More on discrete convexity

Lecture series

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The presentation follows **Gurvich, V. and Naumova, M., 2023. More on discrete convexity.** ***arXiv preprint arXiv:2306.10948***.

Vladimir Gurvich

Mariya Naumova

The similarity between convex functions and submodular discrete functions is actively studied since 1970s:

- L. Lovász,
  Submodular functions and convexity,
  in: A. Bachem, M. Grötschel, B. Korte (Eds.),
  Math. Programming: The State of the Art,
  Bonn 1982, Springer, Berlin (1983) 235–257

- J. Edmonds,
  Submodular functions, matroids, and certain polyhedra, in Combinatorial structures and their Applications,
  Gordon and Breach, New York (1970) 68–87

L. Lovász

J. Edmonds

- S. Fujishige, Submodular functions and optimization, Elsevier, 2005
- K. Murota, Discrete Convex Analysis, Math. Programming, 2003
- E. Boros, K. Elbassioni, V. Gurvich, and L. Khachiyan, An inequality for polymatroid functions and its applications, Discrete Applied Math., 131 (2) (2003) 255–281
- E. Boros, V. Gurvich, and K. Makino, Minimal and locally minimal games and game forms; Discrete Math. 309:13 (2009) 4456–4468

S. Fujishige

K. Murota

E. Boros

L. Khachiyan

K. Elbassioni

K. Makino

V. Danilov

G. Koshevoy

- V. I. Danilov, G. A. Koshevoy, and C. Lang, Gross substitution, discrete convexity, and submodularity. Discret. Appl. Math. 131:2 (2003) 283–298
- S. Fujishige, G. A. Koshevoy, and Y. Sano, Matroids on convex geometries (cg-matroids). Discret. Math. 307:15 (2007) 1936–1950
- G. A. Koshevoy, Discrete convexity and its applications, Combinatorial Optimization - Methods and Applications (2011) 135–163
- K. Murota, A survey of fundamental operations on discrete convex functions of various kinds, Optim. Methods Softw. 36:2-3 (2021) 472–518
- K. Murota, On basic operations related to network induction of discrete convex functions, Optim. Methods Softw. 36:2-3 (2021) 519–559

In all above works matroids played an important role, along with super- and submodularity. The present work is another step in this direction. Here we suggest a simpler approach.

It is well known that each local minimum of a convex function is always its global minimum.

We study some discrete objects that share this property. We provide several examples related to graphs and two-person games in normal form.

Posets

Given a finite partially ordered set (poset) $(\mathcal{P}, \succ)$ and $P, P' \in \mathcal{P}$,

P' is a *successor* of P; notation $P \succ P'$;

P' is an *immediate successor* of P if $P \succ P'$ and $P \succ P'' \succ P'$ for no $P'' \in \mathcal{P}$.

Each poset is defined by its immediate successors: Hasse diagram.

Every successor is realized by a chain of immediate successors.
This chain may be not unique.

Notation $P \succeq P'$ means that either $P \succ P'$ or $P = P'$.

Consider an arbitrary subset (family) $\mathcal{F} \subseteq \mathcal{P}$.

Minima and local minima

Recall that F is a *(local) minimum* of $\mathcal{F}$ if and only if $F \in \mathcal{F}$ but $F' \notin \mathcal{F}$ whenever F' is an (immediate) successor of F.

Since we have space, let us partition this statement into two:

F is a *minimum* of $\mathcal{F}$ if $F \in \mathcal{F}$ but $F' \notin \mathcal{F}$ for any successor F' of F.

F is a *local minimum* of $\mathcal{F}$ if $F \in \mathcal{F}$ but $F' \notin \mathcal{F}$ for any immediate successor F' of F.

We denote by $\mathcal{M} = \mathcal{M}(\mathcal{F}, \mathcal{P}, \succ)$ and by $\mathcal{LM} = \mathcal{LM}(\mathcal{F}, \mathcal{P}, \succ)$, respectively, the set (class) of all minima and all local minima of $\mathcal{F}$ in $(\mathcal{P}, \succ)$.

Obviously, the above definitions imply that $\mathcal{M} \subseteq \mathcal{LM}$.

Hereditary and Convex Discrete Families

A family $\mathcal{F} \subseteq \mathcal{P}$ is called:

- *convex* if $\mathcal{M}(\mathcal{F}) = \mathcal{LM}(\mathcal{F})$;
- *strongly convex* if $\mathcal{F}$ is convex and for any $F \in \mathcal{F}$ and $F' \in \mathcal{M}(\mathcal{F})$ such that $F \succ F'$ there exists an immediate successor P of F such that $P \in \mathcal{F}$ and $F \succ P \succeq F'$.
- *hereditary* if $P \in \mathcal{F}$ whenever P is a successor of some $F \in \mathcal{F}$.
- *weakly hereditary* if $P \in \mathcal{F}$ whenever $F \in \mathcal{F},\ F' \in \mathcal{M}(\mathcal{F})$, and $F \succ P \succeq F'$.

In accordance with the above definitions, the following implications hold:

$$\text{hereditary} \Rightarrow \text{weakly hereditary} \Rightarrow \text{strongly convex} \Rightarrow \text{convex},$$

while all inverse implications fail, as we will show in this paper. Note that the last two concepts become equivalent if family $\mathcal{F}$ has a unique minimum and the first two are equivalent whenever $\mathcal{M}(\mathcal{F}) \supseteq \mathcal{M}(\mathcal{P})$, or more precisely, $\mathcal{M}(\mathcal{F}, \mathcal{P}, \succ) \supseteq \mathcal{M}(\mathcal{P}, \mathcal{P}, \succ)$.

The last concept could be "slightly" modified as follows:

- A family $\mathcal{F}$ is called *very weakly hereditary* if $P \in \mathcal{F}$ whenever $F \in \mathcal{F}$ and $F \succ P \succeq F'$ for some $F' \in \mathcal{M}(\mathcal{F})$.

Then to the above chain of implications we can add the following one:

$$\text{hereditary} \Rightarrow \text{weakly hereditary} \Rightarrow \text{very weakly hereditary} \Rightarrow \text{convex}.$$

Yet, very weakly hereditary $\not\Rightarrow$ strongly convex

We will consider only 3 types of posets: related to

(I) subgraphs (a) induced and (b) on a given vertex set, and

(II) submatrices.

PART I: Graphs and digraphs

Definitions and preliminaries

Given a finite (directed) graph G, we denote by $V(G)$ and $E(G)$ the sets of its vertices and (directed) edges, respectively. Multiple edges are allowed but loops are forbidden.

A (directed) graph G is called: *null-graph* if $V(G) = \emptyset$ and *edge-free* if $E(G) = \emptyset$. The null-graph is unique and edge-free, but not vice versa.

We will consider two partial orders: related to vertices $\succ_V$ and to (directed) edges $\succ_E$. In the first case, $G \succ G'$ if G' is an induced subgraph of G, that is, $V(G') \subseteq V(G)$ and $E(G')$ consists of all (directed) edges of $E(G)$ whose both ends are in $V(G')$.

In the second case, $G \succ G'$ if G' is a subgraph of G defined on the same vertex-set, that is, $V(G') = V(G)$ and $E(G') \subseteq E(G)$.

Given a graph G, which may be directed or not, and a set of its (induced) subgraphs $G_1, \dots, G_n$, define a family $(\mathcal{F}(G), \succ_E)$ (respectively, $(\mathcal{F}(G), \succ_V)$) that consists of all subgraphs G' of G containing as a (induced) subgraph at least one of $G_i, i = 1, \dots, n$.

Lemma

Both families are weakly hereditary. Furthermore, $(\mathcal{F}(G), \succ_V)$ (respectively, $(\mathcal{F}(G), \succ_E)$) is hereditary if and only if $n = 1$ and G_1 is the null-graph (respectively, the edge-free graph).

Proof

Consider a subgraph $G' \in (\mathcal{F}(G), \succ_V)$, (respectively, $G' \in (\mathcal{F}(G), \succ_E)$) that contains (as an induced subgraph) G_i for some $i \in [n] = \{1, \dots, n\}$. Obviously, the above property is kept when we delete a vertex from $V(G') \setminus V(G_i)$ (respectively, an edge $e \in E(G') \setminus E(G_i)$) if any. Obviously, such a vertex (respectively, an edge) exists unless $G = G_i$. Thus, in both cases family $\mathcal{F}(G)$ is weakly hereditary. Obviously, it is hereditary if and only if G_i cannot be reduced. □

Connected graphs

A graph G is called *connected* if for every two distinct vertices $v, v' \in V(G)$ it contains a path connecting v and v'. In particular, the null-graph and the one-vertex graphs are connected, since they do not have two distinct vertices.

$\diamondsuit$ *Order* $\succ_V$

In this case $\mathcal{F} = \mathcal{F}(G)$ is the family of all connected induced subgraphs of a given graph G. Obviously, it is strongly convex. In other words, every connected graph G has a vertex $v \in V = V(G)$ such that $G[V \setminus \{v\}]$ is connected. Indeed, v can be any leaf of a spanning tree of G.

Recall that G' is a *spanning tree* of G if $V(G') = V(G)$, $E(G') \subseteq E(G)$, and G' is a *tree*, that is, connected and has no cycles.

Furthermore, $\mathcal{F}(G)$ is hereditary if and only if G is complete. Otherwise, $\mathcal{F}(G)$ is not even weakly hereditary.

Example

Consider 2-path $G = (v_1, v_2), (v_2, v_3)$. It is connected, but by deleting v_2 we obtain a not connected subgraph induced by $\{v_1, v_3\}$. Thus, $\mathcal{F}(G)$ is not very weakly, hereditary.

Let us modify our convention and assume that the null-graph is not connected. Then, $G' \in \mathcal{M}(G) = \mathcal{LM}(G)$ if and only if $V(G')$ is a single vertex. In this case 2-path $(v_1, v_2), (v_2, v_3)$ is very weakly (but not weakly) hereditary. Indeed, the target vertex may be v_2 but not v_1 or v_3.

To obtain a not even weakly hereditary family $\mathcal{F}(G'')$, consider the 3-path $G'' = (v_1, v_2), (v_2, v_3), (v_3, v_4)$. Then, each target vertex can be obtained by a vertex-eliminating sequence that does not respect connectivity.

♢ *Order* $\succ_E$

Given a connected graph G, family $\mathcal{F} = \mathcal{F}(G)$ consists of all connected subgraphs G' of G with $V(G') = V(G)$.

By definition, all spanning trees of G are in $\mathcal{F}$ and, obviously, they form class $\mathcal{M} = \mathcal{LM}$. By Lemma, family $\mathcal{F}$ is weakly hereditary.

Remark

Let G be a connected graph with weighted edges: $w : E(G) \to \mathbb{R}$. It is well known[1] *that one can obtain a spanning tree of G of maximal total weight by the greedy algorithm, as follows. Delete an edge $e \in E(G)$ such that (i) e belongs to a cycle of G, or in other words, the reduced graph is still connected on $V(G)$, and (ii) e has a minimal weight among all edges satisfying (i). Proceed until such edges exist.*

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[1] ***See***

O. Borůvka, O jistém problému minimálním (About a certain minimal problem), Práce Mor. Přírodověd. Spol. V Brně III (in Czech and German) 3 (1926) 37–58

O. Borůvka, Příspěvek k řešení otázky ekonomické stavby elektrovodních sítí (Contribution to the solution of a problem of economical construction of electrical networks), Elektronický Obzor (in Czech) 15 (1926) 153–154

J. B. Kruskal, On the shortest spanning subtree of a graph and the traveling salesman problem, Proceedings of the American Math. Soc. 7:1 (1956) 48–50

Disconnected graphs

$\diamondsuit$ ***Order*** $\succ_V$

In this case $\mathcal{F} = \mathcal{F}(G)$ is the family of all disconnected induced subgraphs of a given graph G. By convention, the null-graph and one-vertex graphs are connected Hence, class $\mathcal{M}(\mathcal{F})$ consists of all subgraphs of G induced by pairs of non-adjacent vertices. In particular, $\mathcal{F} = \emptyset$ if and only if there is no such pair, that is, graph G is complete.

Proposition

For every graph G, family $\mathcal{F}(G)$ is strongly convex.

Proof

Consider a not connected induced subgraph G' of G and any pair of non-adjacent vertices $v', v'' \in V(G')$. Then, either $V(G') = \{v', v''\}$, in which case $G' \in \mathcal{M}(\mathcal{F}(G))$ is minimal, or we will show that there exists a vertex $v \in V(G') \setminus \{v', v''\}$ such that subgraph G'' induced by $V(G') \setminus \{v\}$ is still not connected. In other words, $\mathcal{F}(G)$ is strongly convex. Assume that G' is not connected and choose two vertices w' and w'' from its distinct connected components. Note that two pairs $\{v', v''\}$ and $\{w', w''\}$ may intersect.

Then, delete $v \in V(G') \setminus \{v', v'', w', w''\}$, if any. The obtained induced subgraph still contains v' and v''. Furthermore, it is not connected, because it still contains w' and w''.

It remains to consider the case when $V(G') = \{v', v''\} \cup \{w', w''\}$. Then, it is not difficult to verify that, after deleting w' or w'', the obtained induced subgraph is not connected . $\square$

Proposition

For every graph G, family $\mathcal{F}(G)$ is very weakly hereditary.

Proof

Consider a not connected induced subgraph G' of G and choose any two vertices $v', v'' \in V(G')$ from distinct connected components of G'. Then, obviously, every induced subgraph G'' of G containing both v' and v'' is in $\mathcal{F}$, that is, not connected. Thus, $\mathcal{F}(G)$ is very weakly hereditary. □

However, family $\mathcal{F}(G)$ is not weakly hereditary for some G.

Example

Consider graph G that consists of a 2-path $(v_1, v_2), (v_2, v_3)$ and an isolated vertex v_0. This graph is disconnected, that is, $G \in \mathcal{F}(G)$, but, by deleting v_0, we obtain a connected graph $G' \notin \mathcal{F}(G)$. Yet, $v_1, v_3 \in V(G')$ and, hence, graph G'' induced by these vertices is in $\mathcal{F}(G)$, moreover, $G'' \in \mathcal{M}(\mathcal{F}(G))$. Thus, $\mathcal{F}(G)$ is not weakly hereditary.

Note that strong convexity holds for $\mathcal{F}(G)$, because one can delete v_2 rather than v_0.

♢ *Order* $\succ_E$

Given a graph G, family $\mathcal{F} = \mathcal{F}(G)$ consists of all disconnected graphs G' such that $V(G') = V(G)$ and $E(G') \subseteq E(G)$. Then, obviously, family $\mathcal{F}(G)$ has a unique minimum unless $|V(G)| = 1$:

$\mathcal{M}(\mathcal{F})$ consists of a unique graph, which is the edge-free graph on $V(G)$. Obviously, deleting edges and keeping the vertex-set respects the non-connectivity. Thus, family $\mathcal{F}$ is hereditary.

Strongly connected directed graphs

A directed graph (digraph) G is called *strongly connected* (SC) if for every two (distinct) vertices of $v, v' \in V(G)$ there is a directed path in G from v to v'.

♢ *Order* $\succ_V$

In this case $\mathcal{F} = \mathcal{F}(G)$ is the family of all SC induced subgraphs of a given digraph G. This family is not convex.

Example

Consider a digraph G that consists of two directed cycles of length at least 3 with a unique common vertex. Clearly, G is a locally minimal SC digraph, $G \in \mathcal{LM}(\mathcal{F}(G))$.

Indeed, G is SC but we destroy this property by deleting any vertex of G. Furthermore, $G \notin \mathcal{M}$, since each of two cycles of G is in $\mathcal{M}$. Thus, $G \in \mathcal{LM} \setminus \mathcal{M}$.

$\diamondsuit$ *Order* $\succ_E$

In this case $\mathcal{F} = \mathcal{F}(G)$ is the family of all SC subgraphs G' of a given digraph G such that $V(G') = V(G)$ and $E(G') \subseteq E(G)$. Note that $\mathcal{F}(G) = \emptyset$ if and only if G is not SC. Obviously, strong connectivity is monotone non-decreasing on 2^E. In other words, for any two subgraphs G' and G'' of G such that $V(G') = V(G'') = V(G)$ and $E(G') \subseteq E(G'') \subseteq E(G)$, we have: G'' is SC on $V(G)$ whenever G' is. By Lemma, family $\mathcal{F}(G)$ is weakly hereditary but not hereditary.

Not strongly connected directed graphs

$\diamondsuit$ ***Order*** $\succ_V$

In this case $\mathcal{F} = \mathcal{F}(G)$ is the family of all not SC induced subgraphs of a given digraph G. It is easily seen that $\mathcal{M}(\mathcal{F})$ consists of all pairs of vertices $v, v' \in V(G)$ such that at least one of two arcs (v, v') or (v', v) is missing in G. There are no such pair in G if and only if $\mathcal{F}(G) = \emptyset$.

An induced subgraph G' of G is not SC, that is, $G' \in \mathcal{F}$, if and only if there exist two (distinct) vertices $v, v' \in V(G')$ such that in G' there is no directed path from v to v'. Furthermore, $G'' \in \mathcal{M}(\mathcal{F})$ if and only if $G'' = G[v, v']$ is induced by distinct two $v, v' \in V(G)$ such that either $(v, v') \notin E(G)$, or $(v', v) \notin E(G)$, or both.

Hence, we can reduce G' to $G[v, v']$ deleting its vertices, except v and v', in any order.

Thus, considered family $\mathcal{F}$ is very weakly hereditary and, hence, convex.

Yet, obviously, it is not hereditary, since an induced subgraph of a not SC digraph can be SC. The simplest examples are two isolated vertices or one arc.

Moreover, $\mathcal{F}$ is not even weakly hereditary. Consider, for example, a directed 3-cycle and one isolated (or pending) vertex v. This digraph is not SC, but after deleting v, we obtain a SC digraph. Meanwhile any 2 vertices of the 3-cycle induce a not SC digraph.

Proposition

Family $\mathcal{F}$ is strongly convex.

Proof

Consider a not SC digraph G' and any its induced subgraph $G'' \in \mathcal{M}(\mathcal{F})$.

As we know, $G'' = G[\{v, v'\}]$ for some $v, v' \in V(G')$ such that either (v, v'), or (v', v), or both are not in $E(G')$. Furthermore, there are $w, w' \in V(G')$ such that G' contains no directed path from w to w'. Note that v and v' as well as w and w' are distinct, while sets $\{v, v'\}$ and $\{w, w'\}$ may intersect. Let us delete vertices of G', except v, v', w, w', one by one in any order getting $G''' = G[\{v, v', w, w'\}]$ at the end. All obtained digraphs remain not CS, since they contain w and w'.

Finally, let us delete w and w' in any order getting $G'' = G[\{v, v'\}] \in \mathcal{M}(\mathcal{F})$. It is easily seen that the previous digraph, $G[\{v, v', w\}]$ or $G[\{v, v', w'\}]$, if any, is nor SC either. $\square$

$\diamondsuit$ *Order* $\succ_E$

In this case $\mathcal{F} = \mathcal{F}(G)$ is the family of all not SC subgraphs G' of a given digraph G such that $V(G') = V(G)$ and $E(G') \subseteq E(G)$. Obviously, all these subgraphs are not SC whenever G is not SC. Thus, family $\mathcal{F}$ is hereditary.

Ternary graphs

A graph is called *ternary* if it contains no induced cycle of length multiple of 3.

Main conjecture: chromatic number of a ternary graph equals 3.

By definition, family $\mathcal{T}$ of ternary graphs is hereditary in order $\succ_V$. In contrast, in order $\succ_E$ this family is not even convex, as the following example shows.

It's a shame since Daniel Král derived the main conjecture assuming this.
Thus, the main conjecture remains open.

Although it was proven[2] that χ is bounded by a constant.

Yet, this constant is much larger than 3.

[2] M.Chudnovsky, A. Scott, P. Seymour, and S. Spirkl, Proof of the Kalai-Meshulam conjecture, Israel Journal of Math. 238 (2020) 639–661.

M. Chudnovsky

A. Scott

P. Seymour

S. Spirkl

Kalai-Meshulam conjecture: $|s| \leq 1$ for a ternary graph, where $s = \sum(-1)^{|S|}$ over all stable sets S of a graph.

G. Kalai

R. Meshulam

Example

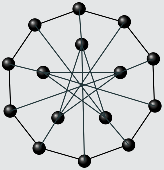
We know[2] that "D. Král asked (unpublished): Is it true that in every ternary graph (with an edge) there is an edge e such that the graph obtained by deleting e is also ternary? This would have implied that all ternary graphs are 3-colourable, but has very recently been disproved; a counterexample was found by M. Wrochna.

D. Král

M. Wrochna

(Take the disjoint union of a 5-cycle and a 10-cycle, and join each vertex of the 5-cycle to two opposite vertices of the 10-cycle, in order.)"

In other words, consider the standard model of the Petersen graph, with two 5-cycles. Then, subdivide every edge of the "outer" cycle by a vertex and connect it with the "opposite" vertex of the "inner" 5-cycle.



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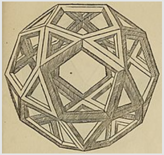
[2] **M. Chudnovsky, A. Scott, P. Seymour, and S. Spirkl, Proof of the Kalai-Meshulam conjecture, Israel Journal of Math. 238 (2020) 639–661.**

Remark

Consider the skeleton graph of the cube. Obviously, an induced 6-cycle appears whenever we delete an edge. However, this graph itself contains two induced 6-cycles.
Also, an induced 6-cycle appears whenever we delete an edge of icosidodecahedron - a polyhedron with twenty triangular faces and twelve pentagonal faces, which has 30 identical vertices, with two triangles and two pentagons meeting at each, and 60 identical edges, each separating a triangle from a pentagon (see Fig.) Yet, this graph itself contains triangles and induced 9-cycles.

Leonardo da Vinci, 1452 – 1519



Icosidodecahedron. Illustration for Luca Pacioli's "Divina proportione" by Leonardo da Vinci

Non-ternary graphs

By definition, non-ternary graph contains an induced cycle of length multiple to 3 (a ternary cycle, for short). Given a graph G, denote by $\mathcal{C}_3(G)$ (respectively, by $\mathcal{IC}_3(G)$) the set of its (induced) ternary cycles and by $\mathcal{F}(G)$ the family of its non-ternary subgraphs. From Lemma we will derive that, with respect to (wrt) both orders $\succ_V$ and $\succ_E$, family $\mathcal{F}(G)$ is weakly hereditary but not hereditary.

$\diamondsuit$ ***Order*** $\succ_V$

In this case, $\mathcal{M}(\mathcal{F}(G)) = \mathcal{IC}_3(G)$. Given an induced subgraph G' of G that contains a ternary cycle $C \in \mathcal{IC}_3(G)$, one can delete a vertex $v \in V(G') \setminus V(C)$ such that the reduced graph G'' still contains C as an induced subgraph unless $G' = C$.

This exactly means that family $\mathcal{F}(G)$ is weakly hereditary.

Obviously, it is not hereditary, since deleting a vertex might destroy all ternary cycles of G'.

$\diamondsuit$ ***Order*** $\succ_E$

In this case, $\mathcal{M}(\mathcal{F}(G))$ is in a one-to-one correspondence with $\mathcal{C}_3(G)$. Recall that $V(G') = V(G)$ for each subgraph $G' \in \mathcal{F}(G)$. Hence, G' consists of a cycle $C \in \mathcal{C}_3(G)$ and several isolated vertices from $V(G) \setminus V(C)$.

Given a non-ternary subgraph G'' of G such that $V(G'') = V(G)$ and G'' contains a (not necessarily induced) ternary cycle $C \in \mathcal{C}_3(G)$, one can delete an edge $e \in V(G'') \setminus V(C)$ such that the reduced graph still contains C unless $G'' = C$. By the main lemma, we have

Family $\mathcal{F}(G)$ is weakly hereditary.

Obviously, it is not hereditary, since deleting an edge might destroy all cycles in G'' of length multiple of 3.

Perfect and imperfect graphs

Definitions and preliminaries

Given a graph G, as usual, $\chi = \chi(G)$ and $\omega = \omega(G)$ denote its chromatic and clique numbers, respectively. Recall that χ is the minimum number of colors in a proper vertex-coloring of G and ω is the number of vertices in a maximum clique of G. Obviously, $\chi(G) \geq \omega(G)$ for every graph G.

Graph G is called *perfect* if $\chi(G') = \omega(G')$ for every induced subgraph G' of G, including G itself.

If we require $\chi(G) = \omega(G)$ only for G then it would be perfect whenever this number is large enough, which is not interesting. Thus, it was wise to require this equality not only for G but for all its induced subgraphs as well.

Thus, by definition, in order $\succ_V$ the family of perfect graphs is hereditary.

This concept was introduced in 1961 by Claude Berge[3] who made the following two conjectures:

C. Berge

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[3] C. Berge, Sur une conjecture relative au problème des codes optimaux, Comm. 13-ème Assemblée Générale de l'URSI, Tokyo, 1961.

For more details see also:

C. Berge, Perfect graphs, in: D. Fulkerson (Ed.), Studies in Graph Theory, Part I, in: M.A.A. Studies in Math., vol. 11, Math. Assoc. Amer., Washington (1975) 1–22.

Which graphs are minimal imperfect?

Odd holes = odd chordless cycles $G = C_{2k+1}$, with $k > 1$.

If $k = 1$ then G is a triangle and $\omega(G) = \chi(G) = 3$.

If $k > 1$ then G is an odd hole and $2 = \omega(G) < \chi(G) = 3$.

Also their complements = odd anti-holes $\overline{G} = \overline{C}_{2k+1}$, with $k \geq 1$.

If $k = 1$ then $\overline{G}$ is a null-graph and $\omega\overline{G} = \chi\overline{G} = 1$.

If $k > 1$ then $\overline{G}$ is an odd antihole and $k = \omega(\overline{G}) < \chi(\overline{G}) = k + 1$.

Perfect Graph Theorem:

G is perfect if (and only if) the complementary graph $\overline{G}$ is perfect.

It was proven in 1972 by Laslo Lovász[4] and since then is called the *Perfect Graph Theorem (PGT)*.

[4] L. Lovász, Normal hypergraphs and the perfect graph conjecture, Discrete Math. 2 (1972) 253–267.

L. Lovász, A characterization of perfect graphs, J. Combinatorial Theory B13 (1972) 95–98.

Strong Perfect Graph Theorem:

Graph G is perfect if and only if it contains no induced odd holes and anti-holes, In other words, odd holes and odd anti-holes are minimal imperfect graphs in order $\succ_V$. This conjecture was proven by M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas in 2002 and published in 2006[5].

Since then this statement is called the *Strong Perfect Graph Theorem (SPGT)*.

A polynomial recognition algorithm for perfect graphs was obtained by M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour, and K. Vušković in 2002 and published in 2005[6].

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[5] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, The strong perfect graph theorem, Ann. Math. 164 (2006) 51–229.

[6] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour, and K. Vušković, Recognizing Berge graphs, Combin. archive 25:2 (2005).

Perfect graphs

♢ ***Order*** $\succ_V$

This family $\mathcal{F}$ is hereditary and $\mathcal{M}(\mathcal{F})$ contains only the null-graph.

♢ ***Order*** $\succ_E$

An edge of a perfect graph G is called *critical* if deletion of it results in an imperfect graph. For example, six edges $(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_1)$, and (v_1, v_3) form a perfect graph in which (v_1, v_3) is a unique critical edge. This concept was introduced by Annegret Wagler[7].

With Stefan Hougary, she proved that a perfect graph has no critical edges if and only if it is *Meyniel*, that is, every odd cycle of length 5 or more (if any) has at least two chords (see Theorem 3.1[7]).

[7] A. Wagler, Critical and anticritical edges in Perfect Graphs, 27th International Workshop on Graph-Theoretic Concepts in Computer Science, LNCS 2204 (2001) 317–327.

There are perfect graphs in which all edges are critical. Some examples were given by the author et al. in 2009[8] and called *Rotterdam* graphs.

Clearly, these graphs are in $\mathcal{LM}(\mathcal{F}) \setminus \mathcal{M}(\mathcal{F})$ and, hence, the considered family, of perfect graphs in order $\succ_E$, is not convex.

Furthermore, E. Boros et V. Gurvich[9] claim that every edge of the complement of a Rotterdam graph is critical too. In other words, a Rotterdam graph becomes imperfect whenever we delete an edge from it or add an edge to it.

Let us note finally that no efficient characterization of the non-critical-edge-free perfect graphs is known, in contrast to the critical-edge-free ones, which are Meyniel. The main result of M. Chudnovsky et al.[10] provides a polynomial recognition algorithm for the former family.

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[8] E. Boros and V. Gurvich, Vertex- and edge-minimal and locally minimal graphs, Discrete Math. 309:12 (2009) 3853–3865. (see Figures 2 and 3)

[9] See Theorem 4 in above.

[10] M. Chudnovsky, G. Cornuéjols, X. Liu, P. Seymour, and K. Vušković, Recognizing Berge graphs, Combin. archive 25:2 (2005).

Imperfect graphs

♢ *Order* $\succ_V$

In this case, by the SPGT, $\mathcal{M}(\mathcal{F}) = \mathcal{LM}(\mathcal{F})$ and this set contains only odd holes and odd anti-holes. Again, by Lemma, $\mathcal{F}$ is weakly hereditary but not hereditary.

♢ *Order* $\succ_E$

In 1972 Elefterie Olaru characterized minimal graphs of this family. He proved that it is convex and $G \in \mathcal{M} = \mathcal{LM}$ if and only if G is an odd hole plus k isolated vertices for some $k \geq 0$.[11] Thus, by Lemma, family $\mathcal{F}$ is weakly hereditary but not hereditary.

Note that the odd anti-holes, except C_5, are not in $\mathcal{LM}$, since each one has an edge whose elimination would result in a graph with an induced odd hole.

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[11] See

E. Olaru, Beitrage zur Theorie der perfekten Graphen, Elektronische Informationsverarbeitung und Kybernetik (EIK) 8 (1972) 147–172.

E. Olaru and H. Sachs, Contributions to a characterization of the structure of perfect graphs, in: C. Berge, V. Chvátal (Eds.), Topics on Perfect Graphs, Ann. Discrete Math. 21 (1984) 121–144.

E. Boros and V. Gurvich, Vertex- and edge-minimal and locally minimal graphs, Discrete Math. 309:12 (2009) 3853–3865.

Graphs with $\chi = \omega$

It is easily seen that $\mathcal{M} = \mathcal{LM}$ for both orders $\succ_V$ and $\succ_E$; in other words, both families are convex.

Indeed, for $\succ_V$ (respectively, for $\succ_E$) sets $\mathcal{M}$ and $\mathcal{LM}$ are equal and contain only the null-graph (respectively, the edge-free graphs) see Propositions 2 and 3 in [**BG09**].

Yet, obviously, deleting a vertex or an edge may fail the equality $\chi = \omega$. Thus, both considered families are not hereditary.

Graphs with $\chi > \omega$

$\diamondsuit$ ***Order*** $\succ_V$

By SPGT, every graph with $\chi > \omega$ contains an odd hole or odd anti-hole as an induced subgraph; in other words, class $\mathcal{M}$ contains only the odd holes and odd anti-holes.

Class $\mathcal{LM}$ is wider; it consists of all so-called *partitionable* graphs defined as follows: Graph G is partitionable if $\chi(G) > \omega(G)$ but $\chi(G') = \omega(G')$ for each induced subgraph G' of G such that $V(G') = V(G) \setminus \{v\}$ for a vertex $v \in V(G)$. Such definition is one of many equivalent characterizations of partitionable graphs; this follows easily from the pioneering results of [**BHT79**, **Pad74**] and it is explicit in [**BGH02**].

Thus, the considered family is not convex.

Remark

*The above characterization of $\mathcal{M}$ is based on SPGT, which is very difficult, while the case of $\mathcal{LM}$ is simple. In contrast, partitionable graphs are much more sophisticated than the odd holes and anti-holes. Although very many equivalent characterizations of partitionable graphs are known (see, for example, [***BHT79**, **BBGMP98**, **CGPW84**, **GT94**, **Pad74***]) yet, their structure is complicated and not well understood. For example, the fact that each partitionable graph contains an induced odd hole or anti-hole is equivalent with the SPGT.*

*The following two questions about partitionable graphs are still open. In addition to the odd holes and odd anti-holes, there is one more partitionable graph G_{17} that has 17 vertices and has no: (i) small transversals and (ii) uncertain edges. It is open whether (i) or (ii) may hold for other partitionable graphs. The conjecture that (i) cannot, if true, would significantly strengthen SPGT; see [***BBGMP98**, **BGH02**, **CGPW84**, **GT94***] for the definitions and more details.*

$\diamondsuit$ ***Order*** $\succ_E$

By Lemma, the corresponding family $\mathcal{F}$ is weakly hereditary: class $\mathcal{M} = \mathcal{LM}$ consists of odd holes with k isolated vertices, for some $k \geq 0$. This follows from Olaru's Theorem [**Ola72**]; see also [**OS84**] and [**BG09**]. Family $\mathcal{F}$ is not hereditary, since obviously, inequality $\chi > \omega$ may turn into equality after deleting an edge.

Kernels in digraphs

Definitions and preliminaries

Given a finite digraph G, a vertex-set $K = K(G) \subseteq V(G)$ is called a *kernel* of G if it is (i) independent and (ii) dominating, that is,

(i) $v, v' \in K(G)$ for no directed edge $(v, v') \in E(G)$ and

(ii) for every $v \in V(G) \setminus K(G)$ there is a directed edge (v, v') from v to some $v' \in K(G)$.

This definition was introduced in 1901 by Charles Bouton [**Bou1901**] for a special digraph (of the popular game of NIM) and then in 1944 it was extended by John Von Neumann and Oskar Morgenstern for arbitrary digraphs in [**NM44**].

J. von Neumann

O. Morgenstern

It is not difficult to verify that an even directed cycle has two kernels, while an odd one has none. This obvious observation was generalized in 1953 by Richardson [**Ric53**] as follows: A digraph has a kernel whenever all its directed cycles are even. The original proof was simplified in [**Isb57**, **HNC65**, **NL71**, **GSNL84**, **BD90**, **DW12**].

Remark

*It is not difficult to verify that a digraph has at most one kernel whenever all its directed cycles are odd [*BG06*]. This claim combined with the Richardson Theorem imply that an acyclic digraph has a unique kernel. The latter statement is important for game theory, allowing to solve finite acyclic graphical zero-sum two-person games. Of course, it has a much simpler direct proof [*NM44*].*

Already in 1973 Vásek Chvátal proved that it is NP-complete to recognize whether a digraph has a kernel[12].

V. Chvátal

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[12] V. Chvátal, On the computational complexity of finding a kernel, Report No. CRM-300, Centre de Recherches Mathématiques, Université de Montréal, 1973.

Kernell-less digraphs

$\diamondsuit$ *Order* $\succ_E$

In this case, given a digraph G, family $\mathcal{F}(G)$ contains only the digraphs G' such that $V(G') = V(G), E(G') \subseteq E(G)$, and G' has no kernel. By Richardson's Theorem, $G' \in \mathcal{M}(\mathcal{F}(G))$ if and only if G' is a directed odd cycle in G (plus the set of isolated vertices $v \in V(G) \setminus V(G')$).

In 1980 Pierre Duchet [**Duc80**] conjectured that every kernel-less digraph G' has an edge $e \in E(G')$ such that the reduced digraph $G'' = G' - e$ (that is, $E(G'') = E(G') \setminus \{e\}$) is still kernel-less unless G'' is an odd cycle plus k isolated vertices for some $k \geq 0$; in other words, family $\mathcal{F}$ of kernel-less digraphs is convex, $\mathcal{M}(\mathcal{F}) = \mathcal{LM}(\mathcal{F})$. This statement, if true, would significantly strengthen Richardson's theorem. Yet, it was shown in [**AFG98**] that a circulant with 43 vertices is a counter-example, a locally minimal but not minimal kernel-free digraph.

Let us recall that a circulant $G = G_n(\ell_1, \ldots, \ell_q)$ is defined as a digraph with n vertices, $V(G) = [n] = \{1, \ldots, n\}$ and nq arcs,
$E(G) = \{(i, i+j) \mid i \in [n], j \in [q] = \{1, \ldots, q\}\}$, where standardly all sums are taken $\mod n$.

Example

([AFG98])

It was shown that a circulant $G_n(1, 7, 8)$ has a kernel if and only if $n \equiv 0 \mod 3$ or $n \equiv 0 \mod 29$. Hence, $G_{43}(1, 7, 8)$ is kernel-less. Yet, a kernel appears whenever an arc of this circulant is deleted. Due to circular symmetry, it is sufficient to consider only three cases and delete one of the arcs (43, 1), (43, 7), or (43, 8). It is not difficult to verify that, respectively, the following three subsets become kernels:

$K_1 = \{1, 5, 10, 14, 16, 19, 25, 28, 30, 34, 39, 43\}$,

$K_7 = \{7, 9, 11, 13, 22, 24, 26, 28, 37, 39, 41, 43\}$,

$K_8 = \{3, 5, 8, 14, 17, 19, 23, 28, 32, 34, 37, 43\} \subseteq \{1, \ldots, 43\} = V$.

Thus, the set of edge-minimal kernel-free digraphs is a proper subset of the locally edge-minimal ones. Although, only one digraph from the difference is known, it seems that the latter class, unlike the former one, is difficult to characterize. For example, it is not known whether a circulant $G_n(\ell_1, \ell_2)$ can be a locally edge-minimal kernel-less digraph, but it is known that it cannot if $n \leq 1,000,000$ [**AFG98**].

$\diamondsuit$ *Order* $\succ_V$

In this case also family $\mathcal{F}$ of the kernel-less digraphs is not convex. Although it seems difficult to characterize (or recognize in polynomial time) both classes $\mathcal{M}(\mathcal{F})$ and $\mathcal{LM}(\mathcal{F})$ of the (locally) vertex-minimal kernel-less digraphs, yet, some digraphs from $\mathcal{LM} \setminus \mathcal{M}$ can be easily constructed; see for example, [**DW12**, **GS82**, **GSNL84**, **GSG07**]. For completeness we provide one more example.

Example

Circulant $G = G_{\mathbf{16}}(1, 7, 8)$ is kernel-less, since 16 is not a multiple of 3 or 29. Yet, a kernel appears whenever we delete a vertex from G. Due to circular symmetry, without loss of generality (wlog) we can delete "the last" vertex, 16, and verify that vertex-set $\{1, 3, 5, 7\}$ becomes a kernel. Hence, $G \in \mathcal{LM}$, but $G \notin \mathcal{M}$, since G contains a directed triangle, $1 + 7 + 8 = 16$, which is kernel-less.

Digraphs with kernels

We will show that in each order $\succ_V$ or $\succ_E$ the corresponding family $\mathcal{F}$ is strongly convex but not weakly hereditary.

Given an arbitrary digraph G, obviously, class $\mathcal{M}(\mathcal{F})$ contains a unique digraph in both cases: the null-graph for $\succ_V$ and the edge-free graph with vertex-set $V(G)$ for $\succ_E$. Thus, by Lemma **??**, both families are not weakly hereditary.

Proposition

Both families $(\mathcal{F}, \succ_V)$ and $(\mathcal{F}, \succ_E)$ are strongly convex.

Proof

Fix a digraph G ith a kernel $K \subseteq V(G)$.

In order $\succ_V$ delete all vertices of $V(G) \setminus K$, if any, one by one. By definition, K remains a kernel in every reduced digraph. This reduction results in an independent set, which is a kernel itself. Now we can delete all vertices one by one getting the null-graph at the end.

In order $\succ_V$, first, we delete all arcs within $V(G) \setminus K$, then all arcs from $V(G) \setminus K$ to K (if any, one by one, in both cases). By definition, K remains a kernel in every reduced digraph. This reduction results in the edge-free digraph on the initial vertex-set $V(G)$. □

On kernel-solvable graphs

A graph is called *kernel-solvable* if every its clique-acyclic orientation has a kernel; see [**BG96**, **BG98**, **BG06**] for the precise definitions and more details. In 1983 Claude Berge and Pierre Duchet conjectured that a graph is kernel-solvable if and only if it is perfect. The "only if part" follows easily from the SPGT (which remained a conjecture till 2002). The "if part" was proven in [**BG96**]; see also [**BG98**, **BG06**]. This proof is independent of SPGT. The family of perfect graphs is hereditary, by definition.

As we know, the family of kernel-less digraphs is not convex wrt order $\succ_E$, in contrast with the family of not kernel-solvable graphs, which is convex [**BG98**].

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