Polynomial algorithms computing two lexicographically safe Nash equilibria in finite two-person games with tight game forms given by oracles

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Abstract

In 1975 the first author proved that every finite tight two-person game form g is Nash-solvable, that is, for every payoffs u and w of two players the obtained game $(g; u, w)$, in normal form, has a Nash equilibrium (NE) in pure strategies. This result was extended in several directions; here we strengthen it further. We construct two special NE realized by a lexicographically safe (lexsafe) strategy of one player and a best response of the other. We obtain a polynomial algorithm computing these lexsafe NE. This is trivial when game form g is given explicitly. Yet, in applications g is frequently realized by an oracle $\mathcal{O}$ such that size of g is exponential in size $|\mathcal{O}|$ of $\mathcal{O}$. We assume that game form $g = g(\mathcal{O})$ generated by $\mathcal{O}$ is tight and that an arbitrary *win-lose game* $(g; u, w)$ (in which payoffs u and w are zero-sum and take only values ± 1) can be solved, in time polynomial in $|\mathcal{O}|$. These assumptions allow us to construct an algorithm computing two (one for each player) lexsafe NE in time polynomial in $|\mathcal{O}|$. We consider four types of oracles known in the literature and show that all four satisfy the above assumptions.

Keywords: Nash equilibrium, Nash-solvability, game form, deterministic graphical game structure, game in normal and in positional form, monotone bargaining, veto voting, Jordan game.
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1. Basic definitions and preliminary results

1.1. Game forms

In this paper we consider finite, not necessarily zero-sum, normal form games of two players, Alice and Bob. They have finite sets of strategies X and Y, respectively.

A *game form* is a mapping $g : X \times Y \to O$, where O is a finite set of outcomes.

Several examples are given in Figure 1, where game forms are represented by tables with rows, columns, and entries labelled by $x \in X$, $y \in Y$, and $o \in O$, respectively.

o_1	o_1
o_2	o_3

g_1

o_1	o_1	o_2	o_2
o_3	o_4	o_3	o_4

g_2

o_1	o_1	o_3
o_1	o_2	o_2
o_3	o_2	o_3

g_3

o_1	o_1	o_3
o_1	o_1	o_2
o_4	o_2	o_2

g_4

o_1	o_2	o_1	o_2
o_3	o_4	o_4	o_3
o_1	o_4	o_1	o_5
o_3	o_2	o_6	o_2

g_5

o_1	o_1
o_1	o_2

g_6

o_1	o_2
o_2	o_1

g_7

o_1	o_1	o_2
o_3	o_4	o_3

g_8

o_1	o_1	o_2
o_4	o_5	o_2
o_4	o_3	o_3

g_9

Fig. 1. Nine game forms. Alice and Bob choose rows and columns, respectively. Forms g_1 - g_6 are tight, forms g_7 - g_9 are not (see section 1.4 for the definition).

Mapping g is assumed to be surjective, but not necessarily injective, that is, an outcome $o \in O$ may occupy an arbitrary array in the table of g.

A pair of strategies (x, y) is called a *situation.* Sets $g(x) = \{g(x, y) \mid y \in Y\}$ and $g(y) = \{g(x, y) \mid x \in X\}$ are called the *supports* of strategies $x \in X$ and $y \in Y$, respectively.

A strategy is called *basic* if its support is not a proper subset of the support of any other strategy. For example, in g_6 the first strategies of Alice and Bob are basic, while the second are not; in the remaining eight game forms all strategies are basic. Moreover, any two strategies of a player, Alice or Bob, have distinct supports.

A situation (x, y) is called *simple* if $g(x) \cap g(y) = \{g(x, y)\}$. For example, all situations of game forms g_1, g_2, g_8, g_9 are simple (such game forms are called *rectangular*); in contrast, no situation is simple in g_7; in g_3 all are simple, except three on the main diagonal; in g_4 all are simple, except the central one; in g_6 all are simple, except one with the outcome o_2.

1.2. Payoffs and games in normal form

Payoffs of Alice and Bob are defined by real valued mappings $u : O \to \mathbb{R}$ and $w : O \to \mathbb{R}$, respectively. We assume that both players are maximizers.

A triplet $(g; u, w)$ defines a *finite two-person game in normal form*, or just a *game*, for short.

Remark 1. *It appears convenient to separate game forms and payoffs studying the normal form games. This approach makes game forms "responsible for generic structural properties of games", which hold for arbitrary payoffs.*

A game $(g; u, w)$ and its payoffs (u, w) are called

- *zero-sum* if $u(o) + w(o) = 0$ for all $o \in O$;
- *win-lose* if, in addition, functions u and w take only two values ± 1.

Alternatively, a win-lose payoff can be specified by a partition $O = O_A \cup O_B$ of all outcomes into two subsets: the outcomes preferred by Alice, O_A, and by Bob, O_B, respectively, that is,

$$u(o) = 1, w(o) = -1 \text{ for } o \in O_A \quad \text{and} \quad u(o) = -1, w(o) = 1 \text{ for } o \in O_B.$$

For a win-lose game notation $(g; O_A, O_B)$ will be used along with $(g; u, w)$.

1.3. Nash equilibria and Nash-solvability

Given a game $(g; u, w)$, a situation (x, y) of its game form $g : X \times Y \to A$ is called a *Nash equilibrium* (NE) if

$$u(g(x, y)) \geq u(g(x', y)), \forall x' \in X, \quad \text{and} \quad w(g(x, y)) \geq w(g(x, y')), \forall y' \in Y;$$

in other words, if neither Alice nor Bob can profit replacing her/his strategy provided the opponent keeps his/her one unchanged.

This concept of solution was introduced in 1950 by John Nash [23, 24].

A game form g is called (*i*) *Nash*, (*ii*) *zero-sum*, (*iii*) *win-lose solvable* if the corresponding game $(g; u, w)$ has a NE for (*i*) all, (*ii*) all zero-sum, (*iii*) all win-lose payoffs (u, w), respectively.

Implications (*i*) $\Rightarrow$ (*ii*) $\Rightarrow$ (*iii*) are obvious. In fact, all three properties are equivalent [11, 13, 15]. For (*ii*) and (*iii*) it was shown earlier by Edmonds and Fulkerson [7]; see also [10]. The list of equivalent properties (*i*), (*ii*) ,(*iii*) can be extended as follows.

1.4. Tight game forms

Mappings $\phi : X \to Y$ and $\psi : Y \to X$ are called *response strategies* of Bob and Alice, respectively. The motivation for this name is clear: a player chooses his/her strategy as a function of a known strategy of the opponent. Standardly, $gr(\phi)$ and $gr(\psi)$ denote the graphs of mappings ϕ and ψ in $X \times Y$.

Game form g is called *tight* if

(j) $g(gr(\phi)) \cap g(gr(\psi)) \neq \emptyset$ for any mappings ϕ and ψ.

It is not difficult to verify that in Figure 1 the first six game forms $(g_1 - g_6)$ are tight, while the last three $(g_7 - g_9)$ are not.

In [7, 10, 11, 13, 17] the reader can find several equivalent properties characterizing tightness. Here we mention some of them.

(jj) For every response strategy $\phi : X \to Y$ there exists a strategy $y \in Y$ such that $g(y) \subseteq g(gr(\phi))$.

(jj′) For every response strategy $\psi : Y \to X$ there exists a strategy $x \in X$ such that $g(x) \subseteq g(gr(\phi))$.

We leave to the careful reader to show that (j) is equivalent to (jj) and to (jj′) as well.

Properties (jj) and (jj′) show that playing a game $(g; u, w)$ with a tight game form g, the players, Bob and Alice, do not need non-trivial response strategies but can restrict themselves by the constant ones, that is, by Y and X, respectively; at least in case of the zero-sum games, $u + w = 0$.

Given a game form $g : X \times Y \to O$, introduce on the ground set O two multi-hypergraphs $A = A(g)$ and $B = B(g)$ whose edges are the supports of strategies of Alice and Bob, respectively:

$$A(g) = \{g(x) \mid x \in X\} \text{ and } B(g) = \{g(y) \mid y \in Y\}.$$

By construction, the edges of A and B pairwise intersect, that is, $g(x) \cap g(y) \neq \emptyset$ for all $x \in X$ and $y \in Y$. Furthermore, g is tight if and only if

(jjj) hypergraphs $A(g)$ and $B(g)$ are *dual*, that is, satisfy also the following two properties:

(a) for every $O' \subseteq O$ such that $O' \cap g(y) \neq \emptyset$ for all $y \in Y$ there exists an $x \in X$ such that $g(x) \subset O'$;
(b) for every $O'' \subseteq O$ such that $O'' \cap g(x) \neq \emptyset$ for all $x \in X$ there exists an $y \in Y$ such that $g(y) \subset O''$.

A multi-hypergraph $\mathcal{H}$ is called *Sperner* if containment $H \subseteq H'$ holds for no two distinct edges $H, H' \in \mathcal{H}$. Given an arbitrary multi-hypergraph $\mathcal{H}$, delete every its edge $H' \in \mathcal{H}$ that strictly contains some other edge $H \in \mathcal{H}$. Furthermore, from each family of edges with the same support delete all but one. This construction results in a unique Sperner hypergraph $\mathcal{H}^0$, which is called the *Sperner reduction* of $\mathcal{H}$. It is easily seen that $\mathcal{H}_1$ and $\mathcal{H}_2$ are dual if and only if $\mathcal{H}_1^0$ and $\mathcal{H}_2^0$ are dual.

Given a tight game form $g : X \times Y \to O$, delete from X and Y all non-basic strategies. Furthermore, consider in X all strategies with equal supports, choose one of them and delete all others; do the same for Y. By construction, the reduced game form $g^0 : X' \times Y' \to O$ has only basic strategies; furthermore, $A(g^0) = A^0(g)$ and $B(g^0) = B^0(g)$ are dual Sperner hypergraphs, which are (unlike g^0) uniquely defined by g .

The following well-known technical statement plays an important role.

Lemma 1. *Given a tight game form g, a basic strategy $x \in X$, and an outcome $o \in g(x)$, there exists a basic strategy $y \in Y$ such that $g(x) \cap g(y) = \{o\}$.*

Proof. Indeed, otherwise for each $y \in Y$ one could choose an outcome $o(y) \in g(y) \cap g(x)$ distinct from o. Then, by (jj′) there exists an $x' \in X$ such that $g(x') \subseteq g(x) \setminus \{o\} \subset g(x)$. Hence, strategy x is not basic, a contradiction. □

The following remarks are obvious: in the statement of the lemma, x and y can be swapped; the obtained situation (x, y), with $g(x, y) = \{o\}$, is simple.

The lemma shows that, given a tight game form g with multi-hypergraphs $A(g)$ and $B(g)$, their Sperner reductions $A^0(g)$ and $B^0(g)$ are dual Sperner hypergraphs uniquely defined by g; in fact, by the basic strategies and simple situations of g.

Furthermore, given a game $(g; u, w)$ with a tight game form g, one can choose an outcome $o = o(x, y) \in g(x) \cap g(y)$ in every not simple situation (x, y) in such a way that none of them is a NE in the obtained game $(g'; u, w)$.

1.5. Tightness and solvability

It is not difficult to see that property (iii) win-lose solvability implies (iv) tightness. In [11] (see also [13]) it was shown that tightness implies (i) Nash-solvability and, hence, all four properties $(i$ - $iv)$ are equivalent.

Several different proofs of $(iv) \Rightarrow (i)$ based on alternative ideas appeared later [2, 3, 6, 17, 19]. Equivalence of (ii), (iii), and (iv) is implicit already in the Edmonds and Fulkerson paper [7] and explicit in [10].

In Section 5 of [13], by means of Lemma 1, implication $(iv) \Rightarrow (i)$ was strengthened as follows:

Proposition 1. *Every game $(g; u, w)$ with a tight game form g has a simple NE (x, y) in basic strategies.* $\square$

In fact, the proof given in [11, 13] allows us to strengthen the results further, specifying more important properties of the obtained NE considered in the next two subsections.

1.6. Lexicographically safe strategies of the players

First, let us remark that without loss of generality (wlog) we can assume that u and w have no ties, or in other words, each of them forms a complete order over O. Clearly, one can get rid of all ties by small perturbations of values of u and w. By definition, the set of NE will be either unchanged or reduced by such perturbations. Since we focus on NE that always exist (for all payoffs, what is called Nash-solvability) we can assume that both payoffs, u and w, have no ties and replace them by complete orders over O.

Consider wlog Alice's payoff u and introduce a lexicographical order $\prec$ over the Boolean 2^O (all subsets of O) based on the priority of eliminating bad outcomes (ones with small values of u). To compare two subsets $O', O'' \subseteq O$ consider two differences $O' \setminus O''$ and $O'' \setminus O'$. Their union $\Delta = (O' \setminus O'') \cup (O'' \setminus O') \neq \emptyset$ is not empty whenever O' and O'' are distinct. Denote by o^* the (unique) outcome in Δ minimizing the value of u.

If $o^* \in O' \setminus O''$ then $O' \prec O''$; if $o^* \in O'' \setminus O'$ then $O'' \prec O'$.

By definition, in this order, a set is smaller than any its subset; in particular, the empty set is the maximum one.

Given g and u, introduce the lexicographical pre-order over Alice's strategies $x \in X$ as follows: the larger is the support $g(x)$ in order $\prec$, the safer is x for Alice, while strategies with the same support are equally safe. (Recall that both players are maximizers.)

Among Alice's strategies those that maximize $g(x)$ will be called her *lexicographically safe (lexsafe)* strategies. It is important to notice that only basic strategies can be lexsafe. Indeed, by definition, x is safer than x' whenever $g(x) \subset g(x')$. It is even more important that Alice's lexsafe strategies are defined by g and u, while Bob's payoff w is irrelevant.

Similarly, using y, w instead of x, u, we define the lexsafe strategies of Bob.

Remark 2. *The concept of a lexsafe strategy is a refinement of the classical concept of a safe (maxmin) strategy. The latter optimizes only the worst case scenario outcome, while the former one optimizes the whole set of outcomes in the lexicographical order defined above.*

1.7. Lexsafe Nash equilibria in games with tight game forms

Recall the proof of $(iv) \Rightarrow (i)$ from [11, 13]. It is constructive: a NE (x, y) is obtained such that y is a lexsafe strategy of Bob, while x is a best response of Alice to this strategy. (Note that some best responses may not fit, since, by the definition of an NE, y must also be a best response to x. However, at least one required response x exists.) Such NE will be called *lexsafe for Bob* and the set of all his lexsafe NE will be denoted by NE-B.

As we already mentioned, there may be several lexsafe strategies but they all are basic and have the same support. Respectively, NE-B may have several equilibria but they all have the same outcome, which will be called the *NE-B outcome.* Indeed, for all Bob's lexsafe NE (x, y) the support $g(y)$ and the best Alice's outcome $g(x, y) \in g(y)$ are the same (that is, well defined). Moreover, they are defined by g and w only, while u is irrelevant.

Note, however, that x, unlike y, may be not basic.

Given a game $(g; u, w)$ with a tight game form g, denote by SNE-B the subset of all simple situations of NE-B. We will show that this subset is not empty and characterize it explicitly.

Consider a not simple $(x, y) \in$ NE-B. By definition, there exist distinct $o, o' \in g(x) \cap g(y)$. Assume wlog that $g(x, y) = o$ and replace it by o'. By assumption, (x, y) is a NE in the original game $(g; u, w)$. Yet, it is no longer a NE in the obtained game. Indeed, in the latter each player can change her/his strategy getting o back, instead of o', and make profit.

Do so for all not simple situations $(x, y) \in$ NE-B. Then, NE-B = SNE-B in the new game $(g'; u, w)$. New game form g' is still tight, since by Lemma 1, $A(g) = A(g')$ and $B(g) = B(g')$. Hence, g' is Nash-solvable, according to [11, 13]. Thus, SNE-B is not empty.

Let us show that SNE-B form a box $X_B^* \times Y_B^*$ in $X \times Y$. Denote by Y_B^* the set of lexsafe strategies of Bob. Recall that all these strategies have the same support. Let o^* be the (unique) NE-B outcome. We already know that there exists a strategy $x \in X$ of Alice such that (x, y^*) is a simple NE for some $y^* \in Y_B^*$. Then, obviously, the same holds for all $y^* \in Y_B^*$, since all these strategies have the same support. Denote by X_B^* the set of all Alice's strategies $x \in X$ having the above property. Clearly, $x' \in X_B^*$ whenever $g(x') \subseteq g(x)$ for some $x \in X_B^*$. Indeed, $g(x) \cap g(y) = \{g(x, y\} = o^*$ implies that $g(x') \cap g(y) = \{g(x', y\} = o^*$, since the latter intersection cannot be empty and is a subset of the former one.

Thus, we proved that SNE-B $= X_B^* \times Y_B^*$.

Note that, unlike Y_B^*, set X_B^* may contain non basic strategies but it must contain some basic strategies, too.

Similarly, we introduce sets NE-A and SNE-A of the lexsafe and simple lexsafe NE of Alice just by swapping y, w and x, u. We can strengthen the main result of [13] as follows.

Theorem 1. *Every game $(g; u, w)$ with a tight game form g has two non-empty sets of simple lexmin equilibria of Alice and Bob, respectively:*

SNE-A $= X_A^* \times Y_A^*$ *and SNE-B* $= X_B^* \times Y_B^*$.

All strategies of X_A^* and Y_B^* and some strategies of X_B^* and Y_A^* are basic. □

Sets NE-A and NE-B may intersect and even coincide. For example, if game $(g; u, w)$ has a unique NE (x, y) then NE-A = NE-B = $\{(x, y)\}$. Another such case is considered in the next subsection.

1.8. Zero-sum payoffs

Consider a zero-sum game $(g; u, w)$ with a game form $g : X \times Y \to O$ and payoffs $u : O \to \mathbb{R}$ and $w : O \to \mathbb{R}$ such that $u(o) + w(o) = 0$ for all $o \in O$. Since $w = -u$, we will restrict ourselves to u only and denote the game by (g, u) rather than $(g; u, w)$. Alice remains maximizer, but Bob becomes minimizer (of u), since he maximizes w. The results that Alice and Bob can guarantee, in the worst case scenario, called *maxmin* and *minmax*, are standardly defined as follows:

$$\max\min = \max_{x \in X} \min_{o \in g(x)} u(o); \ \min\max = \min_{y \in Y} \min_{o \in g(y)} u(o). \tag{1}$$

The strategies $x \in X$ and $y \in Y$ realizing maxmin and minmax in (1) are called *safe*. Denote by X' and Y' the sets of all safe strategies of Alice and Bob, respectively.

A NE in a zero-sum game (g, u) is usually referred to as a *saddle point*. It exists if and only if maxmin = minmax, which holds whenever game form g is tight. The latter result is due to Edmonds and Fulkerson [7]; see also [10].

All situations $(x, y) \in X' \times Y'$ are saddle points. Moreover, payoff function $u(g(x, y))$ is a constant on $X' \times Y'$. This constant is called the *value* of the zero-sum game (g, u).

Wlog we can assume that u have no ties on O. Indeed, if $u(o) = u(o')$ then we just merge these two outcomes in O getting O'. This operation preserves tightness. Applying such merging recursively, we get rid of all ties of u. Then, function g becomes a constant on $X' \times Y'$. In other words, there is a unique saddle point outcome.

Proposition 2. *In a zero-sum game (g, u), all situations of $X' \times Y'$ are simple.*

Proof. Assume for contradiction that $(x, y) \in X' \times Y'$ is not simple, that is, $g(x, y) = o$ and both $g(x)$ and $g(y)$ contains an outcome o' distinct from o. Clearly, o' is better that o either for Alice or for Bob, since the considered game is zero-sum. Hence (x, y) is not a saddle point, which is a contradiction. □

Denote by X'' and Y'' the sets of all lexsafe strategies of Alice and Bob, respectively. Obviously, the following containments hold (and may be strict):

$$X'' \subseteq X' \subseteq X \text{ and } Y'' \subseteq Y' \subseteq Y.$$

As we already mentioned, lexsafe strategies can be viewed as a refinement of safe ones: the former optimize the outcome, while the latter optimize lexicographically the whole set of possible outcomes, in the worst case scenario.

As we know, all lexsafe strategies are basic. In contrast, a safe one may be not basic, however, if $x \in X$ is safe and $g(x') \subseteq g(x)$ then x' is safe too. Similarly for y.

Since $X' \times Y'$ are saddle points and $X'' \times Y'' \subseteq X' \times Y'$, we conclude that $X'' \times Y''$ are saddle points too.

Above observations imply the following characterization of the set of lexsafe NE in the zero-sum case.

Theorem 2. *Equalities NE-A = NE-B = $X'' \times Y''$ hold for every zero-sum game (g, u). This set is not empty whenever g is tight.* □

1.9. Non-zero-sum payoffs. An important example

Yet, in general, sets NE-A and NE-B may be disjoint and a pair of lexsafe strategies of Alice and Bob may be not a NE.

Consider game form g_1 in Figure 1. It is tight. Define payoffs u and w such that $u(o_2) > u(o_1) > u(o_3)$ and $w(o_2) > w(o_3) > w(o_1)$. It is easy to verify that x_1 and y_1 are lexsafe strategies of Alice and Bob, respectively. Yet, situation (x_1, y_1) is not an NE. Alice can improve her result $g_1(x_1, y_1) = o_1$ switching to x_2 and getting $g(x_2, y_1) = o_2$, which is better for her. Thus, two lexsafe strategies, of Alice and Bob, do not form an NE. However, sets NE-A and NE-B are not empty, in accordance with Theorem 1: NE-A = $\{(x_2, y_1)\}$ and NE-B = $\{(x_1, y_2)\}$. The corresponding NE outcomes are o_1 and o_2, respectively.

Note that o_2 is the best outcome for both players. Thus, NE-A is not Pareto-optimal.

Remark 3. *One could conjecture that each player prefers lexsafe NE of the opponent to his/her own. Such a result would be similar to the analogous one from the matching theory; see, for example, [20]. There are two types of stable matchings given by the Gale-Shapley algorithm [9], depending on men propose to women or vice versa. Yet, this conjecture is disproved by the above example.*

1.10. Game forms and game correspondences

A *game correspondence* is defined as an arbitrary mapping $G : X \times Y \to 2^O \setminus \{\emptyset\}$, that is, G assigns a non-empty subset of outcomes to each situation.

Given G, define a game form $g \in G$, choosing an arbitrary outcome $g(x, y) \in G(x, y)$ for each situation (x, y). Conversely, given a game form $g : X \times Y \to O$, define a game correspondence G setting $G(x, y) = g(x) \cap g(y)$. Then, obviously, $g \in G$.

By Lemma 1, if at least one $g^* \in G$ is tight then every $g \in G$ is tight. In this case G is called *tight* too. Furthermore, all $g \in G$ have the same Sperner reduced dual hypergraphs $A^0(g)$ and $B^0(g)$, same simple situations, and for any given payoffs u and w, the same sets of simple situations in NE-A and NE-B.

2. Computing lexsafe NE in polynomial time

If game form $g : X \times Y \to O$ is given explicitly then to find all its NE is simple: one can just consider all situations $(x, y) \in X \times Y$ one by one verifying Nash's definition for each of them. Yet, in applications g is frequently given by an oracle $\mathcal{O}$ such that size of g is exponential in size $|\mathcal{O}|$ of this oracle. Then, the straightforward search for NE suggested above becomes not efficient. Four such oracles will be considered in section 4.

Yet, certain properties required from oracle $\mathcal{O}$ allow us to construct an algorithm computing two lexsafe NE (from NE-A and NE-B, respectively) for every game $(g; u, w)$ with tight game form $g = g(\mathcal{O})$ realized by $\mathcal{O}$, in time polynomial in $|\mathcal{O}|$.

2.1. Requirements to oracles

(I) Oracle $\mathcal{O}$ contains explicitly all outcomes O of g. (Yet, strategies $x \in X$ and $y \in Y$ are implicit in $\mathcal{O}$; moreover, $|X|$ and/or $|Y|$ may be exponential in its size $|\mathcal{O}|$.)

(II) Game form $g = g(\mathcal{O})$ generated by $\mathcal{O}$ is tight.

(III) Every win-lose game $(g(\mathcal{O}); O_A, O_B)$ can be solved in time polynomial in $|\mathcal{O}|$.

Requirement (III) needs a discussion. By tightness of g, exactly one of the following two options holds:

(a) there exists $x \in X$ with $g(x) \subseteq O_A$ (Alice wins);
(b) there exists $y \in Y$ with $g(y) \subseteq O_B$ (Bob wins).

To solve a win-lose game means to determine which option, (a) or (b), holds and to output a winning strategy, x or y, respectively.

Notice that it is possible to output basic winning strategies whenever (III) holds. Indeed, suppose Alice wins and we output her winning strategy x, with $g(x) \subseteq O_A$. Reduce O_A by one output o moving it to O_B, solve the obtained win-lose game, and repeat the procedure for all $o \in O_A$. If Bob wins in all obtained games then x is already a basic winning strategy of Alice. Otherwise we can move an outcome o from O_A to O_B and Alice still wins. Repeating, we obtain a basic winning strategy of Alice (in the original game) in at most $|O_A|$ iterations. The same works for Bob.

3. Computing two lexsafe NE, from NE-A and NE-B, respectively

Let us compute a lexsafe strategy of Bob. Recall that it depends only on his payoff w, while u is irrelevant. Wlog we may assume that $w(o_1) < \ldots < w(o_p)$. (Recall that we can restrict ourselves by strict inequalities; see the beginning of Section 1.6.)

Let us verify whether Bob can exclude his worst k outcomes $O_k = \{o_1, \ldots, o_k\}$. To do so, set $O_A = O_k$, $O_B = O \setminus O_k$, and consider the obtained win-lose game $(g; O_A, O_B)$. Using the oracle, verify if Bob wins. By construction, he does if and only if he can exclude his worst k outcomes O_k in the original game $(g; u, w)$. Applying the dichotomy among $k \in \{1, \ldots, p\}$ obtain the maximal k_0 such that Bob can exclude O_{k_0}. (In particular, $k_0 = 0$ may hold.) This implies that Bob's lexsafe strategy excludes all outcomes from O_{k_0}, but cannot exclude o_{k_0+1} in addition. Hence, we can set for good $O_{k_0} \subseteq O_A$, while $o_{k_0+1} \in O_B$, and repeat the same procedure with the remaining outcomes $O \setminus O_{k_0+1}$. In at most $p = |O|$ iterations, we obtain the unique "lexsafe" subset $O^* \subseteq O$ that Bob is able to exclude. And he must, in order to play "lexsafely". Furthermore, oracle $\mathcal{O}$ will provide, in time polynomial in $|\mathcal{O}|$, the corresponding strategy $y^* \in Y$ with $g(y^*) = O \setminus O^*$. This is Bob's lexsafe strategy. It may be not unique, in contrast to his lexsafe subset $O^* \subseteq O$.

Obviously, y^* is a basic strategy. Indeed, any strategy y^{**} such that $g(y^{**}) \subset g(y^*)$ would be "lexsafer" than y^*.

Note that the above algorithm computes a lexsafe strategy of Bob, given his payoff w and an arbitrary game form g, not necessarily tight. Yet, computing an NE we must use tightness of g, since it is equivalent to Nash-solvability of g.

It was proven in [11] that for any Bob's lexsafe strategy y^*, Alice has a strategy $x^* \in X$ such that the situation (x^*, y^*) is an NE in game $(g; u, w)$ whenever g is tight. Then, in Section 5 of [13], this result was strengthened as follows: a basic such x^* and a simple lexsafe NE (x^*, y^*) exist (that is, $g(x^*) \cap g(y^*) = \{g(x^*, y^*)\}$).

By the definition of an NE, Alice's x^* must be her best response to Bob's y^*, yet, it may be not unique. (Of course, y^* is a best response to x^* as well.)

We still need to construct x^* in time polynomial in $|\mathcal{O}|$. Game form g is tight by assumption (II). By Theorem 1, NE-B contains a simple NE (x^*, y^*) in basic strategies. Strategy y^* and its support $g(y^*)$ are already known. Although x^* is not, but we know that x^* is a best response to y^* and, hence, can compute the corresponding outcome $o^* = g(x^*, y^*) \in g(y^*)$. Since NE (x^*, y^*) is simple, we have $g(x^*) \cap g(y^*) = o^*$. Let $O_B^* = \{o \in O \mid w(o) < w(o^*)\}$ be the set of outcomes that are worse that o^* for Bob. Set $O_A = (O_B^* \setminus g(y^*)) \cup \{o^*\}$, $O_B = O \setminus O_A$, and consider the win-lose game $(g; O_A, O_B)$. Alice wins, since by Theorem 1, NE-B is not empty and, moreover, it contains a simple NE. The oracle outputs (in time polynomial in $|\mathcal{O}|$) a simple winning strategy x^* of Alice. By definition of O_A, all outcomes of $g(x^*)$ are not better than o^* for Bob. Hence, y^* is a best response to x^*. Recall that, vice versa, x^* was defined as a best response to y^*. Thus, (x^*, y^*) is a NE and, by construction, $(x^*, y^*) \in$ NE-B.

The obtained algorithm calls the oracle at most $O(p \log p)$ times, where $p = |O|$, and the oracle answers (solves a suggested win-lose game) in time polynomial in $|\mathcal{O}|$, by assumption (III).

Similarly, a lexsafe strategy from NE-A is computed. Thus, the following statement is proven.

Theorem 3. *Given an oracle $\mathcal{O}$ satisfying requirements (I,II,III), lexsafe NE of Alice and of Bob exist and can be computed in time polynomial in $|\mathcal{O}|$.* $\square$

4. Examples of oracles

Here we consider four types of oracles and verify that all four satisfy requirements (I, II, III).

4.1. Deterministic graphical multi-stage game structures

Let $\Gamma = (V, E)$ be a directed graph (digraph) whose vertices and arcs are interpreted as positions and moves, respectively. Furthermore, denote by V_T the set of terminal positions, of out-degree zero, and by V_A, V_B the positions of positive out-degree, controlled by Alice and Bob, respectively. We assume that $V = V_A \cup V_B \cup V_T$ is a partition. A strategy $x \in X$ of Alice (resp., $y \in Y$ of Bob) is a mapping that assigns to each position $v \in V_A$ (resp., $v \in V_B$) an arbitrary move from this position. An initial position $v_0 \in V_A \cup V_B$ is fixed. Each situation (x, y) defines a unique a walk that begins and v_0 and then follows the decisions made by x and y. This walk $P(x, y)$ is called a *play*. Each play either terminates in V_T or is infinite. In the latter case, it forms a "lasso": first, an initial path, which may be empty, and then a directed cycle (dicycle) repeated infinitely (This holds, because we restrict players by their stationary strategies, that is, a move may depend only on the current position but not on previous positions and/or moves).

The positional structure defined above can also be represented in normal form. We introduce a game form $g : X \times Y \to O$, where, as before, O denotes a set of outcomes. Yet, there are several ways to define this set. One is to "merge" all infinite plays (lassos) and consider them as a single outcome c, thus, setting $O = V_T \cup \{c\}$. This model was introduced by Washburn [28] and called *deterministic graphical game structure* (DGGS).

The following generalization was suggested in [17]. Digraph Γ is called *strongly connected* if for any $v, v' \in V$ there is a directed path from v to v' (and, hence, from v' to v, as well). By this definition, the union of two strongly connected digraphs is strongly connected whenever they have a common vertex. A vertex-inclusion-maximal strongly connected induced subgraph of Γ is called its *strongly connected component* (SCC). In particular, each terminal position $v \in V_T$ is an SCC. It is both obvious and well-known that any digraph $\Gamma = (V, E)$ admits a unique decomposition into SCCs: $\Gamma^o = \Gamma[V^o] = (V^o, E^o)$ for $o \in O$, where O is a set of indices. Furthermore, partition $V = \cup_{o \in O} V^o$ can be constructed in time linear in the size of Γ, that is, in $(|V| + |E|)$. It has numerous applications; see [26, 27] for more details. One more was suggested in [17]. For each $o \in O$, contract the SCC Γ^o into a single vertex v^o. Then, all edges of E^o (including loops) disappear and we obtain an acyclic digraph $\Gamma^* = (O, E^*)$. The set O can be treated as the set of outcomes. Each situation (x, y) uniquely defines a play $P = P(x, y)$. This play either comes to a terminal $v \in V_T$ or forms a lasso. The cycle of this lasso is contained in an SCC o of Γ. Each terminal is an SCC as well. In both cases an SCC $o \in O$ is assigned to the play $P(x, y)$. Thus, we obtain a game form $g : X \times Y \to O$, which is the normal form of the *multi-stage DGGS* (MSDGGS) defined by Γ.

An SCC is called *transient* if it is not a terminal and contains no dicycles. No play can result in such SCC; or in other words it does not generate an outcome. For example, $O = V_T$ in an acyclic digraph, while all remaining SCC are transient.

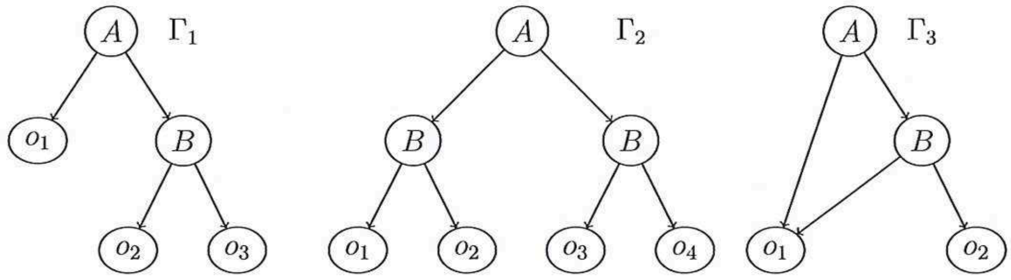
We will demonstrate that both oracles, DGGS and MSDGGS, satisfy the requirements (I, II, III). Indeed, (I) holds since the terminals, as well as SCCs, of a given digraph Γ can be generated in time linear in the size of Γ.

Both requirements, (II) and (III), for both oracles, DGGS and MSDGGS, can be verified simultaneously. Consider the corresponding game forms g' and g and note that g' is obtained from g by merging some outcomes. Namely, all outcomes corresponding to the non-terminal SCCs are replaced by a single outcome c. Obviouslly, merging outcomes respects tightness. Thus, it is enough to verify (II) and (III) for MSDGGSs, By theorem, 1, it is sufficient to prove the win-lose solvability to verify (II). For DGGS it was done in [28]; see also [3, Section 3], [1], [5, Section 12]. The result was extended to MSDGGS in [17]. All proofs were constructive, the corresponding win-lose games were solved in time polynomial in the size of Γ, which implies also (III).

For reader's convenience, we briefly sketch here the proof of (II,III) from [17]. Consider a win-lose game $(g; O_A, O_B)$ with game form $g = g(\mathcal{O})$ generated by a MSDGGS oracle $\mathcal{O}$. We would like to apply the Backward Induction, yet, digraph Γ may have dicycles. So we modify Backward Induction to make it work in presence of dicycles. Recall that O is the set of SCCs of Γ and $\Gamma^* = (O, E^*)$ is acyclic. Consider an SCC $o = \Gamma' = (V', E')$ in Γ that is not terminal, but each move (v', v) from a position $v' \in V'$ either ends in a terminal $v \in V_T$, or stays in Γ', that is, $v' \in V'$. Wlog assume that $o \in O_A$, that is, Alice wins if the play cycles in Γ'. Then Bob wins in a position $v' \in V'$ if and only if he can force the play to come to a terminal $v \in O_B$ and Alice wins in all other positions of V'. Note that for Alice it is not necessary to force the play to come to O_A, it is enough if it cycles in Γ'. Thus, every position of Γ' can be added either to O_A or to O_B. Then we eliminate all edges E' of Γ' and repeat until the initial position v_0 of Γ is evaluated. This procedure proves solvability of game form $g(\mathcal{O})$ and solves a win-lose game $(g; O_A, O_B)$ in time linear in the size of $\mathcal{O} = \Gamma$.

Acyclic deterministic graphical game structures

A game form is called *rectangular* if all its situations are simple. It is shown in [12] that a game form g is generated by a DGGS whose graph is a tree if and only if g is tight and rectangular. Two examples, Γ_1 and Γ_2 are given in Figure 2. They generate game forms g_1 and g_2; see also Figure 1. More



o_1	o_1
o_2	o_3

g_1

o_1	o_1	o_2	o_2
o_3	o_4	o_3	o_4

g_2

o_1	o_1
o_1	o_2

g_6

Fig. 2. Acyclic deterministic graphical game structures and corresponding game forms

examples can be found in Section 3 of [15], where the above characterization is extended to the n-person case.

Acyclic DGGS Γ_1 in Figure 2 generates game form g_1; see also Figure 1. Recall game $(g_1; u, w)$ from Section 1.9, with

$$u(o_2) > u(o_1) > u(o_3) \text{ and } w(o_2) > w(o_3) > w(o_1).$$

Notice that the Backward Induction NE (see [8, 21] and also [16]) is NE-A and is not Pareto-optimal. In general, this NE may differ from both, NE-A and NE-B.

Acyclic DGGS Γ_3 in Figure 2 generates game form g_6; see also Figure 1.

Clearly, in absence of dicycles in Γ, the concepts of DGGS and MSDGGS coincide. It is also clear that an acyclic DGGS is a special case of MSDGGS. Thus, properties (I, II, III) required from an oracle hold for both.

Cyclic deterministic graphical game structures

The outcomes of MSDGGS are its SCCs. In particular, each terminal position is an outcome. Let us now assume that every simple dicycle is a separate outcome (and each terminal remains an outcome as well). Such DGGSs, called *cyclic*, were studied in [5]; see also several earlier papers cited there. Cyclic DGGS can also serve as oracles generating game forms; see examples in Figures 1 and 2 of [5]; compare examples 3 and 4 in Figure 2 there with game forms g_4 and g_5 in Figure 1.

Game forms generated by the cyclic DGGS may be not tight; see Figure 1 in [5]. In other words, property (II) fails for the corresponding oracles, in general. Yet, it holds in some important special cases.

A digraph $G = (V, E)$ is called *symmetric* if $(v, v') \in E$ whenever $(v', v) \in E$. Cyclic DGGS on symmetric digraphs are called *symmetric*. Symmetric Cyclic DGGSs satisfying (II) (and called *solvable*) are explicitly characterized in [5]; see sections 1-3 and Theorems 1-3.

It follows from the the results of [5] that (III) also holds for solvable cyclic symmetric DGGS. Hence, Theorem 3 is applicable.

4.2. Jordan oracle; choosing Battlefields in Wonderland

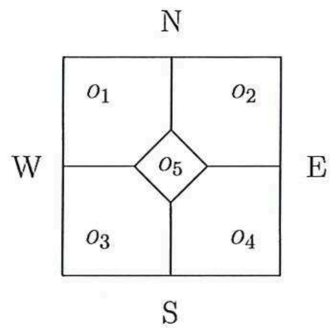
Wonderland is a subset of the plane homeomorphic to the closed disc. Wlog, we can consider a square Q with the sides N, E, S, W, as in Figures 3 and 4. Let us partition Q into areas $O = \{o_1, \ldots, o_p\}$ each of which is homeomorphic to the closed disc, too. Every two distinct areas $o_i, o_j \in O$ are either disjoint or intersect in a set homeomorphic to a closed interval that contains more than one point. Equivalently, we can require that the borders of the areas in O form a regular graph of degree 3. (Note that four vertices of the square are not vertices of this graph.) Two examples are given in Figures 3 and 4.

Remark 4. *Consider game form g_5 in Figure 1 and merge outcomes o_5 and o_6 in it getting g'_5. (This operation respects tightness). Note that $g'_5 \in G$, where G is the game correspondence given in Figure 3. See also Figure 4 of [5], where g_5 also appears as the normal form of a cyclic game form.*

The following interpretation was suggested in [19]. Two players, Alice Tweedledee and Bob Tweedledum, agreed to have a battle. The next thing to do is to agree on a battlefield, which should be an area $o \in O$. The strategies $x \in X$ of Alice (resp., $y \in Y$ of Bob) are all inclusion-minimal subsets $x \subseteq O$ (resp., $y \subseteq O$) connecting W and E (resp., N and S). Any two such subsets x and y intersect, by the Jordan curve theorem. The intersection may contain several areas of O. Hence, a game correspondence $G : X \times Y \to 2^O \setminus \{\emptyset\}$ is defined. It is easy to show that G is tight. To do so, choose an arbitrary $g \in G$ and consider a win-lose game given by a partition $O = O_A \cup O_B$. Again, by the Jordan curve theorem, from the following two options exactly one holds:

(a) areas from O_A connect W and E;
(b) areas from O_B connect N and S.

The above observations imply that Jordan oracle $\mathcal{O}$ satisfies requirements (I) and (II). It remains to verify (III), that is, we have to decide, in time polynomial in $p = |O|$, whether (a) or (b) holds and find some corresponding x or y, respectively. To do so, consider all areas from O_B boarding N, then



o_1	o_2	o_1	o_2
o_3	o_4	o_4	o_3
o_1	o_4	$o_1 o_4 o_5$	o_5
o_3	o_2	o_5	$o_2 o_3 o_5$

Fig. 3. The Jordan game correspondence of the map of Wonderland.

add all areas from O_B boarding these areas, etc. Such iterations will stop in time linear in p either reaching S (then, obviously, (b) holds) or not (then, (a) holds by the Jordan curve theorem). Moreover, in the first case we obtain a set of areas y' from O_B connecting N and S; in the second case - a set of areas x' from O_A connecting W and E. The former strategy y' is obtained explicitly; the latter one, x', is also easy to construct. To do so, denote by O'_B the set of areas obtained in the course of iterations. It does not reach S. Hence, the areas from O_A that border O'_B connect W and E, by the Jordan curve theorem. This case is realized in Figure 4; Alice wins.

Then, in linear time we can reduce y' (or resp. x') to an inclusion-minimal set y (resp., x) connecting N and S (resp. W and E), thus, getting the basic strategies of Bob (resp., Alice). To do so, we eliminate areas from y' (resp., x') one by one until (b) (resp., (a)) still holds.

Remark 5. *We required inclusion-minimality for the subsets $x, y \in O$ just to reduce the number of strategies (which may still remain exponential in p). This requirement can be waved.*

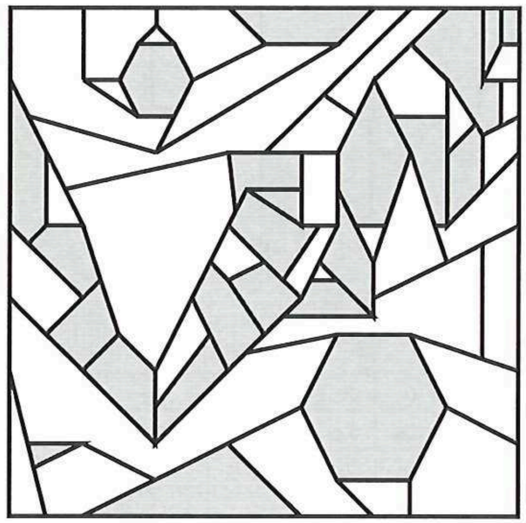


Fig. 4. Gray and white areas are in O_A and O_B, respectively. Alice wins.

4.3. Monotone bargaining schemes

The following oracle was introduced in [19]. Two players, Alice and Bob, possess items $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$, respectively. Both sets are ordered: $a_1 \prec \ldots \prec a_m$ and $b_1 \prec \ldots \prec b_n$. Both players know both orders.

The direct product $O = A \times B = \{(a, b) \mid a \in A, b \in B\}$ is the set of *outcomes.*

Alice's strategies are monotone non-decreasing mappings $x : A \to B$ (that is, $x(a) \geq x(a')$ whenever $a > a'$) showing that she is ready to exchange a for $x(a)$ for any $a \in A$. Similarly, Bob's strategies are monotone non-decreasing mappings $y : B \to A$ (that is, $x(a) \geq x(a')$ whenever $a > a'$) showing that he is ready to exchange b for $y(b)$ for any $b \in B$.

It is not difficult to compute the cardinalities of the sets of strategies and outcomes:

$$|X| = \binom{m+n-1}{m}, \ |Y| = \binom{m+n-1}{n}; \quad |O| = |A \times B| = mn. \qquad (2)$$

Given a situation (x, y), an outcome $(a, b) \in O$ is called a *deal* (in this situation) if $x(a) = b$ and $y(b) = a$. Denote by $G(x, y) \subseteq O$ the set of all deals in situation (x, y). We will show that $G(x, y) \neq \emptyset$. Yet, it may contain several deals.

This construction is called a *monotone bargaining (MB) scheme.* It can be viewed as an oracle $\mathcal{O}$ generating game correspondence $G : X \times Y \to 2^O \setminus \{\emptyset\}$. By (2), requirement (I) holds for $\mathcal{O}$.

Note that $G = G_{m,n}$ is uniquely defined by m and n. Any $g \in G$ is called a *MB game form.*

For example, if $m = n = 3$ then $|X| = |Y| = 3$ and we obtain game form g_4 in Figure 1; game correspondence $G(x, y)$ is given in Figure 1a of [19].

The following interpretation was suggested in [19]. Alice and Bob are dealers possessing the sets of objects A and B, respectively, and a deal $(a, b) \in A \times B$ means that they exchange a and b. They may be art-dealers, car dealers; or one of them may be just a buyer with a discrete budget. For example, $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$ may be paintings or sculptures ordered in accordance with their age (not price or value).

To any pair of mappings $x : A \to B$ and $y : B \to A$ (not necessarily monotone non-decreasing) let us assign a bipartite digraph $\Gamma = \Gamma(x, y)$ on the vertex-set $A \cup B$ as follows: $[a, b)$ (respectively, $[b, a)$) is an arc of $\Gamma(x, y)$ whenever $x(a) = b$ (respectively, $y(b) = a$).

Some visualization helps. Embed $\Gamma(x, y)$ into a plane; putting ordered A and B in two parallel columns. Two arcs corresponding to x may have a common head, but not tail. Furthermore, they cannot cross whenever mapping x is monotone non-decreasing. Similarly for y.

By construction, digraph Γ is bipartite, with parts A and B. Hence, every dicycle in Γ is even. There is an obvious one-to-one correspondence between the dicycles of length 2 (or 2-*dicycles*, for short) in $\Gamma(x, y)$ and the deals of $G(x, y)$.

Proposition 3. *For each situation (x, y) its digraph $\Gamma(x, y)$ contains at least one dicycle of length 2 (a deal) and cannot contain longer dicycles.*

Proof. For any initial vertex $v \in A \cup B$, strategies x and y uniquely define an infinite walk from v, which is called a play. Since sets A and B are finite and there are no terminals, this play is a *lasso*: it consists of an initial directed path, which may be empty, and a dicycle C repeated infinitely. Furthermore, C must be a 2-dicycle whenever mappings x and y are monotone non-decreasing. Indeed, if C is longer than 2 then either x, or y, or both are not monotone, since crossing arcs appear. □

Consider a win-lose MB game $(g; O_A, O_B)$, where $g = g(\mathcal{O})$ is an MB game form generated by an MB scheme $\mathcal{O}$. As we already mentioned, requirement

(I) holds for $\mathcal{O}$. The following statement shows that (II) and (III) hold as well.

Proposition 4. *Game form $g = g(\mathcal{O})$ is tight and each win-lose MB game $(g; O_A, O_B)$ can be solved in time polynomial in $|\mathcal{O}| = mn$.*

The first part was already proven in [19]. Yet, here we provide a much shorter proof.

Proof. For the sake of simplicity, we will slightly abuse notation writing that both directed edges $[a, b)$ and $[b, a)$ are in O_A or in O_B whenever the corresponding deal (a, b) is in O_A or in O_B, respectively.

Consider complete bipartite symmetric digraph Γ on $m + n$ vertices $A = \{a_1, \dots, a_m\}, B = \{b_1, \dots, b_n\}$, and with $2mn$ directed edges $\{[a_i, b_j), [b_j, a_i) \mid i = 1, \dots, m; j = 1, \dots, n\}$. The following two statements are obvious.

(a) Alice wins if she has a monotone non-decreasing strategy $x^* : A \to B$ such that $[a, x^*(a)) \in O_A$ for all $a \in A$.

(b) Bob wins if he has a monotone non-decreasing strategy $y^* : B \to A$ such that $(b, y^*(b)) \in O_B$ for all $b \in B$.

Indeed, it is easily seen that x^* and y^* are the winning strategies of Alice and Bob, respectively. It is enough to show that $g(x^*, y) \in O_A$ for any $y \in Y$. Recall the proof of Proposition 3: Fix x^*, choose an arbitrary $y \in Y$, and consider the play $P = P(x^*, y)$ beginning from an arbitrary initial position $v \in A \cup B$. By Proposition 3, P is a lasso resulting in a 2-cycle (a, b). The corresponding deal $(a, b) \in O_A$, in case (a), by the choice of x^*, and Alice wins. Similarly, $g(x, y^*) \in O_B$ in case (b) for any $x \in X$, by the choice of y^*, and Bob wins.

Obviously, (a) and (b) cannot hold simultaneously, since otherwise $(a, b) \in Q_A \cup Q_B$, which is a contradiction, since $O = Q_A \cup Q_B$ is a partition.

Let us show that either (a) or (b) holds, that is, g is tight, which implies (II). The proof will be constructive: we obtain either x^* satisfying (a) or y^* satisfying (b) in time polynomial in mn, which in its turn implies (III). We will construct a play P by the following greedy iterative algorithm. Let $a^1 = a_1$ be an initial position of P. (We use superscripts to number iterations.) If $[a^1, b) \in O_B$ for all $b \in B$ then Bob wins. (His winning strategy y^* is defined by: $y^*(b) = a^1$ for all $b \in B$. Then $[y^*(b), b) \in O_B$ for all $b \in B$ and (b) holds) Otherwise, denote by b^1 the (unique) minimal $b \in B$ such that $[a^1, b) \in O_A$. Then, by definition, $[b^1, a^1) \in O_A$ too. Furthermore, by this choice of b^1, we have: $[b, a^1) \in O_B$ for all $b \prec b^1$, while $[b^1, a^1) \in O_A$.

If $[b^1, a) \in O_A$ for all $a \succeq a^1$ then Alice wins. (Her winning strategy x^* is defined by: $x^*(a) = b^1$ for all $a \in A$. Then $[a, x^*(a)) \in O_A$ for all $a \in A$.) Otherwise, denote by a^2 the (unique) minimal $a \in A$ such that $[b^1, a) \in O_B$.

Then, by definition, $[a^2, b^1) \in O_B$ too. Furthermore, by the choice of a^2, we have: $[a, b^1) \in O_A$ for all $a \prec a^2$, while $[a^2, b^1) \in O_B$.

The general k-th step of this greedy recursion is as follows.

If $[a^k, b) \in O_B$ for all $b \succeq b^{k-1}$ then Bob wins. (His winning strategy y^* is defined by: $y^*(b) = a^i$ for each b such that $b^i \succ b \succeq b^{i-1}$, for $i = 1, \ldots, k$, assuming conventionally that $b \succ b^0$ holds for all $b \in B$).

Otherwise, denote by b^k the (unique) minimal $b \in B$ such that $b \succ b^{k-1}$ and $[a^k, b) \in O_A$. Then $[b^k, a^k) \in O_A$ too.

Furthermore, by the choice of b^k, we have: $[b, a^k) \in O_B$ for all b such that $b^k \succ b \succeq b^{k-1}$, while $[b^k, a^k) \in O_A$.

If $[b^k, a) \in O_A$ for all $a \succeq a^k$ then Alice wins. (Her winning strategy x^* is defined by: $x^*(a) = b^j$ for each a such that $a^{j+1} \succ a \succeq a^j$, for $j = 1, \ldots, k$, assuming conventionally that $a^{k+1} \succ a$ holds for all $a \in A$.)

Otherwise, denote by a^{k+1} the (unique) minimal $a \in A$ such that $[b^k, a) \in O_B$. Then $[a^{k+1}, b^k) \in O_B$, too.

Furthermore, by the choice of a^{k+1}, we have: $[a, b^k) \in O_A$ for all a such that $a^{k+1} \succ a \succeq a^k$, while $[a^{k+1}, b^k) \in O_B$.

Thus, after each iteration a^k (resp., b^k) both Alice and Bob have winning moves in all positions $a \prec a^k$ and $b \preceq b^{k-1}$ (resp., $a \preceq a^k$ and $b \prec b^k$). Since sets A and B are finite, the procedure will stop on some iteration either $a^{k^*} \prec a_m$ or $b^{k^*} \prec b_n$, indicating that Bob or, respectively, Alice wins.

Furthermore, we obtain his or her winning strategy in time linear in mn.

The following slightly different procedure can be applied too. First, we start looking for a winning strategy x^* for Alice. Consider successively $a_1, a_2, \ldots$ and construct (again recursively and greedily) her monotone non-decreasing strategy x^* as follows: $x^*(a_i) = b^i$ such that $[a_i, b^i) \in O_A$, $b^i \succeq b^{i-1}$, and b^i is the minimal element of B satisfying these two properties. If this will work for all $i = 1, \ldots, m$ then Alice wins and we obtain her winning strategy x^* satisfying (a). Otherwise, if the procedure stops on some $i < m$ (no required b^i exists for a_i) then Bob wins. His winning strategy y^* satisfying (b) is defined as follows: $y^*(b) = a^i$ for all b such that $b^{i-1} \preceq b \prec b^i$, where a_i is the smallest a such that $x^*(a) = b^i$, for $j = 1, 2, \ldots$. By convention, $b^0 \prec b$ for all $b \in B$. □

Thus, requirements (I,II,III) hold for the MB schemes and, hence, Theorem 3 is applicable.

4.4. *Veto voting schemes*

Two voters (players), Alice and Bob choose among candidates (options, outcomes) $O = \{o_1, \ldots, o_p\}$. They are assigned some positive integer veto powers and given μ_A and μ_B veto cards, respectively. Each candidate $o \in O$ is assigned an integer positive veto resistance λ_o. We assume that

$$\mu_A + \mu_B + 1 = \lambda_{o_1} + \ldots + \lambda_{o_p}. \tag{3}$$

A strategy of a voter is an arbitrary distribution of her/his veto cards among the candidates. Given a pair of strategies x and y, a candidate $o \in O$ who got at least λ_o veto cards (from Alice and Bob together) is vetoed. From the set $G(x, y)$ of all not vetoed candidates one $g(x, y) \in G(x, y)$ is elected. By (3), $G(x, y) \neq \emptyset$. Thus, we obtain a *veto voting* (VV) scheme $\mathcal{O}$, VV game form $g = g(\mathcal{O})$, and VV game correspondence $G = G(\mathcal{O})$; see Chapter 6 of [22],[25],[14] for more details. By construction, VV schemes are oracles satisfying (I).

For example, game form g_3 in Figure 1 corresponds to the VVS defined by

$$\mu_A = \mu_B = \lambda_{o_1} = \lambda_{o_2} = \lambda_{o_3} = 1.$$

Let us show that requirements (II) and (III) also hold for VV schemes.

Proposition 5. *Each game form g defined by a VV scheme satisfying (3) is tight. Every win-lose game $(g; O_A, O_B)$ can be solved in time polynomial in $|\mathcal{O}| = \log(\mu_A \mu_B \prod_{o \in O} \lambda_o)$.*

Proof. To see this, consider a win-lose game $(g; O_A, O_B)$. By (3), from two options, (a) Alice can veto O_B and (b) Bob can veto O_A, exactly one holds. Alice and Bob win in case of (a) and (b), respectively. Given numbers μ_A, μ_B, and λ_o, $o \in O$, one can decide whether (a) or (b) holds. In each case the winning strategy of Alice or Bob is straightforward: just veto all opponent's candidates, O_B or O_A, respectively. $\square$

Thus, the VV oracles satisfy (I,II,III) and Theorem 3 is applicable.

4.5. Tight game correspondences and forms of arbitrary monotone properties

The most general setting is defined as follows. Given a finite ground set O, consider a family of its subsets $\mathcal{P} \subseteq 2^O$. Standardly, we call $\mathcal{P}$ a *property* and say that a subset $O' \subseteq O$ satisfies $\mathcal{P}$ or not whenever $O' \in \mathcal{P}$ or $O' \notin \mathcal{P}$, respectively. Property $\mathcal{P}$ is called *inclusion monotone non-decreasing*, or simply *monotone*, for short, if $O'' \in \mathcal{P}$ implies $O' \in \mathcal{P}$ whenever $O'' \subseteq O' \subseteq O$. We restrict ourselves to monotone properties.

Define the sets of strategies X of Alice and Y of Bob as follows: $x \in X$ (resp., $y \in Y$) is any [inclusion minimal] subset $O' \subseteq O$ such that $O' \in \mathcal{P}$ (resp., $O \setminus O' \notin \mathcal{P}$).

The restriction in brackets does not matter, it can be waved or kept. In the latter case, sets X and Y may be significantly reduced.

Define a game correspondence $G = G(\mathcal{P})$ by setting $G(x, y) = x \cap y$ for any $x \in X$ and $y \in Y$. It is both obvious and well-known that $G(x, y) \neq \emptyset$ and, moreover, G is tight. Hence, any game form $g \in G$ is tight too.

Thus, requirements (I) and (II) hold automatically whenever a monotone property $\mathcal{P}$ is given by an oracle $\mathcal{O}(\mathcal{P})$. Yet, (III) must be required in addition. In other words, $\mathcal{O}(\mathcal{P})$ must be a polynomial membership oracle, that is, for a given subset $O' \subseteq O$, it must decide if $O \in \mathcal{P}$ in time polynomial in $|O| + |\mathcal{O}(\mathcal{P})|$.

It is easily seen that this general setting includes in particular all four examples of oracles given in this section before. Many other examples can be found in [4, 18].

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