega)}{a_1 \exp(-i\omega) + b_1 + c_1 \exp(i\omega)} \exp(-ik\omega)d\omega,\ k \in \mathbb{Z}.$$

After replacing the variable $z = \exp(i\omega)$ we obtain:

$$C_k = \frac{1}{2\pi i}\oint \frac{a_0 + b_0 z + c_0 z^2}{a_1 + b_1 z + c_1 z^2}\, z^{-(k+1)}\, dz. \tag{4}$$

where the integral is taken along the unit circumference on the complex plane. The calculation of the integral reduces to the calculation of the sum of residues inside the unit circle.

Let us assume now that the coefficients: $a_1,\ b_1,\ c_1 \neq 0$ (so named up-wind schemes, where $c_1 = c_0 = 0$, will be considered separately in section 4). Then the roots of the denominator are nonzero. Let's also assume that the roots of the square polynomial in the denominator are simple, i.e. $\alpha \neq \beta$. Then, it can be represented as $c_1(z-\alpha)(z-\beta)$, and the whole fraction can be decomposed into the simplest: $R(z) = C + \frac{A}{z-\alpha} + \frac{B}{z-\beta}$. Here $C = c_0/c_1$. Next, we choose the normalization (gauge) of the rational function $c_1 = 1$.

The poles of the function $R(z)$ can be localized at three points: $0,\ \alpha,\ \beta$. The points α and β can be located both inside and outside of the unit circle. Let's start from the point $z = 0$, where the residue does not depend on the location of the remaining roots $\alpha \neq \beta$.

The first term in the function R gives a residue $C = \delta_0^k$ (Kronecker delta). The other

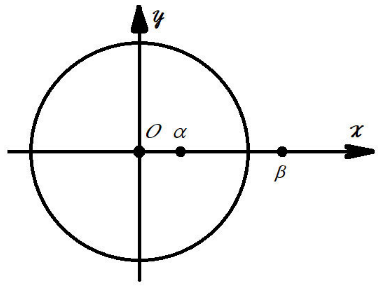

Figure 2: Case 2. Both roots of the denominator are real and $|\beta|> 1 > |\alpha|> 0$.

two terms in R, that are holomorphic in the neighborhood of zero, and we transform:

$$R(z) = C + \frac{A}{-\alpha}\frac{1}{1-z/\alpha} + \frac{B}{-\beta}\frac{1}{1-z/\beta} = C + \frac{A}{-\alpha}\sum_{m=0}^{\infty}\left(\frac{z}{\alpha}\right)^m + \frac{B}{-\beta}\sum_{m=0}^{\infty}\left(\frac{z}{\beta}\right)^m.$$

Therefore,

$$\mathbf{res}_0\left[R(z)z^{-k-1}\right] = \begin{cases} -A\alpha^{-k-1} - B\beta^{-k-1} & \Leftarrow \quad k > 0 \\ C - A\alpha^{-1} - B\beta^{-1} & \Leftarrow \quad k = 0 \\ 0 & \Leftarrow \quad k < 0 \end{cases} \tag{5}$$

Then we consider the residues in the points α, β. There are several variants of their location.

Case 1. Let us assume that both roots of the denominator $\alpha \neq \beta$ are real and modulo less than 1, see Fig. 1.

The residue at zero, for integral (4) is equal to the summand $s_0 = C + \frac{A}{-\alpha} + \frac{B}{-\beta}$ for $k = 0$.

The residues at the points α, β are equal to $A\alpha^{-1-k}$ and $B\beta{-1-k}$, correspondingly.

Therefore, the sum of these three residues is described by the formula:

$$C_k = \mathbf{res}_\Sigma\left[R(z)z^{-(k+1)}\right] = \begin{cases} 0 & \Leftarrow \quad k > 0 \\ C & \Leftarrow \quad k = 0 \\ A\alpha^{-k-1} + B\beta^{-k-1} & \Leftarrow \quad k < 0 \end{cases} \tag{6}$$

For all $k \in \mathbb{Z}$, the sum of the residues (6) differs in sign from the residue of the function $R(z)z^{-(k+1)}$ at infinity. The non-negativity of the Fourier coefficients is guaranteed if all five constants: A, B, C, α, β are non-negative. But this condition is not necessary. For example, for $A = 10$, $B = -1$, $C = 1$, $\alpha = 2$, $\beta = -1$ all Fourier coefficients C_k, according to (6), are non-negative.

Case 2. Let us consider now the case $0 < |\alpha|< 1 < |\beta|$, see Fig. 2. Since the residue of the function $R(z)z^{-(k+1)}$ at the point $z = \alpha$ is equal to $A\alpha^{-(k+1)}$ for all k, the sum

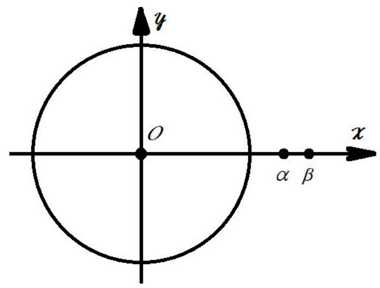

Figure 3: Case 3. Both roots of the denominator are large: $|\alpha|,\ |\beta|> 1$.

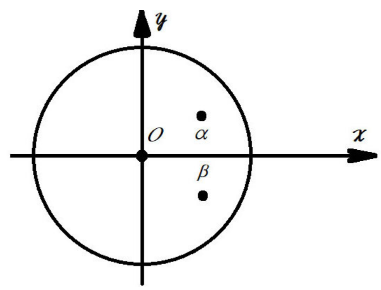

Figure 4: Case 4. The roots of the denominator are conjugate: $\alpha = \bar\beta$ and $|\alpha|= |\beta|< 1$.

of the residues of the function at poles 0 and α is equal to

$$C_k = \mathbf{res}_\Sigma \left[R(z)z^{-(k+1)}\right] = \begin{cases} -A\alpha^{-k-1} & \Leftarrow \quad k>0 \\ C - A\alpha^{-1} & \Leftarrow \quad k=0 \\ B\beta^{-k-1} & \Leftarrow \quad k<0 \end{cases} \tag{7}$$

The values are non-negative for all k iff $A,\ \alpha,\ \beta \geq 0 \geq B$ and $C - B\beta^{-1} \geq 0$.

Case 3. $1 < |\alpha|,\ |\beta|$. It is insignificant in this case: are the roots real or not real. In this case, inside the unit circle, there is a unique pole $z = 0$ of the function $R(z)z^{-k-1}$. The formula (5) is applicable.

Case 4. If the points $\alpha,\ \beta$ are complex, then it follows from the condition of the realness of the fraction in (4) that $\alpha = \bar\beta,\ A = \bar B$. Here $|\alpha|= |\beta|< 1$. Formula (6) may be used here, and we obtain $A\alpha^{-k-1} + B\beta^{-k-1} = 2\Re\left(A\alpha^{-k-1}\right)$. Since $\alpha \notin \mathbb{R}$, these values cannot take only non-negative values for all $k < 0$. Therefore, the scheme is not monotonic.

Thus, a complete classification of the cases (the case of two multiple roots $\alpha = \beta < 1$ is not difficult, too) is obtained when the linear implicit difference scheme on the stencil turns out to be monotonic.

3. Examples

Example 1. The Crank – Nicolson scheme for the diffusion with a linear source (the telegraph equation, the Klein – Gordon equation):

$$\frac{u_j^{n+1}-u_j^n}{\tau} = D\frac{\left(u_{j-1}^{n+1}-2u_j^{n+1}+u_{j+1}^{n+1}\right)+\left(u_{j-1}^{n}-2u_j^{n}+u_{j+1}^{n}\right)}{2h^2} + a\frac{u_j^{n+1}+u_j^n}{2}$$

can be rewritten as:

$$-(\nu/2)u_{j+1}^{n+1}+(1+\nu-\vartheta)u_j^{n+1}-(\nu/2)u_{j-1}^{n+1}-(\nu/2)u_{j+1}^{n}(-1+\nu-\vartheta)u_j^{n}-(\nu/2)u_{j-1}^{n}=0,$$

where $\nu = D\tau/h^2 > 0,\ \vartheta = a\tau/2$, or

$$u_{j+1}^{n+1}+b_1u_j^{n+1}+u_{j-1}^{n+1} = u_{j+1}^{n}+b_0u_j^{n}+u_{j-1}^{n},$$

where $b_1 = 2(\vartheta-1-\nu)/\nu$, $b_0 = 2(\vartheta+1-\nu)/\nu$.

Discriminant $\mathcal{D}$ of the polynomial $\lambda^2+b_1\lambda+1$ is equal to

$$b_1^2-4 = 4\nu^{-2}\left[(\vartheta-1-\nu)^2-\nu^2\right] = 4\nu^{-2}(\vartheta-1)(\vartheta-1-2\nu).$$

If the discriminant $\mathcal{D}$ is negative, then case 4 takes place and the scheme Crank – Nicolson is not monotonic.

It is necessary and sufficient for the discriminant positivity that these brackets are the same sign. It is true if

i) $\vartheta \le 1$, and therefore $b_1 < 0$, or if ii) $(\vartheta-1) \ge 2\nu$, and then $b_1 > 0$.

For the variant i) we obtain

$$\beta = \left(-b_1+\sqrt{b_1^2-4}\right)/2 > 1 > \alpha = \left(-b_1-\sqrt{b_1^2-4}\right)/2 > 0.$$

Therefore, this finite-difference scheme is included into case 2, and, therefore, we will determine the coefficients $A,\ B$.

Since

$$R(z) = \frac{1+b_0z+z^2}{1+b_1z+z^2} = 1+\frac{(b_0-b_1)z}{1+b_1z+z^2} = C+\frac{A}{-\alpha}\frac{1}{1-z/\alpha}+\frac{B}{-\beta}\frac{1}{1-z/\beta},$$

the following relations for the coefficients are fulfilled:

$$A\beta+B\alpha=0,\ A+B=b_0-b_1=4/\nu \Rightarrow A(\alpha-\beta)=4\alpha/\nu \Rightarrow$$

$$\Rightarrow A = 4\alpha/[\nu(\alpha-\beta)] < 0,\ B = 4\beta/[\nu(\beta-\alpha)] > 0.$$

The monotonicity condition for $k=0$: $C-A/\alpha = 1+4/[\nu(\beta-\alpha)] > 0$ is fulfilled, too.

As about variant ii), when ($\vartheta \ge 1+2\nu$), and, therefore,

$$\beta = \left(-b_1-\sqrt{b_1^2-4}\right)/2 < -1 < \alpha = \left(-b_1+\sqrt{b_1^2-4}\right)/2 < 0,$$

the scheme is not monotonic.

Thus, the Crank — Nicolson scheme for the diffusion equation is monotonic iff $a\tau \le 2$. In particular, the inequality is fulfilled, if $a \le 0$. In the case $a > 0$ we obtain the constraint for the temporal step τ.

Example 2. Explicit Euler scheme for the diffusion equation:

$$\frac{u_j^{n+1} - u_j^n}{\tau} = D\frac{u_{j-1}^n - 2u_j^n + u_{j+1}^n}{h^2} + au_j^n$$

is monotonic, iff the inequality $1 - 2\nu + \vartheta \ge 0$ is fulfilled.

Example 3. Implicit Euler scheme for the above equation:

$$\frac{u_j^{n+1} - u_j^n}{\tau} = D\frac{u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}}{h^2} + au_j^{n+1}$$

may be rewritten as

$$-\nu u_{j-1}^{n+1} + (1 + 2\nu - \vartheta)u_j^{n+1} - \nu u_{j+1}^{n+1} = u_j^n.$$

Therefore,

$$R(z) = \frac{-1/\nu}{z^2 + b_1 z + 1}, \quad \text{where } b_1 = \frac{\vartheta - 1 - 2\nu}{\nu}.$$

If $b_1^2 < 4$, i.e.

$$(\vartheta - 1 - 2\nu)^2 < 4\nu^2 \Leftrightarrow (\vartheta - 1)(\vartheta - 1 - 4\nu) < 0 \Leftrightarrow \vartheta \in (1,\, 1 + 4\nu),$$

then the discriminant for the denominator of $R(z)$ is negative, the roots of the denominator are complex and conjugate, and the scheme is not monotonic.

Next, we investigate the following variants, when the discriminant is non-negative

$$i)\; \vartheta - 1 \le 0 \Rightarrow \vartheta - 1 < 4.$$

Here $b_1 < 0$, and therefore,

$$\beta = \left(-b_1 + \sqrt{b_1^2 - 4}\right)/2 > 1 > \alpha = \left(-b_1 - \sqrt{b_1^2 - 4}\right)/2 > 0.$$

Let us determine the coefficients $A,\ B$ in the representation

$$R(z) = \frac{-1/\nu}{z^2 + b_1 z + 1} = \frac{A}{z - A} + \frac{B}{z - B} \Rightarrow A + B = 0,\ A\beta + B + \alpha = 1/\nu.$$

Therefore, $B(\alpha - \beta) = 1/\nu \Rightarrow B < 0$, and the scheme is not monotonic.

$$ii)\; \vartheta - 1 \ge 4\nu \Rightarrow \vartheta - 1 > 0.$$

Here $b_1 > 0$, and therefore, the roots are negative:

$$\beta = \left(-b_1 - \sqrt{b_1^2 - 4}\right)/2 < -1 < \alpha < \left(-b_1 + \sqrt{b_1^2 - 4}\right)/2 < 0.$$

The Fourier coefficients C_k have different signs, and the scheme is not monotonic.

Thus, the implicit Euler scheme for the equation is not monotonic for any parameters.

Example 4. Compact implicit scheme for the diffusion equation (the scheme is absolutely stable) [3]:

$$a_0 = c_0 = 2(6\nu - 1),\ a_1 = c_1 = 2(6\nu + 1),\ b_0 = -4(6\nu + 5),\ b_1 = -4(6\nu - 5).$$

We divide the coefficients on $a_1 = c_1$ to exchange the gauge:

$$a_0 = c_0 = \frac{6\nu - 1}{6\nu + 1},\ a_1 = c_1 = 1,\ b_0 = \frac{-2(6\nu + 5)}{6\nu + 1},\ b_1 = \frac{-2(6\nu - 5)}{6\nu + 1}.$$

The roots of the denominator in $R(z)$ are real, iff $b_1^2 \geq 4$, i.e.

$$(6\nu - 5)^2 \geq (6\nu + 1)^2 \Leftrightarrow \nu \leq 1/3.$$

The function $b_1(\nu)$ on the segment $[0,\ 1/3]$ is positive, and therefore, $b_1 \geq 2$. The discriminant of the denominator is non-negative, and therefore its roots are real. Let us represent the function:

$$R(z) = a_0 + \frac{Pz}{z^2 + b_1 z + 1} = a_0 + \frac{A}{z - \alpha} + \frac{B}{z - \beta},$$

where $P = b_0 - a_0 b_1 = \frac{-144\nu}{(6\nu+1)^2} < 0$. According to the Viet theorem $\alpha + \beta = -b_1 < 0,\ \alpha\beta = 1 > 0$, and therefore,

$$0 > \alpha > -1 > \beta.$$

The scheme must be included into the Case 2, and it is not monotonic.

Example 5. Let us consider the Crank – Nicolson scheme,

$$\frac{u_j^{n+1} - u_j^n}{\tau} + \frac{V}{4h}\left(u_{j+1}^{n+1} - u_{j-1}^{n+1} + u_{j+1}^n - u_{j-1}^n\right) = 0,$$

approximating the linear transfer equation:

$$\partial_t u + V \partial_x u = 0.$$

Let's rewrite the finite-difference equation:

$$u_{j+1}^{n+1} + b_1 u_j^{n+1} - u_{j-1}^{n+1} = u_{j+1}^n + b_0 u_j^n - u_{j-1}^n,$$

where $b_0 = b_1 = \nu^{-1}$. Here

$$R(z) = \frac{1+\nu^{-1}-z^2}{-1+\nu^{-1}+z^2} = -1 + \frac{2\nu^{-1}}{-1+\nu^{-1}+z^2},$$

and

$$\beta = \left(-\nu^{-1} - \sqrt{\nu^{-2}+4}\right)/2 < -1 < 0 < \alpha = \left(-\nu^{-1} + \sqrt{\nu^{-2}+4}\right)/2 < 1.$$

The root α is negative and according to (7) the scheme is not monotonic.

Example 6. Let us consider the compact scheme:

$$\left(u_{j-1}^{n+1} + 4u_j^{n+1} + u_{j+1}^{n+1}\right) - \left(u_{j-1}^{n} + 4u_j^{n} + u_{j+1}^{n}\right) + \mu\left(u_{j+1}^{n} + u_{j+1}^{n+1} - u_{j-1}^{n} - u_{j-1}^{n+1}\right) = 0,$$

where $\mu = \frac{3\tau V}{2h}$, that approximating the linear transfer equation.

Remark. Symbol of the resolving operator (see (3)) for the scheme

$$\sigma(\omega) = \frac{2\cos(\omega) + 4 - 3i\nu\sin(\omega)}{2\cos(\omega) + 4 + 3i\nu\sin(\omega)},$$

where $\nu = V\tau/h$ is dimension-less Courant parameter, is very closed to the symbol for the resolving operator of the differential equation, i.e. to $\exp(-i\nu\omega)$ near $\omega = 0$. Really, both modulo are equal to 1 (and therefore, the scheme is absolutely stable), and the arguments satisfy to the relation

$$\arg\sigma_{scheme} - \arg\sigma_{etalon} = \mathbf{O}\left(\omega^5\right)$$

as $\omega \to 0$.

The scheme may be rewritten as the following:

$$(1+\mu)u_{j-1}^{n+1} + 4u_j^{n+1} + (1-\mu)u_{j+1}^{n+1} = (1-\mu)u_{j-1}^{n} + 4u_j^{n} + (1+\mu)u_{j+1}^{n},$$

and therefore,

$$R(z) = \frac{(1-\mu) + 4z + (1+\mu)z^2}{(1+\mu) + 4z + (1-\mu)z^2}.$$

When $|\mu| < 1$ both roots of the denominator are negative:

$$\beta < -1 < \alpha < 0,$$

and the scheme is not monotonic.

When $\mu > 1$ we obtain

$$\beta > 1 > -0,5 > \alpha > -1,$$

and, according to (7), the scheme is not monotonic.

The similar answer we obtain for $\mu < -1$, where

$$1 > \alpha > 0 > -1 > \beta.$$

Thus, in any case, according to formula (7), the scheme is not monotonic.

4. One-side implicit schemes

Let us consider scheme (5) at $c_0 = c_1 = 0$. Here, the normalization $c_1 = 1$ is impossible. Such a scheme is obtained, in particular, as an "up-wind" scheme for a first-order equations, [1], [2], [4], [6], [7]. We obtain here the non-degeneracy condition for the implicit finite difference scheme: $|a_1|< |b_1|$. The residue at zero for integral (4), for

$$R(z) = \frac{a_0 + b_0 z}{a_1 + b_1 z} = C + \frac{A}{z - \alpha},$$

where

$$C = \frac{b_0}{b_1},\ A = \frac{a_0}{b_0} - \frac{a_1}{b_1},\ \alpha = -\frac{a_1}{b_1} \Rightarrow \frac{A}{-\alpha} = \frac{a_0 b_1}{b_0 a_1} - 1,$$

is equal to

$$\mathbf{res}_0 \left[R(z) z^{-k-1}\right] = \begin{cases} -A\alpha^{-k-1} & \Leftarrow \quad k > 0 \\ C - A\alpha^{-1} & \Leftarrow \quad k = 0 \\ 0 & \Leftarrow \quad k < 0 \end{cases}. \tag{8}$$

According to non-degeneracy condition, $|\alpha|< 1$, and we add the residue in the point $z = \alpha$. Therefore

$$C_k = res_\Sigma \left[R(z) z^{-(k+1)}\right] = \begin{cases} -A\alpha^{-k-1} & \Leftarrow \quad k > 0 \\ C - A\alpha^{-1} & \Leftarrow \quad k = 0 \\ 0 & \Leftarrow \quad k < 0 \end{cases} \tag{9}$$

The inequalities $-A\alpha^{k-1} \geq 0$ for all positive k is equivalent to the inequality $A \leq 0 \leq \alpha$.

Example 1. The linear transfer equation may be approximated by the following onside (up-wind) implicit scheme:

$$\frac{u_j^{n+1} - u_j^n}{\tau} + V \frac{u_j^{n+1} - u_{j-1}^{n+1} + u_j^n - u_{j-1}^n}{2h} = 0$$

or

$$-\nu u_{j-1}^{n+1} + u_j^{n+1}(2 + \nu) = \nu u_{j-1}^n + u_j^{n+1}(2 - \nu),$$

i.e.

$$a_0 = \nu,\ b_0 = 2 - \nu,\ a_1 = -\nu,\ b_1 = 2 + \nu \Rightarrow C = \frac{2 - \nu}{2 + \nu},\ \alpha = \frac{\nu}{2 + \nu} > 0,\ A = \frac{2 + \nu}{\nu - 2},$$

and the inequality $A \leq 0$ is fulfilled iff $|\nu|\leq 2$. Then $C > 0$, too. The scheme is monotonic, iff $|\nu|\leq 2$.

Example 2. Let us consider the compact up-wind scheme for the same differential equation:

$$\frac{u_j^{n+1}+u_{j-1}^{n+1}-u_j^n-u_{j-1}^n}{2\tau}+V\frac{u_j^{n+1}-u_{j-1}^{n+1}+u_j^n-u_{j-1}^n}{2h}=0,$$

approximating the linear transfer equation on the left side. It can be transformed into the dimension-less form:

$$(1-\nu)u_{j-1}^{n+1}+(1+\nu)u_j^{n+1}=(1+\nu)u_{j-1}^n+(1-\nu)u_j^n,$$

where $\nu = V\tau/h$ is the Courant parameter of the difference scheme. Thus,

$$a_0=b_1=1+\nu\ b_0=a_1=1-\nu.$$

Since

$$R(z)=\frac{\kappa+z}{1+\kappa z}=1+\frac{\kappa-\kappa^{-1}}{1+\kappa z},$$

we obtain that the monotonicity condition is fulfilled, iff $\kappa \geq 1$, i.e. $1 \geq \nu \geq 0$. It is usual condition for up-wind finite-difference schemes.

5. Non-linear schemes

On the set of explicit difference schemes (1), algebraic addition operation:

$$u_j^{n+1}=H_j\left(\vec{u}^n\right)+G_j\left(\vec{u}^n\right),$$

as well as superposition operation:

$$v_j^{n+1}=H_j\left(\vec{u}^n\right),$$

$$u_j^{n+1}=G_j\left(\vec{v}^{n+1}\right),$$

may be defined.

If both schemes are monotonic, then their sum and superposition are monotonic, too.

Implicit finite-difference non-linear scheme

$$F\left(u_{j-1}^n,\,u_j^n,\,u_{j+1}^n,\,u_{j-1}^{n+1},\,u_j^{n+1},\,u_{j+1}^{n+1}\right)=0, \tag{10}$$

approximating nonlinear partial differential equations on a grid of N knots, on each time step leads to the need to solve a system of N nonlinear equations with N variables. Usually, there are many solutions of such systems.

For the implementation of the finite-difference scheme is useful at each time step to use the following method: first, calculate the first guess with a suitable explicit difference scheme:

$$v_j^{n+1}=G_j\left(\vec{u}^n\right), \tag{11}$$

and then apply linearization. The desired solution is represented as $\vec{u}^{n+1} = \vec{v}^{n+1} +$

$\vec{\epsilon}^{n+1}$, where $\epsilon_j \ll 1,\ j = 1, \ldots, N$. Substituting in (10), we obtain the system of linear algebraic equations for small deviations ϵ_j. The matrix of the system is three-diagonal. It is composed of the first derivatives of the function F with respect to the last three variables at the points u^n_{j-1}, u^n_j, u^n_{j+1}, v^{n+1}_{j-1}, v^{n+1}_j, v^{n+1}_{j+1}. If both the schemes: explicit one (11) and the described linear implicit scheme are monotonic, then the resulting nonlinear implicit scheme $\vec{u}^n \mapsto \vec{u}^{n+1}$ is also monotonic.

If a system of conservation laws is approximated, then, as numerical experiments show, monotony takes place until the solution approaches a gradient catastrophe.

6. Conclusion

Necessary and sufficient conditions for the monotonicity of linear implicit finite-difference schemes on 2×2 and 2×3 stencils are formulated. The key role in this criterion is played by the location of the roots of the denominator of the fraction - i.e of the symbol of the finite-difference operator. This approach can be applied to schemes with wider stencils. This criterion is not sophisticated and can be used in conjunction with other conditions, for example, with conditions of a high order of approximation that are used in compact finite-difference schemes.

We are going to consider the corresponding compact implicit schemes for quasi-linear problems in a separate publication.

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