More on discrete convexity

Lecture series

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Complete edge-chromatic graphs

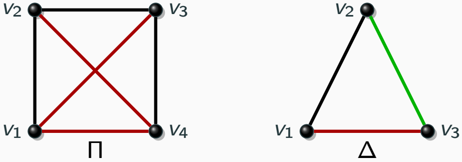
Definitions and preliminaries

A d-graph $\mathcal{G} = (V; E_1, \dots, E_d)$ is a complete graph whose edges are colored by d colors $I = [d] = \{1, \dots, d\}$, or in other words, are partitioned into d subsets some of which might be empty. These subsets are called *chromatic components*. For example, we call G a 2- or 3-graph if G has only 2, respectively, 3, non-empty chromatic components. According to this definition order $\succ_E$ makes no sense for d-graphs, so we will restrict ourselves by $\succ_V$.

The following 2-graph Π and 3-graph Δ will play an important role:

$\Pi = (V; E_1, E_2)$, where $V = \{v_1, v_2, v_3, v_4\}$, $E_1 = \{(v_1, v_2), (v_2, v_3), (v_3, v_4)\}$, and $E_2 = \{(v_2, v_4), (v_4, v_1), (v_1, v_3)\}$;

$\Delta = (V; E_1, E_2, E_3)$, where $V = \{v_1, v_2, v_3\}$, $E_1 = \{(v_1, v_2)\}, E_2 = \{(v_2, v_3)\}$, and $E_3 = \{(v_3, v_1)\}$.



2-graph Π and 3-graph Δ

Note that both chromatic components of Π are isomorphic to P_4 and that Δ is a 3-colored triangle.

Let us also remark that, formally, d-graphs $\Pi(d)$ (respectively, $\Delta(d)$) defined for every integer $d \geq 2$ (respectively, $d \geq 3$), while $d-2$ (respectively, $d-3$) of their chromatic components are empty. We will omit argument d assuming that it is a fixed parameter.

Both d-graphs Π and Δ were first considered, for different reasons, in 1967 by Tibor Gallai in his fundamental paper [**Gal67**]. Δ-free d-graphs are sometimes called *Gallai's* graphs.

T. Gallai

The Π- and Δ-free d-graphs have important applications to d-person positional games and to read-once Boolean functions in case $d = 2$ [**BG06**, **BG09**, **BGM09**, **BGM14**, **GG11**, **Gur77**, **Gur78**, **Gur82**, **Gur84**, **Gur09**, **Gur11**].

See the reference list at the end of the file.

Substitution for graphs and d-graphs

Given a graph G', a vertex $v \in V(G)$, and graph G'' such that $V(G') \cap V(G'') = \emptyset$, substitute G'' for v in G' connecting a vertex $v'' \in V(G'')$ with a vertex $v' \in V(G')$ if and only if v and v' were adjacent in G'. Denote the obtained graph by $G = G'(v \to G'')$ and call it the *substitution* of G'' for v in G' or simply the *substitution* when arguments are clear from the context.

Substitution $\mathcal{G} = \mathcal{G}'(v \to \mathcal{G}'')$ for d-graphs is defined in a similar way. We will see that many important classes of graphs and d-graphs are closed wrt substitution. This will be instrumental in our proofs.

Complementary connected d-graphs

We say that a d-graph $\mathcal{G}$ is *complementary connected (CC)* if the complement of each chromatic component of $\mathcal{G}$ is connected on V, in other words, if for each two vertices $u, w \in V$ and color $i \in [d] = \{1, \ldots, d\}$ there is a path between u and w without edges of E_i.

By convention, the null-d-graph and one-vertex d-graph are not CC. It is easily seen that there is no CC d-graph with two vertices and that Δ (respectively, Π) is a unique CC d-graph with three (respectively, four) vertices.

It is also easily seen that Π and Δ are minimal CC d-graphs, that is, they do not contain induced CC subgraphs. Moreover, there are no others.

Theorem

*([**Gur78**, **Gur84**]). Every CC d-graph contains* Π *or* Δ*.* $\square$

Remark

In case of Π*, that is, for* $d = 2$*, the result was obtained earlier* [**CLB81**, **Sei74**, **Sum71**, **Sum73**].
It was one of the problems on the 1970 Moscow Mathematics Olympiad, which was successfully solved by seven high-school students [**GT86**].

As far as I know, D.P. Sumner was the first who discovered[13] *that* Π *is a unique minimal CC d-graph.*

[13] *D.P. Sumner, Indecomposable graphs, Ph.D. Thesis, Univ. of Massachusetts, Amherst, 1971.*

D. Sumner

CASE $d = 2$. ***Sketch of the proof***

For $d = 2$, two edge sets are complementary. Hence, CC means just bi-connected, that is, both edges and non-edges are connected on V.

Assume for contradiction that a CC 2-graph has no Π.

Lemma

For any (v_1, v_2) of color 1 there is a v_3 such that (v_1, v_3) and (v_2, v_3) are of color 2.

Proof

Consider a shortest path of color 2 from v_1 to v_2. It exists, since color 2 is connected on V.

If the length is 2 then the statement holds.

If the length is larger then Π exists. $\square$

Then, proceed as follows.

By the lemma, there exists a v_4 such that (v_3, v_4), (v_2, v_4) are of color 1. Then (v_1, v_4) is color 1, too, since otherwise these 4 vertices induce a Π.

Then, there exists a v_5 such that, etc.

We conclude that there are only two Π-free CC 2-graphs and both are infinite. $\square$

This theorem can be strengthened as follows:

Theorem

*([**BG09**]). Every CC d-graph $\mathcal{G}$, except Π and Δ, contains a vertex v such that the reduced d-graph $\mathcal{G}[V \setminus \{v\}]$ is still CC.* $\square$

This statement was announced in [**Gur78**, **Gur84**] and proven in [**BG09**], see also [**Gur09**]. It implies that, by deleting vertices one by one, we can reduce every CC d-graph to a copy of Π or Δ, keeping CC.

In other words, family $\mathcal{F} = \mathcal{F}^{CC}$ of CC d-graphs is convex and the class $\mathcal{M} = \mathcal{LM}$ contains only 2-graph(s) Π and 3-graph(s) Δ.

Let us show that $\mathcal{F}_d = \mathcal{F}_d^{CC}$ is not strongly convex, already for $d = 2$.

Since chromatic components of a d-graph may be empty, this also shows that F_d^{CC} is not strongly convex, for any d.

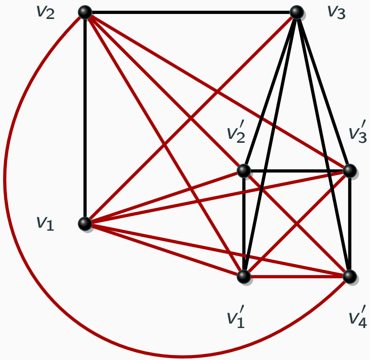
It is both known and easily seen [**Gur09**] that the family of CC graphs, as well as CC d-graphs, is closed wrt substitution.

Moreover, $\mathcal{G} = \mathcal{G}'(v \to \mathcal{G}'')$ is CC if and only if $\mathcal{G}'$ is CC.

Example

Consider 2-graph Π on the vertex-set $\{v_1, v_2, v_3, v_4\}$ and substitute for v_4 another 2-graph Π' on the vertex-set $\{v_1', v_2', v_3', v_4'\}$. Obtained 2-graph $\mathcal{G} = \Pi(v_4 \to \Pi')$ is CC, since Π and Π' are CC and family $\mathcal{F}_d$ is closed wrt substitution. Furthermore, it is easy to verify that the CC property disappears if we delete v_1, v_2 or v_3 from $\mathcal{G}$. Thus, we cannot reduce $\mathcal{G}$ to Π' keeping CC, which means that family $\mathcal{F}_2^{CC}$ of the CC 2-graphs is not strongly convex.

However, we can reduce $\mathcal{G}$ to Π keeping CC, in agreement with convexity of $\mathcal{F}_2^{CC}$.



Not CC d-graphs

Let us denote this family by $\mathcal{F}_d = \mathcal{F}_d^{not-CC}$ and show that it is strongly convex but not weakly hereditary if $d > 1$. Of course, $\mathcal{F}_1$ is hereditary.

A two-vertex d-graph, that is, a single edge, is CC only if $d = 1$.

Proposition

Assume that $d > 1$ and $|V| \geq 2$. Each not CC d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$ contains a vertex $v \in V$ such that the sub-d-graph $\mathcal{G}[V \setminus v]$ is still not CC.

Proof

Since $\mathcal{G}$ is not CC, there is a color $i \in [d]$ such that graph $\overline{G_i} = (V, \overline{E_i})$ is not connected. As we know, one can eliminate vertices of V one by one keeping this property until V is reduced to two vertices. However, the obtained not CC d-graph is still not minimal, since by convention, the null-d-graph and a one-vertex d-graph are not CC either. Thus, the null-d-graph is the only (locally) minimal CC d-graph. So, by the last two steps, we reduce the obtained two-vertex not CC d-graph to a one-vertex d-graph and then to the null-d-graph. Both are not CC. Thus, family $\mathcal{F}_d$ is strongly convex for all d. $\square$

Yet, the family of not CC d-graphs is not weakly hereditary whenever $d > 1$.

Example

Let us add a vertex v_0 to Π or Δ and connect it to all other vertices by edges of the same color. Clearly, the obtain d-graph is not CC if $d > 1$. Yet, deleting vertex v_0 from it we obtain Π or Δ, which are both CC. Thus, family $\mathcal{F}_d$ is not hereditary.

Moreover, it is not weakly hereditary, by Lemma, unless $d = 1$.

Digraph containing Δ are not CIS. This is Δ-conjecture; see below.

CIS property of d-graphs

Given a d-graph $\mathcal{G} = (V; E_1, \ldots, E_d)$, choose a maximal independent set $S_i \subseteq V$ in every graph $G_i = (V, E_i)$ and denote by $\mathcal{S} = \{S_i \mid i \in [d] = \{1, \ldots, d\}\}$ the obtained collection; furthermore set $S = \bigcap_{i=1}^{d} S_i$. Obviously, $|S| \leq 1$ for every $\mathcal{S}$. Indeed, if $v, v' \in S$ then $(v, v') \notin E_i$ for all $i \in [d]$, that is, this edge has no color.

We say that $\mathcal{G}$ has the CIS property and call $\mathcal{G}$ a *CIS d-graph* if $S \neq \emptyset$ for every selection $\mathcal{S}$. CIS d-graphs were introduced in 2006 in [**ABG06**]; see also [**BG09**, **BGM14**, **BGZ09**, **DLZ04**, **DLZ04a**, **Gur84**, **Gur09**, **Gur11**, **WZZ09**, **Zan95**].

For $d = 2$, a 2-graph consists of two complementary graphs G_1 and G_2 on the same vertex-set V. In this case CIS property means that in G_i every maximal clique C intersects every maximal stable set S for $i = 1, 2$. (This explains the name CIS.) Obviously, C and S may have at most one vertex in common.

Not CIS d-graphs

It is easy to verify that d-graphs Π and Δ are not CIS, while every their sub-d-graph is CIS, in other words, Π and Δ are minimal not CIS d-graphs. Moreover, they are also locally minimal and there are no other.

Theorem

*([**ABG10**]). Every not CIS d-graph $\mathcal{G} = (V; E_1 \dots, E_d)$, except Π and Δ, has a vertex $v \in V$ such that the reduced d-graph $\mathcal{G}[V \setminus \{v\}]$ is not CIS.* $\square$

In other words, for any $d \geq 2$, family $\mathcal{F}_d = \mathcal{F}_d^{not-CIS}$ of not CIS d-graphs is convex and class $\mathcal{M}(\mathcal{F}_d) = \mathcal{LM}(\mathcal{F}_d)$ consists of only of Π if $d = 2$ and of Π and Δ if $d > 2$.

Interestingly, family $\mathcal{F}_d^{CC}$ of CC d-graphs with $d > 1$ has the same properties: it is convex and class $\mathcal{M}(\mathcal{F}_d^{CC}) = \mathcal{LM}(\mathcal{F}_d^{CC})$ contains only Π and Δ.

However, these two families differ, $F_d = \mathcal{F}_d^{CC} \neq \mathcal{F}_d^{not-CIS} = F'_d$.

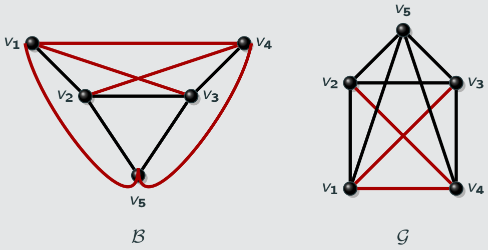
Moreover, both differences $\mathcal{F}_d \setminus \mathcal{F}'_d$ and $\mathcal{F}'_d \setminus \mathcal{F}_d$ are not empty, already for $d = 2$.

Example

Consider the bull-graph (also called A-graph) is self-complementary; hence, the corresponding bull 2-graph $\mathcal{B}$ given on Figure **??** is both CC and CIS. Thus, $\mathcal{B} \in \mathcal{F}_\mathbf{2} \setminus \mathcal{F}'_\mathbf{2}$.

Consider 2-graph Π colored by colors 1 and 2; add to it a new vertex v_5 and connect it to four vertices of Π by four edges of the same color, say 1. It is easily seen that the obtained 2-graph $\mathcal{G}$ is not CC and not CIS. Thus, $\mathcal{G} \in \mathcal{F}'_\mathbf{2} \setminus \mathcal{F}_\mathbf{2}$.

The above two examples also show that both set-differences are not empty for every $d \geq 2$, since chromatic components may be empty.



2-graphs $\mathcal{B}$ and $\mathcal{G}$

For each $d \geq 2$ both families $\mathcal{F}_d$ of CC d-graphs and $\mathcal{F}'_d$ of not CIS d-graphs are convex, and $\mathcal{F}_d$ is not strongly convex, already for $d = 2$. It remains only to prove that $\mathcal{F}'_2$ not strongly convex. It was shown in [**ABG10**] that CIS d-graphs are closed wrt substitution.

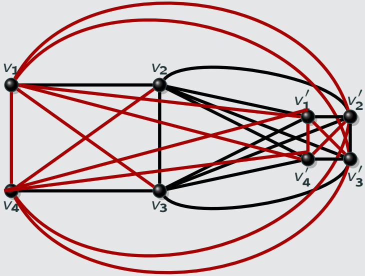
Example

Consider the bull 2-graph $\mathcal{B}$ defined by the edges

$(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_2, v_5), (v_3, v_5)$ of color 1,

$(v_2, v_4), (v_4, v_1), (v_1, v_3), (v_1, v_5), (v_4, v_5)$ of color 2.

Note that vertices $\{v_1, v_2, v_3, v_4\}$ induce a Π. As we already mentioned, $\mathcal{B}$ is a CIS 2-graph, while Π is not. Let us substitute v_5 in $\mathcal{B}$ by 2-graph Π' defined by the edges:
$(v'_1, v'_2), (v'_2, v'_3), (v'_3, v'_4)$ of color 1,
$(v'_2, v'_4), (v'_4, v'_1), (v'_1, v'_3)$ of color 2.



2-graph $\mathcal{B}'$

The resulting 2-graph $\mathcal{B}' = \mathcal{B}(v_5 \to \Pi)$ is not CIS. Indeed, it is easily seen that two disjoint vertex-sets $C = \{v_2, v_3, v_2', v_3'\}$ and $S = \{v_1, v_4, v_1', v_4'\}$ form in $\mathcal{B}'$ maximal cliques of colors 1 and 2, respectively.

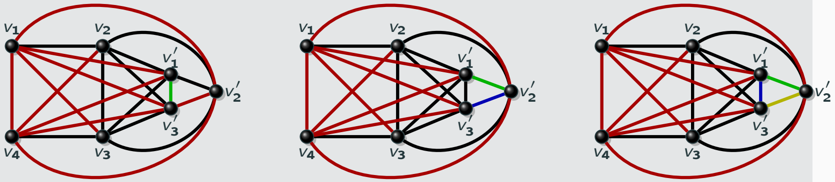
In contrast, we obtain a CIS 2-graph, by substituting v_5 in $\mathcal{B}$ by a proper sub-2-graph $\mathcal{G}$ of Π'. Indeed Π' is minimal not CIS, hence, $\mathcal{G}$ is CIS, $\mathcal{B}$ is CIS too, and CIS d-graphs are closed wrt substitution.

Summarizing, we conclude that 2-graph $\mathcal{B}'$ is not CIS, but one obtains a CIS sub-2-graph by deleting any vertex $v \in \{v_1', v_2', v_3', v_4'\}$ from $\mathcal{B}'$. Hence, if we want to stay in $\mathcal{F}'(\mathcal{B}')$, we can only delete a vertex from $V(\Pi) = \{v_1, v_2, v_3, v_4\}$, keeping Π' but destroying Π. Thus, $\mathcal{B}'$ cannot be reduced to Π within $\mathcal{F}'(\mathcal{B}')$, which means that family $\mathcal{F}_2'$ is not strongly convex.

However, in agreement with convexity of family $\mathcal{F}_2'(\mathcal{B}')$, one can reduce $\mathcal{B}'$ to Π' staying within this family.

Similarly, we can substitute v_5 in $\mathcal{B}$ by Δ, on vertices v_1', v_2', v_3' edge-colored arbitrarily. In particular, we can use colors 1, or 2, or any other. Again, it is not difficult to verify that the resulting d-graph graph $\mathcal{B}'' = \mathcal{B}(v_5 \to \Delta)$ is not CIS. (No CIS d-graph with Δ is known, see Section 36.) Yet, deleting a vertex of Δ from $\mathcal{B}''$ we obtain a CIS sub-d-graph of $\mathcal{B}''$. Indeed, Δ is minimal not CIS, hence, any its sub-d-graph is CIS, bull 2-graph $\mathcal{B}$ is CIS too, and CIS d-graphs are closed wrt substitution. Hence, if we want to stay in $\mathcal{F}_3(\mathcal{B}'')$ then $\mathcal{B}''$ can be reduced to Δ induced by v_1', v_2', v_3', but not to Π, induced by $\{v_1, v_2, v_3, v_4\}$, This disproves the strong convexity of $\mathcal{F}_3(\mathcal{B}'')$.

Example



Three versions of 3-graph $\mathcal{B}''$

Π- *and* Δ*-free* d*-graphs*

Clearly, the family of Π- and Δ-free d-graphs is hereditary;

class $\mathcal{M} = \mathcal{LM}$ contains only null-d-graph.

We know that for two different convex families, CC and not CIS d-graphs, class $\mathcal{M} = \mathcal{LM}$ contains only d-graphs Π and Δ.

Hence, the following three properties of a d-graph $\mathcal{G}$ are equivalent:

(i) $\mathcal{G}$ is Π- and Δ-free;

(ii) $\mathcal{G}$ contains no CC sub-d-graph;

(iii) $\mathcal{G}$ contains only CIS subgraphs.

Canonical decomposition of Π- *and* Δ-*free* d-*graphs*

This result allows us to construct a unique one-to-one correspondences between

(j) Π- and Δ-free d-graphs;

(jj) vertex d-colored rooted trees;

(jjj) tight and rectangular game forms of d players.

LEMMA. [**Gur78**, **Gur84**]. For each Π- and Δ-free d-graph $G = (V; E_1, \dots, E_d)$ there exists a unique $i \in [d] = \{1, \dots, d\}$ such that the complement of the i-th chromatic component $G_i = (V; E_i)$ is not connected on V.

Then, partition this G_i into connected components. Operation $\oplus i$.

Each one is still Π- and Δ-free.

Hence, there exists a unique $i \in [d] = \{1, \dots, d\}$... Note that $j \neq i$, because G_i was complementary connected, by construction. Etc.

Thus, we obtain the unique desired decomposition: a tree whose vertices are labelled by $[d] = \{1, \dots, d\}$.

For examples, see figures in [**Gur09**], and [**ABG06**].

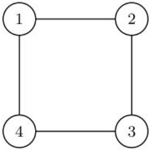
In its turn, this result enables us to characterize read-once Boolean functions, when $d = 2$ [**GG11**, **Gur77**, **Gur78**, **Gur84**, **Gur91**]

and normal forms of graphical d-person games modelled by trees [**ABG06**, **Gur78**, **Gur82**, **Gur84**, **Gur09**, **Gur11**].

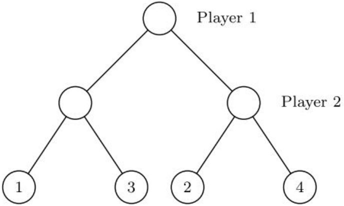
Π- and Δ-free graphs;

corresponding game structures: trees whose vertices (positions) are labelled by $[d] = \{1, \dots, d\}$;

games in normal form, that is game forms that are tight and rectangular.



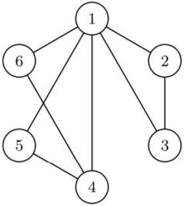
$$\mathcal{S}_1 = \{(1,3)(2,4)\}$$
$$\mathcal{S}_2 = \{(1,2)(2,3)(3,4)(4,1)\}$$



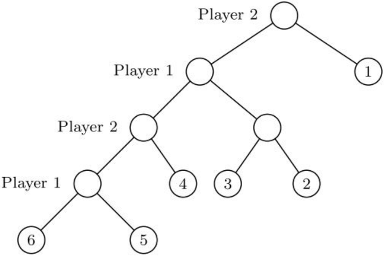
	12	23	34	41
13	1	3	3	1
24	2	2	4	4

$$F_1 = F = 13 \vee 24$$
$$F_2 = F^d = (1 \vee 3)(2 \vee 4) = 12 \vee 13 \vee 34 \vee 41$$

A P_4-free graph and the corresponding positional and normal game forms



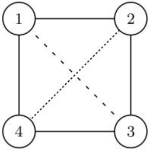
$$\mathcal{S}_1 = \{(1)(2,4)(3,4)(2,5,6)(3,5,6)\}$$
$$\mathcal{S}_2 = \{(1,2,3)(1,4,5)(1,4,6)\}$$



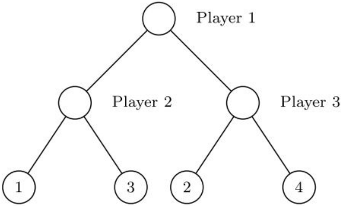
	1	24	34	256	356
123	1	2	3	2	3
145	1	4	4	5	5
146	1	4	4	6	6

$$F_1 = 1(23 \vee 4(5 \vee 6)) = 123 \vee 145 \vee 146$$
$$F_2 = 1 \vee (2 \vee 3)(4 \vee 56) = 1 \vee 24 \vee 34 \vee 256 \vee 356$$

Another P_4-free graph and the corresponding positional and normal game forms



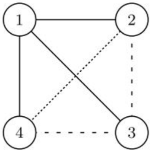
$$
\begin{aligned}
\mathcal{S}_1 &= \{(13)(24)\} \\
\mathcal{S}_2 &= \{(124)(234)\} \\
\mathcal{S}_3 &= \{(123)(134)\}
\end{aligned}
$$



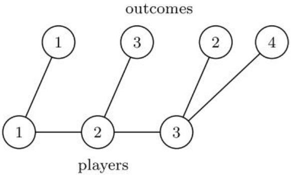
13			24	
1	1	124	2	4
3	3	234	2	4
1	1		1	1
2	3		2	3
3	4		3	4

$$
\begin{aligned}
F_1 &= 13 \vee 24 &&= 13 \vee 24 \\
F_2 &= (1 \vee 3)24 &&= 124 \vee 234 \\
F_3 &= 13(2 \vee 4) &&= 123 \vee 134
\end{aligned}
\qquad
\begin{aligned}
F_{23} &= (1 \vee 3)(2 \vee 4) &&= 12 \vee 23 \vee 34 \vee 41 \\
F_{13} &= 13 \vee 2 \vee 4 &&= 13 \vee 2 \vee 4 \\
F_{12} &= 1 \vee 3 \vee 24 &&= 1 \vee 3 \vee 24
\end{aligned}
$$

A Π- and Δ-free 3-colored *d*-graph and corresponding positional and normal game forms



$$\mathcal{S}_1 = \{(1)(234)\}$$
$$\mathcal{S}_2 = \{(13)(124)\}$$
$$\mathcal{S}_3 = \{(123)(134)\}$$



1			234	
1	1	13	3	3
1	1	124	2	4
1	1		1	1
2	3		2	3
3	4		3	4

$$\begin{aligned} F_1 &= 1 \vee 324 &&= 1 \vee 234 \\ F_2 &= 1(3 \vee 24) &&= 13 \vee 124 \\ F_3 &= 13(2 \vee 4) &&= 123 \vee 134 \end{aligned}$$

$$\begin{aligned} F_{23} &= 1(3 \vee 2 \vee 4) &&= 12 \vee 13 \vee 14 \\ F_{13} &= 1 \vee 3(2 \vee 4) &&= 1 \vee 23 \vee 34 \\ F_{12} &= 1 \vee 3 \vee 24 &&= 1 \vee 3 \vee 24 \end{aligned}$$

Another Π- and Δ-free 3-colored d-graph and the corresponding positional and normal game forms

More on CIS d-graphs

Several examples of CIS d-graphs can be found in [**ABG06**].

For example, each Π- and Δ-free d-graph is CIS.

(Furthermore, according to Δ-conjecture, no CIS d-graph contains Δ; see Section 36 below.)

We have no efficient characterization or recognition algorithm for CIS d-graphs, even for $d = 2$. The problem looks difficult because family $\mathcal{F}_2^{CIS}$ of CIS 2-graphs is not hereditary.

For example, bull 2-graph is CIS, but deleting its "top vertex" we obtain sub-2-graph Π, which is not CIS.

Moreover, family $\mathcal{F}_2^{CIS}$ of CIS 2-graph is not convex.

Example

For any integer $n > 2$, we will construct a 2-graph $\mathcal{G}_n$ such that $\mathcal{G}_n \in \mathcal{LM}(\mathcal{F}_{\mathbf{2}}^{CIS}) \setminus \mathcal{M}(\mathcal{F}_{\mathbf{2}}^{CIS})$.

To do so, consider complete bipartite $n \times n$ graph $K_{n,n}$, its line graph $G_n = L(K_{n,n})$, and its complement $\overline{G_n}$. These two complementary graphs on the same vertex-set form the required 2-graph $\mathcal{G}_n = \mathcal{L}(\mathcal{K}_{(n,n)}$ for each $n > 2$; see **[ABG06]** as an example for $n = 3$.

It is easy to verify that $\mathcal{G}_n$ is a CIS 2-graph, but for each $v \in V(\mathcal{G}_n)$ the reduced sub-2-graph $\mathcal{G}_n[V \setminus v]$ is not CIS.

Due to symmetry, it is enough to check this claim for just one arbitrary $v \in V(\mathcal{G}_n)$. Thus, $L(\mathcal{G}_n)$ is a locally minimal CIS 2-graph. Yet, it is not minimal, since only the null-2-graph is.

Moreover, every 2-graph with at most 3 vertices is CIS and the only minimal not CIS 2-graph is Π.

Δ-CONJECTURE.

A d-graph (with $d > 2$) is not CIS whenever it contains a Δ.

Δ-free d-graphs are called Gallai's.

Hence, the family of Π-free not CIS d-graphs, with $d > 2$, is weakly hereditary.

It can be proven without Δ-conjecture [**ABG09**].

Furthermore, the family of Π- and Δ-free d-graphs is hereditary, by definition.

There d-graphs are in one-to-one correspondence with the positional d-person game structures modeled by trees [**Gur82**, **Gur09**, **Gur82**].

No efficient characterization of locally minimal CIS d-graphs is known. However, Δ-conjecture, if true, would allow us to reduce arbitrary d to $d = 2$; see subsection about Δ below.

Let us note finally that a 2-graph $\mathcal{G} = (V; E_1, E_2)$ is CIS whenever each maximal clique of its chromatic component $G_i = (V, E_i)$ has a simplicial vertex, for $i = 1$ or $i = 2$ [**ABG06**]. Hence, every 2-graph $\mathcal{G}$ is a subgraph of a CIS 2-graph $\mathcal{G}'$. This is easy to verify [**ABG06**].

Note, however, that the size of $\mathcal{G}'$ is exponential in the size of $\mathcal{G}$.

Thus, CIS d-graphs cannot be described in terms of forbidden subgraphs, already for $d = 2$. This is not surprising, since this family is not hereditary.

Given a CIS d-graph $\mathcal{G}$ and a partition $\mathcal{P}$ of its colors $[d] = \{1, \ldots, d\}$ into δ non-empty subsets such that $2 \leq \delta \leq d$, merging colors in each of this subsets we obtain a δ-graph $\mathcal{G}' = \mathcal{G}'(\mathcal{G}, \mathcal{P})$, which we will call the *projection* of $\mathcal{G}$ wrt color-merging $\mathcal{P}$. It is not difficult to verify (see [**ABG06**] for details) that:

- if $\mathcal{G}$ is CIS then $\mathcal{G}'$ is, but not vice versa;
- if $\mathcal{G}'$ contains a Δ then G does, but not vice versa;
- if $\mathcal{G}'$ contains a Π then $\mathcal{G}$ contains a Π or Δ.

We can reformulate the first two claims as follows:

$\mathcal{G}'$ is CIS or, respectively, Gallai's whenever $\mathcal{G}$ is.

For $\delta = 2$ the first claim implies that merging an arbitrary set of chromatic components of a CIS d-graph results in a CIS graph.

MODULAR DECOMPOSITION of Gallai's (Δ-free) d-graphs; Λ-decomposition.

THEOREM. Each Δ-free d-graphs is a modular decomposition of 2-graphs.

This statement follows from the results of Gyárfás and Simonyi [**GS04**], which, in their turn, are based on the results of Cameron, Edmonds, and Lovasz [**CE97**, **CEL86**], Möring [**Mor85**], and Gallai [**Gal67**]; see more details in [**ABG06**, **ABG10**] and [**Gur09**]. It is based on the following main Lemma that Gyárfás and Simonyi [**GS04**] attribute to Gallai [**Gal67**].

LEMMA. Each Δ-free d-graph has a not connected chromatic component.

Then, the connected components of this chromatic component form a modular decomposition in which d is reduced by 1.

Indeed, fix any two such connected components. All edges between them are of the same color, since a Δ would appear otherwise.

The operation of substitution $G = G'(v \to G'')$ for graphs and $\mathcal{G} = \mathcal{G}'(v \to \mathcal{G}'')$ d-graphs was already defined above.

It can be similarly introduced for multi-variable functions and for many other objects; see [**Mor85**] for more details; In this paper G'' and $\mathcal{G}''$ are referred to as *modules* and substitution as *modular decomposition*.

We say that a family $\mathcal{F}$ of graphs or digraphs is exactly closed wrt substitution if $G \in \mathcal{F}$ if and only if $G', G'' \in \mathcal{F}$ and, respectively, $\mathcal{G} \in \mathcal{F}$ if and only if $\mathcal{G}', \mathcal{G}'' \in \mathcal{F}$. It is both known and easy to verify that the following families are exactly closed wrt substitution: perfect, CIS, CC, and , P_4-free graphs; CIS, CC, Gallai (Δ-free), Π- and Δ-free d-graphs.

Recall that a graph G is called *CIS* if $C \cap S \neq \emptyset$ for every maximal clique C and maximal stable set S of G. Given a CIS (respectively, CC) 2-graph $\mathcal{G} = (V; E_1, E_2)$, each of its two chromatic components $G_i = (V, E_i)$, $i = 1, 2$ is a CIS (respectively, CC) graph. See more about CIS and CC graphs in [**ABG06**, **ABG10**, **BG09**, **BGM14**, **BGZ09**, **DLZ04**, **DLZ04a**, **Gur84**, **Gur09**, **Gur11**, **WZZ09**, **Zan95**].

Perfect graphs are closed wrt complementation, by the Perfect Graph Theorem [**Lov72a**, **Lov72b**]. Obviously, CIS, CC, and P_4-free graphs also have this property, just by definition.

Let $\mathcal{F}$ be a family of Gallai d-graphs $\mathcal{G} = (V; E_1, \ldots, E_d)$ such that the family $\mathcal{F}'$ of their chromatic components $G_i = (V, E_i)$, $i \in [d] = \{1, \ldots, d\}$ is (i) closed wrt complementation and (ii) exactly closed wrt substitution. For example, family $\mathcal{F}'$ that contains only perfect, or CIS, or CC, or P_4-free graphs has properties (i) and (ii).

Every d-graphs $\mathcal{G} \in \mathcal{F}$ with $d \geq 3$ can be decomposed by two non-trivial d-graphs, that is, $\mathcal{G} = \mathcal{G}'(v \to \mathcal{G}'')$, where d-graphs $\mathcal{G}'$ and $\mathcal{G}''$ are distinct from $\mathcal{G}$ and from the trivial one-vertex d-graph.

As a corollary, we conclude that Gallai's d-graphs whose chromatic components have properties (i) and (ii) can be decomposed by the 2-colored such d-graphs. In other words, for Gallai's d-graphs, the case of arbitrary d can be reduced to $d = 2$. This statement follows from the results of Gyárfás and Simonyi [**GS04**], which, in their turn, are based on the results of Cameron, Edmonds, and Lovasz [**CE97**, **CEL86**], Möring [**Mor85**], and Gallai [**Gal67**]; see more details in [**ABG06**, **ABG10**] and [**Gur09**].

For example, the modular decomposition of Π- and Δ-free d-graphs has important applications in theory of positional (graphical) n-person games modelled by trees; in particular, it is instrumental in characterizing the normal forms of these games [**ABG06**], [**Gur75**], [**Gur78**], and [**Gur82**, **Gur84**, **Gur09**].

see the corresponding figures again.

Δ-conjecture All CIS d-graphs are Gallai's, or in other words, d-graphs containing Δ are not CIS.

This conjecture was suggested in [**Gur78**] and it remains open. Some partial results were obtained in [**Gur78**]. In particular:

(i) Every Π- and Δ-free d-graph is CIS.

(ii) It is sufficient to prove Δ conjecture for 3-graphs; then, it follows for d-graphs with arbitrary d.

The second claim was proven by Andrey Gol'berg (private communications) in 1975 as follows: Consider the projection $\mathcal{G}' = \mathcal{G}'(\mathcal{G}, \mathcal{P})$ of $\mathcal{G}$ wrt a color-merging $\mathcal{P}$. As we know, $\mathcal{G}'$ is CIS whenever G is. Suppose Δ-conjecture fails for $\mathcal{G}$, in other words, $\mathcal{G}$ is CIS but not Gallai, that is, it contains a Δ, say Δ_0. Consider a color-merging $\mathcal{P}$ with $\delta = 3$ such that three colors of Δ_0 are still pairwise distinct in $\mathcal{P}$. Then, projection $\mathcal{G}' = \mathcal{G}'(\mathcal{G}, \mathcal{P})$ still contains a *Delta*, but $\mathcal{G}'$ is a CIS 3-graph. Thus, Δ-conjecture fails for 3-graphs too.

According to the previous theorem, each CIS d-graph is a modular decomposition of CIS 2-graphs, modulo Δ-conjecture. If it holds, studying CIS d-graphs is reduced to studying CIS graphs. Yet, the latter is still difficult.

If Δ-conjecture fails then here are CIS d-graphs with Δ.

Let us note that in case of perfect, CC or P_4-free chromatic components of a d-graph, we still have to require that it is Δ-free; yet, in case of CIS this requirement may be waved, modulo Δ-conjecture; see more details in [**ABG06**, **BGZ09**, **DLZ04**, **DLZ04a**, **Gur09**, **Gur11**].

Finite two-person normal form games and game forms

In this section we consider matrices, that is, mappings $M : I \times J \to R$, whose rows $I = \{i_1, \ldots, i_n\}$ and columns $J = \{j_1, \ldots, j_m\}$ are the strategies of Alice and Bob, respectively, while R may vary: it is real numbers $\mathbb{R}$ or their pairs $\mathbb{R}^2$ in case of matrix and bimatrix games, respectively, and $R = \Omega = \{\omega_1, \ldots, \omega_k\}$ is a finite set of outcomes in case of game forms. In all cases,
$\mathcal{P} = \mathcal{P}(M) = I \cup J$ is the ground set and *succ* is the containment order over $\mathcal{P}$; in other words, $\mathcal{P}(M)$ consists of all submatrices of M. By convention, we identify all elements of $\mathcal{P}$ with $I = \emptyset$ or $J = \emptyset$: they correspond to the empty submatrix, which is the unique minimum in $(\mathcal{P}, \succ)$.

Saddle point free matrices

In this case $R = \mathbb{R}$ is the set of real numbers. We assume that Alice is the maximizer and she controls the rows, while Bob is the minimizer and he controls the columns.

An entry of M is a *saddle point* (SP) if and only it is minimal in its row and maximal in its column (not necessarily strictly, in both cases). It is well known that a matrix has a SP if and only if its maxmin and minmax are equal. Obviously, a 2×2 matrix M has no SP if and only if one of its diagonals is strictly larger than the other, that is, $[r_{i_1,j_1}; r_{i_2,j_2}] \cap [r_{i_1,j_2}; r_{i_2,j_1}] = \emptyset$.

In 1964 Lloyd Shapley **[Sha64]** proved that a matrix has a SP if (but not only if) every its 2×2 submatrix has a SP. In other words, for the family $\mathcal{F}$ of all SP free matrices, class $\mathcal{M}(\mathcal{F})$ of the minimal SP free matrices consists of the 2×2 SP free matrices.

L. Shapley

Proof of Shapley's theorem

Let, $v = Val[M]$.

Let, j_1 be the index of the column with minimum number r of entries greater than v (assume $r > 0$).

Let, i_1 be a row position where, $a(i_1, j_1) > \omega$, then, $\exists j_2$ such that $a(i_1, j_2) \leq v$.

Since the column indexed by j_2 has at least r positions with entry value $> v$, too many to be paired off against the $r - 1$ remaining entries $> v$ of the other column j_1.

Hence, $\exists i_2$ column position such that $a(i_2, j_2) > v > a(i_2, j_1)$.

Since the 2×2 submatrix:

$$\begin{array}{ccc} & (i_1) & (i_2) \\ (j_1) & > v & \leq v \\ (j_2) & \leq v & > v \end{array}$$

clearly has no saddle-point, the assumption $r > 0$ must be incorrect.

This result was strengthen as follows:

Theorem

*([**BGM09**]). Every SP free matrix of size larger that 2×2 has a row or column such that it can be deleted and the remaining matrix is still SP free.* $\square$

In other words, $\mathcal{LM}(\mathcal{F}) = \mathcal{M}(\mathcal{F})$, that is, family $\mathcal{F}$ is convex.

Yet, F is not strongly convex, as the following example shows.

Example

Consider the following 4×4 0,1-matrix M:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

The following observations are easy to verify:

Matrix M is SP free and it contains two (locally) minimal SP free 2×2 sabmatrices: the first one, $M_{\mathbf{1}}$, upper left, is determined by the first two rows and columns of M, while the second one, $M_{\mathbf{2}}$, lower right, is determined by the last two rows and columns of M.

Furthermore, if we eliminate one of the last (respectively, first) two rows or columns of M, then a SP appears (respectively, it does not) in the obtained submatrix. Hence, keeping SP freeness one can reduce M to $M_{\mathbf{2}}$ but not to $M_{\mathbf{1}}$. The first claim is in agreement with convexity, while the second one disproves strong convexity of the family of SP free matrices.

Moreover, no SP appears when we delete the first two rows and columns from M in an arbitrary order, thus reducing M to $M_{\mathbf{2}}$. In other words, family $\mathcal{F}(M)$ is very weakly hereditary, not only convex. We leave open, if this hold for property $\mathcal{F}$ in general.

Matrices with saddle points

Proposition

Given a matrix M with a SP, family $\mathcal{F} = \mathcal{F}(M)$ of all submatrices of M with a SP is strongly convex.

Proof

Obviously, class $\mathcal{M} = \mathcal{M}(\mathcal{F}(M))$ consists of all 1×1 submatrices (that is, entries) of M. By assumption $M \in \mathcal{F}$; let (i^*, j^*) be a SP in M. Obviously, it remains a SP when we delete from M a row distinct from i^* or a column distinct from j^*. Hence, $\mathcal{M} = \mathcal{L}\mathcal{M}$, or in other words, $\mathcal{F}$ is convex.

To prove strong convexity, fix an arbitrary entry (i_0, j_0) in M and reduce M deleting successively its rows, except i_0, i^*, and columns, except j_0, j^*. As we already mentioned, (i^*, j^*) remains a SP. Consider three cases:

If $i_0 = i^*$ and $j_0 = j^*$, we arrive to the 1×1 submatrix $(i_0, j_0) = (i^*, j^*)$.

If $i_0 \neq i^*$ and $j_0 \neq j^*$ we arrive to the 2×2 submatrix formed by these two rows and columns. Note that (i^*, j^*) is still a SP. Then delete row i^* getting a 1×2 submarix. Clearly, it has a SP, which may be not (i^*, j^*), yet. Finally, we delete column j^* getting (i_0, j_0).

If $i_0 = i^*$ or $j_0 = j^*$ but not both, then we arrive to a 1×2 or 2×1 submatrix that consists of (i_0, j_0) and (i^*, j^*). Note that (i^*, j^*) is still a SP. Then we delete it getting (i_0, j_0) in one step. $\square$

However, family $\mathcal{F}(M)$ is not weakly hereditary.

Example

Consider matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It has two saddle points, both in the first column, but, by deleting this column, we obtain a SP free matrix.

Absolutely determined matrices

A matrix is called *absolutely determined* if every its submatrix has a SP. By Shapley's theorem [**Sha64**], it happens if and only if each 2×2 submatrix has a SP. This condition can by simplified in case of symmetric matrices [**GL89**, **GL90**, **GL92**].

By definition the considered family is hereditary.

Nash equilibria free bimatrices

A bimatrix game (A, B) is defined as a pair of mappings $a : I \times J \to \mathbb{R}$ and $b : I \times J \to \mathbb{R}$ that specify the utility (or payoff) functions of Alice and Bob, respectively. Now both players are maximizers.

A situation $(i, j) \in I \times J$ is called a *Nash equilibrium* (NE) if no player can improve the result by choosing another strategy provided the opponent keeps the same strategy, that is, if $a(i, j) \geq a(i', j) \forall i' \in I$ and $b(i, j) \geq b(i, j') \forall j' \in J$.

In other words, i is a best response to j for Alice and j is a best response to j for Bob.

Clearly, Nash equilibria generalize saddle points, which correspond to the zero-sum case: $a(i,j) + b(i,j) = 0$ for all $i \in I$ and $j \in J$.

However, unlike SP free games, the minimal NE-free bimatrix games may be larger than 2×2. Let us recall an example from [**GL90**]. Consider a 3×3 bimatrix game (A, B) such that

$$\begin{aligned}
b(i_1, j_1) > b(i_1, j_2) \geq b(i_1, j_3), \\
b(i_2, j_3) > b(i_2, j_1) \geq b(i_2, j_2), \\
b(i_3, j_2) > b(i_3, j_3) \geq b(i_3, j_1);
\end{aligned}$$

$$\begin{aligned}
a(i_2, j_1) > a(i_1, j_1) \geq a(i_3, j_1), \\
a(i_1, j_2) > a(i_3, j_2) \geq a(i_2, j_2), \\
a(i_3, j_3) > a(i_2, j_3) \geq a(i_1, j_3).
\end{aligned}$$

Naturally, for situations in the same row (respectively, column) the values of b (respectively, a) are compared, since Alice controls rows and has utility function a, while Bob controls columns and has utility function b.

It is easy to verify that:

$b(i_1, j_1)$ is the unique maximum in row i_1 and

$a(i_1, j_1)$ is a second largest in the column j_1.

$b(i_2, j_3)$ is the unique maximum in row i_2 and

$a(i_2, j_3)$ is a second largest in column j_3;

$b(i_3, j_2)$ is the unique maximum in row i_3 and

$a(i_3, j_2)$ is a second largest in column j_2;

$a(i_2, j_1)$ is the unique maximum in column j_1 and

$b(i_2, j_1)$ is a second largest in row i_2;

$a(i_1, j_2)$ is the unique maximum in column j_2 and

$b(i_1, j_2)$ is a second largest in row i_1;

$a(i_3, j_3)$ is the unique maximum in column j_3 and

$b(i_3, j_3)$ is a second largest in row i_3.

Consequently, this game is NE-free, since no situation is simultaneously the best in its row wrt b and in its column wrt a. Yet, if we delete a row or column then a NE appears. For example, let us delete i_1. Then the situation (i_3, j_2) becomes a NE. Indeed, $b(i_3, j_2)$ is the largest in the row i_3 and $a(i_3, j_2)$ is a second largest in the column j_2, yet, the largest, $a(i_1, j_2)$, was deleted. Similarly, situations $(i_1, j_1), (i_2, j_3), (i_1, j_2), (i_3, j_3), (i_2, j_1)$ become NE after deleting lines i_2, i_3, j_1, j_2, j_3, respectively.

Thus, (A, B) is a locally minimal NE-free bimatrix game. Moreover, it is also minimal. Indeed, one can easily verify that all 2×2 subgames of (A, B) have a NE and, of course, 1×2, 2×1, and 1×1 games always have it.

In general, the following criterion of local minimality holds:

Theorem

*(**[BGM09]**) A bimatrix game (A, B) is a locally minimal NE-free game if and only if it satisfies the following four conditions:*

(i) it is square, that is, $|I| = |J| = k$*;*

(ii) there exist two one-to-one mappings (permutations) $\sigma : I \to J$ *and* $\delta : J \to I$ *such that their graphs,* $gr(\sigma)$ *and* $gr(\delta)$*, are disjoint in* $I \times J$*, or in other words, if* $(i, \sigma(i)) \neq (\delta(j), j)$ *for all* $i \in I$ *and* $j \in J$*;*

(iiia) $a(\delta(j), j)$ *is the unique maximum in column* j *and a second largest (though not necessarily unique) in row* $\delta(j)$*;*

(iiib) $b(i, \sigma(i))$ *is the unique maximum in row* i *and a second largest (though not necessarily unique) in column* $\sigma(i)$*.* □

Thus, we have a simple explicit characterization of the class $\mathcal{LM}$ of the locally minimal NE-free bimatrix games. However, not each such game is minimal. Indeed, mappings σ and *delta* define $2k$ entries of a $k \times k$ bimatrix locally minimal NE-free game. Yet, it may contain some smaller $k' \times k'$ NE-free subgames; see [**BGM09**] for more details.

Thus, the family of NE-free bibatrix games is not convex. Interestingly, in this case class $\mathcal{M}$ is more complicated than $\mathcal{LM}$. In contrast to the latter, no good characterization of the former is known.

It is not difficult to verify that in the zero-sum case k cannot be larger than 2.

In contrast, bimatrix games with NE are similar to matrix games with SP. This family is weakly hereditary but not hereditary. The proof from the previous subsection can be applied in this case.

Tightness

Given a finite set of outcomes Ω, let X and Y be finite sets of strategies of Alice and Bob, respectively. A mapping $g : X \times Y \to \Omega$ is called a *game form*. One can view a game form as a game without payoffs, which are not given yet.

A game form is called *tight* if its rows and columns form dual hypergraphs. Several equivalent definitions of tightness can be found, for example, in [**GN21**, **GN22**]. Nine examples of game forms are given below; the first six are tight, the last three are not.

ω_1	ω_1
ω_2	ω_3

g_1

ω_1	ω_1	ω_2	ω_2
ω_3	ω_4	ω_3	ω_4

g_2

ω_1	ω_1	ω_3
ω_1	ω_2	ω_2
ω_3	ω_2	ω_3

g_3

ω_1	ω_1	ω_3
ω_1	ω_1	ω_2
ω_4	ω_2	ω_2

g_4

ω_1	ω_2	ω_1	ω_2
ω_3	ω_4	ω_4	ω_3
ω_1	ω_4	ω_1	ω_5
ω_3	ω_2	ω_6	ω_2

g_5

ω_1	ω_1
ω_1	ω_2

g_6

ω_1	ω_2
ω_2	ω_1

g_7

ω_1	ω_1	ω_2
ω_3	ω_4	ω_3

g_8

ω_1	ω_1	ω_2
ω_4	ω_5	ω_2
ω_4	ω_3	ω_3

g_9

Nine game forms. Forms g_1 - g_6 are tight, forms g_7 - g_9 are not.

Tightness is equivalent with SP-solvability [**EF70**, **Gur73**] and with NE-solvability [**Gur75**, **Gur89**, **GN22**].

Not tight game forms

Since tightness and SP-solvability are equivalent, the Shapley Theorem implies that all minimal not tight game forms are of size 2×2. Two sets of outcome corresponding to two diagonals are disjoint. There are three such game forms:

ω_1	ω_2
ω_2	ω_1

ω_1	ω_2
ω_3	ω_1

ω_1	ω_2
ω_3	ω_4

Three not tight 2×2 game forms

This result was strengthened in [**BGM09**], where it was shown that these three game forms are the only locally minimal not tight ones. In other words, family $\mathcal{F}$ of not tight game forms is convex and class $\mathcal{M}(\mathcal{F}) = \mathcal{LM}(\mathcal{F})$ consists of the above three game forms.

Here we will strengthen this result further as follows.

Theorem

Family $\mathcal{F}$ of not tight game forms is strongly convex but not weakly hereditary.

Proof

Obviously, a game form with a single outcome is tight.

Consider game forms with two outcomes $\omega_1 = a$ and $\omega_2 = b$. Obviously, such game form g is tight if and only if one of the following cases holds:

Case (rca): Game form g contains an a-row and an a-column, that is, there exists an $x_0 \in X$ and $y_0 \in Y$ such that $g(x, y) = a$ whenever $x = x_0$ or $y = y_0$.

Case (rcb): Game form g contains a b-row and a b-column.

Case (rab): Game form g contains an a-row and a b-row.

Case (cab): Game form g contains an a-column and a b-column.

Example

Family $\mathcal{F}$ is not weakly hereditary, consider the following 4×4 game form:

a	b	a	b
b	a	a	b
a	a	a	b
b	b	b	a

Obviously, it is not tight but becomes tight if we delete the last rows or column (or two last rows or columns). In contrast, every other, not last, row or column can be deleted and the the obtained reduced game form remains not tight.

This example proves that family $\mathcal{F}$ is not weakly hereditary, yet, it does not disprove strong convexity. Actually, family $\mathcal{F}$ is strongly convex. To show this, consider a not tight game form $g : X \times Y \to \Omega$ of size larger than 2×2 and fix a 2×2 not tight subform $g^* : X^* \times Y^* \to \Omega$ in it, that is, $|X^*| = |Y^*| = 2$, $X^* \subseteq X,\ Y^* \subseteq Y$, and at least one of these two containments is strict. Wlog, we can assume that g^* is formed by the first two rows and columns of g.

It is enough to show that one can delete a row $x \in X \setminus X^*$ or a column $y \in Y \setminus Y^*$ such that the reduced game form g' is still not tight.

It is both obvious and well-known that merging outcomes respects tightness. Hence, wlog, we can assume that $\Omega = \{\omega_1, \omega_2\} = \{a, b\}$ consists of two outcomes, and that g^* is

a	b
b	a

It is also clear that adding an a-row or a-column to a tight game form respects its tightness. In other words, deleting an a-row or an a-column from a not tight game form g respects its non-tightness. By symmetry, the same holds for b-rows and b-columns as well.

Assume for contradiction that after deleting a row $x \in X \setminus X^*$, or a column from $y \in Y \setminus Y^*$ from g, the obtained reduced subform g' is tight. Then, as we know, one of the cases: (rca), (rcb), (rab), (cab) holds. Assume wlog that we delete a column (rather than a row) and consider all four cases.

In case (rca) g' has an a-column $x \in X \setminus X^*$. Deleting this column from g we obtain a not tight game form, which is a contradiction. By symmetry, case (rca) resolved too.

Case (cab) is trivial, since then both g' and g are tight, which is a contradiction.

Thus, only case (rab) remains, and we can assume that after deleting each column $y \in Y \setminus Y^*$ from g the obtained reduced submatrix g' has both a- and b-rows. This is possible (only) when g contains 4 rows and 3 columns.

Yet, by symmetry, we can also assume that after deleting each row $x \in X \setminus X^*$ from g the obtained reduced submatrix g'' has both a- and b-columns.

This is already impossible. To see it, consider the 4×4 game form shown in Fig. 1 in which we can assign a or b arbitrarily to every symbol $*$. It is not difficult to verify that every such assignment results in contradiction: the obtained subform is either tight, or not tight and among its last two rows and columns there is at least one deleting which results in a subform that is still not tight. $\square$

a	b	a	b
b	a	a	b
a	a	$*$	$*$
b	b	$*$	$*$

Figure 1. Case (rab), a contradiction.

Tight game forms

Tightness is not hereditary. For example, game form g_3 is tight but after deleting the last column one obtains g_8, which is not tight.

Moreover, it was shown in [**BGM09**] that the family $\mathcal{F}$ of tight game forms is not convex. Class $\mathcal{M}(\mathcal{F})$ contains only 1×1 game forms, while $\mathcal{LM}(\mathcal{F})$ is a complicated class, which seems difficult to characterize; only some necessary and some sufficient conditions are obtained in [**BGM09**]. Note that $g_3 \notin \mathcal{LM}(\mathcal{F})$, since deleting its row keeps tightness.

Totally tight game forms

A game form is called *totally tight (TT)* if every its subform is tight; for example, g_3 in Figure above is TT.

Proposition

*([**BGMP10**]). Tightness of all* 2×2 *subforms already implies total tightness.* $\square$

Sketch of the proof This result is implied by the following two criteria of solvability [**Gur73**, **Sha64**]. The first states that a game has a SP if (but not only if) every its 2×2 subgame has a SP [**Sha64**]. The second claims that a game form is *zero-sum-solvable* if and only if it is tight [**Gur73**]; see more details in [**Gur89**, **GN21**, **GN22**]. Let us note that the second result is implicit already in [**EF70**]. $\square$

It is easily seen that the next three properties of the 2×2 game forms are equivalent:

(i) tightness, (ii) total tightness, (iii) presence of a constant row or column.

As we know, there are only three not tight 2×2 game forms.

Thus, family of the TT game forms is characterized by these three forbidden subforms. Hence, this family is hereditary.

It is also known that a game form is TT if and only if it is *acyclic*, that is, for arbitrary payoffs of two players the obtained game has no *improvement cycle*; see [**BGMP10**] for the proof and precise definitions.

An explicit recursive characterization of the TT game forms is also given in [**BGMP10**].

Not totally tight game forms

As we already mentioned, a game form is not TT if and only if it contains at least one of the three not tight 2×2 subforms given in Figure above. Thus, family of not TT game forms is weakly hereditary. Obviously, it is not hereditary. Indeed, by deleting a row or column from a 2×2 game form one obtains a 1×2 or 2×1 game form, which is tight.

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