Discrete Applied Mathematics 340 (2023) 53–68

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

Computing lexicographically safe Nash equilibria in finite two-person games with tight game forms given by oracles

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A R T I C L E I N F O

Article history:
Received 3 January 2022
Received in revised form 13 June 2023
Accepted 20 June 2023
Available online 14 July 2023

Keywords:
Nash equilibrium
Nash-solvability
Game form
Tightness
Deterministic graphical game structure
Game in normal and in positional form
Monotone bargaining
Veto voting
Jordan game

A B S T R A C T

In 1975 the first author proved that every finite tight two-person game form g is Nash-solvable, that is, for every payoffs u and w of two players the obtained normal form game $(g; u, w)$ has a Nash equilibrium (NE) in pure strategies. Several proofs of this theorem were obtained later. Here we strengthen the result and give a new proof, which is shorter than previous ones. We show that game $(g; u, w)$ has two types of NE, realized by a lexicographically safe (lexsafe) strategy of one player and some special best response of the other. The proof is constructive, we obtain a polynomial algorithm computing these lexsafe NE. This is trivial when game form g is given explicitly. Yet, in applications g is frequently realized by an oracle $\mathcal{O}$ such that size of g is exponential in the size $|\mathcal{O}|$ of $\mathcal{O}$. We assume that game form $g = g(\mathcal{O})$ generated by $\mathcal{O}$ is tight and that an arbitrary ± 1 *game* $(g; u^0, w^0)$ (in which payoffs u^0 and w^0 are zero-sum and take only values ± 1) can be solved in time polynomial in $|\mathcal{O}|$. These assumptions allow us to compute two (one for each player) lexsafe NE in time polynomial in $|\mathcal{O}|$. These NE may coincide. We consider four types of oracles known in the literature and show that all four satisfy the above assumptions.

1. Introduction

Here we outline main results. Precise definitions will be given later.

Consider a finite n-person game *in normal form* representing it as a pair (g, u), where u is the *payoff* function of n players and g is the so-called *game form*. Respectively, the latter can be viewed as a game without payoffs, which are not given yet. Such approach is standard and convenient: game form g "is responsible" for structural properties of game (g, u), which hold for any payoff u. For example, game form g is called *Nash-solvable* if game (g, u) has a Nash equilibrium (NE) in pure strategies for every payoff u.

In 1950 Nash proved that every n-person normal form game has a NE in *mixed strategies* [39,40]. Yet, there are large families of games solvable in *pure strategies*, for example, *finite n-person positional (graphical) games with perfect information*. Every such game structure Γ uniquely defines a finite n-person game form $g(\Gamma)$ that is Nash-solvable. In Section 8.3 we consider this class of Nash-solvable game forms. Furthermore, we expand the set of outcomes including not only terminal positions but also other strongly connected components of Γ. Doing so, we also expand substantially the corresponding family of game forms, which remain Nash-solvable, but only in case of two players, $n = 2$.

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https://doi.org/10.1016/j.dam.2023.06.030
0166-218X/

Yet, perfect information is only sufficient but not necessary for Nash-solvability. A concept of *tightness* fits much better. This property is of algebraic nature. It was introduced in [20,21] and in the latter paper it was shown that a finite two-person game form is Nash-solvable if and only if it is *tight*. Note that already for $n = 3$ tightness is neither necessary nor sufficient for Nash-solvability. These results were obtained in [21,23]; several different proofs were given later [3,5,11,27,30]. Tightness remains necessary (and, of course, sufficient) for Nash-solvability in the zero-sum case too. This result was obtained earlier: it follows easily from the so-called *Bottleneck Extrema Theorem* by Edmonds and Fulkerson [12]; see also [20].

Here we suggest a new (and much simpler) proof of the general result. We introduce a concept of *lexicographically safe (lexsafe)* pure strategy of a player in a given game (g, u). This is a refinement of the standard concept of a safe (maxmin) strategy that maximizes the worst possible outcome, while the lexsafe strategy realizes the lexicographical maximum of all possible outcomes. Thus, the lexsafe strategies are most safe, but may be not rational. (For comparison, recall that NE may be not Pareto optimal.) One can view this as a price of stability.

We prove that a NE appears whenever one player applies a lexsafe strategy, while the opponent chooses some special *best response* to it. Yet, if both players choose their lexsafe strategies then the obtained pair may be not a NE.

Thus, there are two types of NE: lexsafe for one or for the other player. These NE may coincide. For example, it happens in the zero-sum case, or when the considered game has a unique NE.

By definition, lexsafe strategies of a player do not depend on the payoff of the opponent; the player may be just unaware of it. This is an interesting property important for applications.

In the proof given in [21,23] the lexsafe strategies were implicitly constructed by an iterative algorithm increasing strategies in a lexicographical order. Here we suggest a simple polynomial algorithm searching for a lexsafe strategy of a player and for a corresponding NE. Such algorithm is obvious when a game form g is given explicitly. Yet, in applications g is frequently given by an oracle $\mathcal{O}$, which size may be logarithmic in the size of g. We assume that this oracle solves in polynomial time any two-person game (g, u^0) in which payoff u^0 is zero-sum and takes only values ± 1; oracle $\mathcal{O}$ tells us who wins and determines a winning strategy. Based on this assumption, we provide an algorithm computing a lexsafe NE in an arbitrary game (g, u) in time polynomial in the size of $\mathcal{O}$.

In the last section we consider four examples of such oracles from different areas of game theory and show that all four satisfy the above assumption.

2. Basic definitions

2.1. Game forms

In this paper we consider finite, not necessarily zero-sum, normal form games of two players, Alice and Bob. They have finite sets of strategies X and Y, respectively. A *game form* is a mapping $g : X \times Y \to \Omega$, where Ω is a finite set of outcomes. Several examples are given in Fig. 1, where game forms are represented by tables with rows, columns, and entries labeled by $x \in X$, $y \in Y$, and $\omega \in \Omega$, respectively.

Mapping g is assumed to be surjective, but not necessarily injective, that is, an outcome $\omega \in \Omega$ may occupy an arbitrary array in the table of g.

A pair of strategies (x, y) is called a *situation* (term "strategy profile" is also used in literature). Sets $g(x) = \{g(x, y) \mid y \in Y\}$ and $g(y) = \{g(x, y) \mid x \in X\}$ are called the *supports* of strategies $x \in X$ and $y \in Y$, respectively.

A strategy is called *minimal* if its support is not a proper superset of the support of any other strategy. For example, in g_6 the first strategies of Alice and Bob are minimal, while the second are not; in the remaining eight game forms all strategies are minimal. Moreover, any two strategies of a player have distinct supports, for every game form, except g_7.

A situation (x, y) is called *simple* if $g(x) \cap g(y) = \{g(x, y)\}$. For example, all situations of game forms g_1, g_2, g_8, g_9 are simple (such game forms are called *rectangular*); in contrast, no situation is simple in g_7; in g_3 all are simple, except three on the main diagonal; in g_4 all are simple, except the central one; in g_6 all are simple, except one with the outcome ω_2.

2.2. Payoffs and games in normal form

Payoffs of Alice and Bob are defined by real valued mappings $u : \Omega \to \mathbb{R}$ and $w : \Omega \to \mathbb{R}$, respectively. We assume that both players are maximizers. Triplet $(g; u, w)$ defines a *finite two-person game in normal form*, or just a *game*, for short. Game $(g; u, w)$ and payoffs (u, w) are called:

- *zero-sum* if $u + w = 0$, that is, $u(\omega) + w(\omega) = 0$ for all $\omega \in \Omega$;
- *zero-sum* ± 1 (or just ± 1, for short) if, in addition, functions u and w take only two values 1 and -1.

Alternatively, a ± 1 payoff can be given by a partition $\Omega = \Omega_A \cup \Omega_B$, where Ω_A and Ω_B are the outcomes preferred by Alice and by Bob, respectively:

$$u(\omega) = 1, w(\omega) = -1 \text{ for } \omega \in \Omega_A \text{ and } u(\omega) = -1, w(\omega) = 1 \text{ for } \omega \in \Omega_B.$$

For a ± 1 game notation $(g; \Omega_A, \Omega_B)$ will be used along with $(g; u, w)$.

ω_1	ω_1
ω_2	ω_3

g_1

ω_1	ω_1	ω_2	ω_2
ω_3	ω_4	ω_3	ω_4

g_2

ω_1	ω_1	ω_3
ω_1	ω_2	ω_2
ω_3	ω_2	ω_3

g_3

ω_1	ω_1	ω_3
ω_1	ω_1	ω_2
ω_4	ω_2	ω_2

g_4

ω_1	ω_2	ω_1	ω_2
ω_3	ω_4	ω_4	ω_3
ω_1	ω_4	ω_1	ω_5
ω_3	ω_2	ω_6	ω_2

g_5

ω_1	ω_1
ω_1	ω_2

g_6

ω_1	ω_2
ω_2	ω_1

g_7

ω_1	ω_1	ω_2
ω_3	ω_4	ω_3

g_8

ω_1	ω_1	ω_2
ω_4	ω_5	ω_2
ω_4	ω_3	ω_3

g_9

Fig. 1. Nine game forms. Alice and Bob choose rows and columns, respectively. Forms g_1 - g_6 are tight, forms g_7 - g_9 are not; see Section 2.4 for the definition.

2.3. Nash equilibria and saddle points

Given a game $(g; u, w)$, a situation (x, y) of its game form $g : X \times Y \to A$ is called a *Nash equilibrium* (NE) if

$$u(g(x, y)) \geq u(g(x', y)), \forall x' \in X, \quad \text{and} \quad w(g(x, y)) \geq w(g(x, y')), \forall y' \in Y;$$

that is, if neither Alice nor Bob can profit replacing her/his strategy provided the opponent keeps his/her one unchanged, or in other words, if x is a best response for y and y is a best response for x. Note that a best response may be not unique.

This concept of solution was introduced in 1950 by John Nash [39,40]. In the zero-sum case, a NE is called *a saddle point*. The latter concept was known much earlier; the former one is its natural extension to the non-zero-sum case.

Recall that a zero-sum game (g, u, w) has a saddle point if and only if maxmin and minmax are equal:

$$\max\min = \max_{x \in X} \min_{\omega \in g(x)} u(\omega); \quad \min\max = \min_{y \in Y} \max_{\omega \in g(y)} u(\omega); \tag{1}$$

furthermore, maxmin $<$ minmax if and only if $(g; u, w)$ has no saddle point.

Remark 1. In [39,40] solvability in mixed strategies is studies. In contrast, we restrict the players to their pure strategies. Such approach is considered, for example, in [1–5,7–9,14,21,23,26,27,30,32,34,36,37,42,44,45,47].

2.4. Solvability of game forms

A game form g is called: (i) *Nash-*, (ii) *zero-sum-*, (iii) ± 1*-solvable* if the corresponding game $(g; u, w)$ has a NE for (i) all, (ii) all zero-sum, (iii) all zero-sum ± 1 payoffs, respectively.

Implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious. In fact, all three properties are equivalent [21,23,25]. For (ii) and (iii) it was shown earlier by Edmonds and Fulkerson [12]; see also [20]. The list of equivalent properties (i), (ii), (iii) was extended in [21] as follows.

2.5. Replacing payoffs by preferences and eliminating ties

Given a game $(g; u, w)$, we can assume wlog that payoffs $u : \Omega \to \mathbb{R}$ and $w : \Omega \to \mathbb{R}$ have no ties. Indeed, one can get rid of all ties by arbitrarily small perturbations of values of u and w. In accordance with definition, the set of NE will be either unchanged or reduced by such perturbations. We focus on Nash-solvability (in pure strategies), that is we study conditions that guarantee the existence of a NE for arbitrary payoffs u and w. Hence, we can wlog assume that both,

u and w, have no ties and replace them by linear orders $\succ_A$ and $\succ_B$ over the set of outcomes Ω, which are called the *preferences* of Alice and Bob, respectively. Thus, game $(g; u, w)$ can be replaced by $(g; \succ_A, \succ_B)$ and it is enough to study Nash-solvability of the latter. Although some NE of $(g; u, w)$ may disappear in $(g; \succ_A, \succ_B)$, yet, Nash-solvability holds or fails for both games simultaneously.

Remark 2. Above arguments would fail in the case of mixed strategies.

2.6. Tight game forms

Mappings $\phi : X \to Y$ and $\psi : Y \to X$ are called *response strategies* of Bob and Alice, respectively. The motivation for this name is clear: a player chooses his/her strategy as a function of a known strategy of the opponent. Standardly, $gr(\phi)$ and $gr(\psi)$ denote the graphs of mappings ϕ and ψ in $X \times Y$.

Game form $g : X \times Y \to \Omega$ is called *tight* if

(I) $g(gr(\phi)) \cap g(gr(\psi)) \neq \emptyset$ for any mappings ϕ and ψ.

It is not difficult to verify that in Fig. 1 the first six game forms $(g_1 - g_6)$ are tight, while the last three $(g_7 - g_9)$ are not.

In [12,20,21,23,27] the reader can find several equivalent properties characterizing tightness. Here we recall some of them.

(II-A) For every response strategy $\phi : X \to Y$ there exists a strategy $y \in Y$ such that $g(y) \subseteq g(gr(\phi))$.

(II-B) For every response strategy $\psi : Y \to X$ there exists a strategy $x \in X$ such that $g(x) \subseteq g(gr(\phi))$.

It is not difficult to see that (I) and (II-A) are equivalent [20,23]. Then, by transposing g, we conclude that (I) and (II-B) are equivalent as well. Hence, all three properties are equivalent. One can verify this for nine examples $g_1 - g_9$.

Properties (II-A) and (II-B) show that playing a zero-sum game $(g; u, w)$ with a tight game form g Bob and Alice do not need non-trivial response strategies but can restrict themselves by the standard ones, that is, by Y and X, respectively.

Given a game form $g : X \times Y \to \Omega$, introduce on the ground set Ω of the outcomes two multi-hypergraphs $\mathcal{A} = \mathcal{A}(g)$ and $\mathcal{B} = \mathcal{B}(g)$ whose edges are the supports of strategies of Alice and Bob, respectively:

$$\mathcal{A}(g) = \{g(x) \mid x \in X\} \text{ and } \mathcal{B}(g) = \{g(y) \mid y \in Y\}.$$

Recall that distinct edges of a multi-hypergraph may contain one another or even coincide. Obviously, the edges of $\mathcal{A}$ and $\mathcal{B}$ pairwise intersect, that is, $g(x) \cap g(y) \neq \emptyset$ for all $x \in X$ and $y \in Y$. Furthermore, g is tight if and only if

(III) hypergraphs $\mathcal{A}(g)$ and $\mathcal{B}(g)$ are *dual*, that is, satisfy also the following two (equivalent) properties:

(III-A) for every $\Omega_A \subseteq \Omega$ such that $\Omega_A \cap g(y) \neq \emptyset$ for all $y \in Y$ there exists an $x \in X$ such that $g(x) \subseteq \Omega_A$;
(III-B) for every $\Omega_B \subseteq \Omega$ such that $\Omega_B \cap g(x) \neq \emptyset$ for all $x \in X$ there exists an $y \in Y$ such that $g(y) \subseteq \Omega_B$.

Remark 3. Verification of tightness of an explicitly given game form is an important open problem. No polynomial algorithm is known. A quasi-polynomial one was suggested in [13]; see also [29].

3. Tightness and solvability

Let us recall an old theorem.

Theorem 1 ([21,23])**.** *The following properties of a game form are equivalent:* (i) *Nash-,* (ii) *zero-sum-,* (iii) ± 1*-solvability, and (iv) tightness.*

Proof. As we already mentioned, implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are obvious.

Also $(iii) \Rightarrow (iv)$ is easily seen. Indeed, assume for contradiction that a game form g is not tight. Then, there exists a response strategies $\phi : X \to Y$ and $\psi : Y \to X$ of Bob and Alice such that $g(gr(\phi)) \cap g(gr(\psi)) = \emptyset$. Then, we can partition Ω into two sets of outcomes Ω_A and Ω_B (winning for Alice and Bob, respectively) in such a way that $g(gr(\phi)) \subseteq \Omega_B$ and $g(gr(\psi)) \subseteq \Omega_A$. (Note that for tight g this would not be possible.) Then, $-1 = maxmin < minmax = 1$ in the obtained ± 1 game $(g; \Omega_A, \Omega_B)$ and, hence, it has no saddle point.

The inverse implication $(iii) \Leftarrow (iv)$, (as well as $(ii) \Leftarrow (iv)$, which looks stronger) are proven similarly; see [12,20]. Assume that a zero-sum game $(g; u, w)$ has no saddle point. Then, (1) fails and maxmin < minmax. Consider arbitrary best response strategies $\phi : X \to Y$ and $\psi : Y \to X$ of Bob and Alice, respectively. Obviously, $g(gr(\phi)) \cap g(gr(\psi)) = \emptyset$ and, thus, g is not tight.

The last claim means that a tight game form is SP-solvable. Moreover, it has a simple SP situation in minimal strategies [23].

To finish the proof of the theorem it only remains to show implication $(i) \Leftarrow (iv)$, that is, tightness implies Nash-solvability. First, this was done in [21], then, with more details, in [23]. Several different proofs appeared later [3,5,11,27,30]. In the next section we suggest a new (and shortest) proof based on an important general property of dual multi-hypergraphs.

4. Lexicographical theorem for dual multi-hypergraphs

4.1. Summary

Let $\mathcal{A}$ and $\mathcal{B}$ be an arbitrary pair of finite dual multi-hypergraphs on a common ground set Ω. Each of them may have embedded or equal edges. An edge is called *containment minimal* (or just *minimal*, for short) if it is not a strict superset of another edge. (Note that minimal edges may still be equal.) If $\mathcal{A}$ and $\mathcal{B}$ are dual then

(j) $A \cap B \neq \emptyset$ for every pair $A \in \mathcal{A}$ and $B \in \mathcal{B}$;

(jj) if A is minimal then for every $\omega \in A$ there exists a (minimal) $B \in \mathcal{B}$ such that $A \cap B = \{\omega\}$.
We will extend claim (jj) as follows. A linear order $\succ$ over Ω uniquely defines a lexicographic order $\succ_\ell$ over the power set 2^Ω.

(jjj-A) Let A be a lexicographically maximal (lexmax) edge of $\mathcal{A}$. Then, edge A is minimal in $\mathcal{A}$ and for every $\omega \in A$ there exists a (minimal) edge $B \in \mathcal{B}$ such that $A \cap B = \{\omega\}$ and $\omega \succeq \omega'$ for each $\omega' \in B$.

By swapping $A, \mathcal{A}$ and $B, \mathcal{B}$, we obtain the dual statement (jjj-B).

These two statements form the *lexicographical theorem* for dual multi-hypergraphs. To formulate it accurately, we will need a few definitions.

4.2. Lexicographical orders over the subsets.

A linear order $\succ$ over a set Ω uniquely determines a lexicographical order $\succ_\ell$ over the power set 2^Ω (of all subsets of Ω) as follows. Roughly speaking, the more small elements are out of a set — the better it is. In particular, $\Omega' \succ_\ell \Omega''$ whenever $\Omega' \subset \Omega''$ and, hence, the empty set $\emptyset \subset \Omega$ is the best in 2^Ω.

Remark 4. Also $\{\omega'\} \succ_\ell \{\omega', \omega''\}$ for any $\omega', \omega'' \in \Omega$ and order $\succ$, although set $\{\omega', \omega''\}$ gives a chance for a better outcome ω'' if $\omega' \prec \omega''$; see game form g_6 in Fig. 1 and Section 6.2 for more details.

More precisely, to compare two arbitrary subsets $\Omega', \Omega'' \subseteq \Omega$ consider their symmetric difference $\Delta = (\Omega' \setminus \Omega'') \cup (\Omega'' \setminus \Omega')$. Clearly, $\Delta \neq \emptyset$ if and only if sets Ω' and Ω'' are distinct. Let ω be the minimum with respect to $\succ$ element in Δ. If $\omega \in (\Omega' \setminus \Omega'')$ then $\Omega'' \succ_\ell \Omega'$; if $\omega \in (\Omega'' \setminus \Omega')$ then $\Omega' \succ_\ell \Omega''$.

We can reformulate this definition equivalently as follows. Without loss of generality (wlog), set $\Omega = \{\omega_1, \dots, \omega_p\}$ and assume that $\omega_1 \prec \dots \prec \omega_p$; assign the negative weight $w(\omega_i) = -2^{k-i}$ to every $\omega_i \in \Omega$, and set $w(S) = \sum_{\omega \in S} w(\omega)$ for each subset $S \subseteq \Omega$. Then, $\Omega' \succ_\ell \Omega''$ if and only if $w(\Omega') > w(\Omega'')$.

Denote by *supp*(S) the 0,1-vector $(s_1, \dots, s_k)$, where $s_i = 1$ if and only if $\omega_i \in S$. Then obviously, $\Omega' \succ_\ell \Omega''$ if and only if *supp*(Ω') is less than *supp*(Ω'') in the standard lexicographical order.

4.3. Dual multi-hypergraphs

Two finite multi-hypergraphs $\mathcal{A}$ and $\mathcal{B}$ on the common ground set Ω are called *dual* if (j) holds: $A \cap B \neq \emptyset$ for every pair $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and also (jv-A) for each $B^T \subseteq \Omega$ such that $B^T \cap B \neq \emptyset$ for every $B \in \mathcal{B}$ there exists an $A \in \mathcal{A}$ such that $A \subseteq B^T$.

If (j) and (jv-A) both hold we say that $\mathcal{A}$ is dual to $\mathcal{B}$ and use notation $\mathcal{A} = \mathcal{B}^d$. Swapping $A, \mathcal{A}$ and $B, \mathcal{B}$ in (jv-A) we obtain (jv-B) and an equivalent definition of duality, that is, (j) and (jv-A) hold if and only (j) and (jv-B) hold. In other words, $\mathcal{A} = \mathcal{B}^d$ if and only if $\mathcal{B} = \mathcal{A}^d$. So we just say that multi-hypergraphs $\mathcal{A}$ and $\mathcal{B}$ are dual.

Remark 5. Dual multi-hypergraphs have numerous applications and appear in different areas under different names, such as "clutters" and "blockers" [12] or DNFs and CNFs of monotone Boolean functions [10].

4.4. Lexicographical theorem

Claims (j) and (jj) are well-known [10]. Actually, (j) is required by the definition of duality and (jj) is obvious. Indeed, if (jj) fails then edge A cannot be minimal, since its proper subset $A \setminus \{\omega\}$ would still intersect all $B \in \mathcal{B}$.

Our main result is statement (jjj-A). Fix an arbitrary order $\succ$ over Ω and find a lexmax edge $A^L \in \mathcal{A}$, that is, one maximal with respect to the lexicographical order $\succ_\ell$ over 2^Ω. Note that such A^L may be not unique but all lexmax edges are equal. The lexicographic theorem is formulated as follows:

Theorem 2. *A lexmax edge A^L is minimal in $\mathcal{A}$. Furthermore, for every $\omega^* \in A^L$ there exists a (minimal) edge $B^M \in \mathcal{B}$ such that $A^L \cap B^M = \{\omega^*\}$ and $\omega^* \succ \omega$ for each $\omega \in B^M \setminus \{\omega^*\}$.*

Proof. A lexmax edge must be minimal, since a set is strictly less than any its proper subset in order $\succ_\ell$.

Assume for contradiction that there exists an $\omega^* \in A^L$ such that for every (minimal) $B \in \mathcal{B}$ satisfying (jj), $B \cap A^L = \{\omega^*\}$, there exists an $\omega \in B$ such that $\omega \succ \omega^*$. Clearly, this assumption holds for every $B^0 \in \mathcal{B}$ if it holds for each minimal $B^0 \in \mathcal{B}$. Let us show that it contradicts the lexmaximality of A^L. To do so partition all edges $B \in \mathcal{B}$ into two types:

(a) there is an $\omega \in B \cap A^L$ distinct from ω^*;
(b) $B \cap A^L = \{\omega^*\}$.

In case (b), by our assumption, there is an $\omega \in B$ such that $\omega \succ \omega^*$.

In both cases, (a) and (b), choose the specified ω from B, thus, getting a transversal B^T. By (jv-A), there exists an $A \in \mathcal{A}$ such that $A \subseteq B^T$ and, hence, $A \succeq_\ell B^T$. Furthermore, by construction, $B^T \succ_\ell A^L$. Indeed, $\omega^* \notin B^T$ and it is replaced in B^T by some larger elements, $\omega \succ \omega^*$, in case (b), while all other elements of B^T, if any, belong to $A^L \setminus \{\omega^*\}$, according to case (a).

Thus, by transitivity, $A \succ_\ell A^L$. Yet, by assumption of the theorem, A^L is a lexmax edge of $\mathcal{A}$, which is a contradiction. □

4.5. Sperner hypergraphs

A multi-hypergraph is called *Sperner* if no two of its distinct edges contain one another; in particular, they cannot be equal. In this case, we have a hypergraph rather than multi-hypergraph. For a multi-hypergraph there exists a unique dual Sperner hypergraph. If $\mathcal{A}$ and $\mathcal{B}$ are dual and Sperner then $\mathcal{A}^{dd} = \mathcal{A}$ and $\mathcal{B}^{dd} = \mathcal{B}$; furthermore $\cup_{A\in\mathcal{A}} A = \cup_{B\in\mathcal{B}} B = \Omega$. In general, for multi-hypergraphs, $\cup_{A\in\mathcal{A}} A$ and $\cup_{B\in\mathcal{B}} B$ may be different subsets of Ω.

Remark 6. Here we assume that the reader is familiar with basic notions related to monotone Boolean functions, in particular, with DNFs and duality. An introduction can be found in [10, Sections 1, 3 and 4].

It is well known [10] that (dual) multi-hypergraphs are in one-to-one correspondence with (dual) monotone DNFs: (prime) implicants of the latter correspond to (minimal) edges of the former. Furthermore, Sperner hypergraphs correspond to irredundant DNFs. However, we do not restrict ourselves to this case. Although the lexicographical theorem would not lose much but its applications to Nash-solvability would.

5. Determining edges A^L and B^M of Theorem 2 in polynomial time.

5.1. Preliminaries

Edges A and B mentioned in (jjj-A) can be found in polynomial time. The problem is trivial when multi-hypergraphs $\mathcal{A}$ and $\mathcal{B}$ are given explicitly. We will solve it when only $\mathcal{A}$ is given, and not explicitly, but by a polynomial containment oracle. For an arbitrary subset $\Omega_A \subseteq \Omega$ this oracle answers in polynomial time the question $Q(\mathcal{A}, \Omega_A)$: whether Ω_A contains an edge $A \in \mathcal{A}$.

By duality of $\mathcal{A}$ and $\mathcal{B}$, we have $A \not\subseteq \Omega_A$ for all $A \in \mathcal{A}$ if and only if $B \subseteq \Omega_B = \Omega \setminus \Omega_A$ for some $B \in \mathcal{B}$. In other words, question $Q(\mathcal{A}, \Omega_A)$ is answered in the negative if and only if $Q(\mathcal{B}, \Omega_B)$ is answered in the positive. Thus, we do not need two separate oracles for $\mathcal{A}$ and $\mathcal{B}$; it is sufficient to have one, say, for $\mathcal{A}$.

5.2. Determining a lexmax edge $\mathcal{A}^L$ in polynomial time.

Recall that multi-hypergraph $\mathcal{A}$ may contain several lexmax edges A^L, but they are all equal. Fix an arbitrary linear order $\succ$ over Ω. Wlog we can assume that $\Omega = \{\omega_1, \dots, \omega_p\}$ and $\omega_1 \prec \cdots \prec \omega_p$.

Step 1: Consider $\Omega_t^1 = \{\omega_t, \dots, \omega_p\}$ and, by asking question $Q(\mathcal{A}, \Omega_t^1)$ for $t = 1, \dots, p$, find the maximum t_1 for which the answer is still positive. Then, ω_{t_1} belongs to $\mathcal{A}^L$, while $\omega_1, \dots, \omega_{t_1-1}$ do not.

Step 2: Consider $\Omega_t^2 = \{\omega_{t_1}, \omega_{t_1+t}, \dots, \omega_p\}$ and, by asking question $Q(\mathcal{A}, \Omega_t^2)$ for $t = 1, \dots, p - t_1$, find the maximum t_2 for which the answer is still positive. Then, $\omega_{t_1}, \omega_{t_1+t_2} \in \mathcal{A}^L$, while $\omega_t \notin \mathcal{A}^L$ for any other $t < t_1 + t_2$.

Step 3: Consider $\Omega_t^3 = \{\omega_{t_1}, \omega_{t_1+t_2}, \omega_{t_1+t_2+t}, \dots, \omega_p\}$ and, by asking question $Q(\mathcal{A}, \Omega_t^3)$ for $t = 1, \dots, p - (t_1 + t_2)$, find the maximum t_3 for which the answer is still positive. Then, $\omega_{t_1}, \omega_{t_1+t_2}, \omega_{t_1+t_2+t_3} \in \mathcal{A}^L$, while $\omega_t \notin \mathcal{A}^L$ for any other $t < t_1 + t_2 + t_3$; etc.

This procedure will produce a lexmax edge A^L in at most p polynomial iterations. Note that on each step i we can speed up the search of t_i by applying dichotomy.

5.3. Determining an edge B^M from Theorem 2 .

First, find a lexmax edge $A^L \in \mathcal{A}$ and choose an arbitrary $\omega^* \in A^L$. We look for an edge $B^M \in \mathcal{B}$ such that $A^L \cap B^M = \{\omega^*\}$ and $\omega^* \succ \omega$ for every $\omega \in B^M \setminus \{\omega^*\}$. In other "words",

$$B^M \subseteq \Omega_B = \Omega \setminus [(A^L \setminus \{\omega^*\}) \cup \{\omega \mid \omega \succ \omega^*\}].$$

By Theorem 2, such B^M exists and, hence, the oracle answers $Q(\mathcal{B}, \Omega_B)$ in the positive, or equivalently, $Q(\mathcal{A}, \Omega \setminus \Omega_B)$ in the negative. We could take any $B^M \in \mathcal{B}$ such that $B^M \subseteq \Omega_B$. Yet, multi-hypergraph $\mathcal{B}$ is not given explicitly. To get B^M we need "to minimize" Ω_B. To do so, let us delete its elements one by one in some order until we obtain a minimum set Ω_B^* for which the answer to $Q(\mathcal{A}, \Omega \setminus \Omega_B^*)$ is still negative, that is, answers to $Q(\mathcal{A}, \Omega \setminus (\Omega_B^* \setminus \{\omega\}))$ become positive for every $\omega \in \Omega_B^*$. Then, we set $B^M = \Omega_B^*$. Again we can speed up the procedure by applying dichotomy.

Note that the above reduction procedure may be not unique, since we can eliminate elements of $\Omega \setminus \Omega_B$ in an arbitrary order. Thus, in contrast to A^L, there may be several not equal edges B^M satisfying all conditions of Theorem 2.

6. Lexicographically safe NE in games with tight game forms

6.1. Summary

First, we apply Theorem 2 to finish the proof of Theorem 1. It remains to show that $(i) \Leftarrow (iv)$, that is, tightness implies Nash-solvability. In other words, a game $(g; u, w)$ has a NE for any payoffs u and w whenever game form g is tight. The proof is constructive: we will obtain two special types of NE.

Given g and u, choose a lexmax strategy $x \in X$ of Alice. By Theorem 2, there is a strategy $y \in Y$ of Bob such that (x, y) is a NE. By definition, y must be a best response to x such that x is also a best response to y. By Theorem 2, the obtain situation (x, y) is simple and both strategies, x and y are minimal. More precisely, x must be minimal, while y can be chosen minimal. These NE will be called *lexsafe NE of Alice* and the set of these NE will denoted by NE-A. Similarly, we define a set NE-B of Bob's *lexsafe NE*.

Remark 7. We assume that both players are maximizers and adjective "lexsafe" can be replaced by "lexmax". If both players are minimizers then it can be replaced by "lexmin". In the zero-sum case Alice is the maximizer, while Bob is the minimizer. Flexible term lexsafe may replace both, lexmax or lexmin.

Results of Section 5 provide a polynomial algorithm determining at least one NE from NE-A and at least one from NE-B (which may coincide) in a given game $(g; u, w)$ with a tight game form g. This is trivial when g is given explicitly. Yet, the algorithm works when one of two multi-hypergraphs $\mathcal{A}(g)$ or $\mathcal{B}(g)$ is given by a polynomial containment oracle.

6.2. Lexicographically safe strategies of players

Given g and preference $\succ_A$ of Alice, let us introduce the lexicographical pre-order over Alice's strategies $x \in X$ as follows. Consider lexicographical order $\succ_A^\ell$ over 2^Ω defined by the linear order $\succ_A$ over Ω. The larger is the support $g(x) \subseteq \Omega$ in order $\succ_A^\ell$, the safer is strategy x for Alice, while strategies with the same support are equally safe. Alice's strategies that maximize support $g(x)$ in order $\succ_A^\ell$ will be called her lexmax (or lexsafe) strategies.

In particular, all lexmax strategies have the same support.

Furthermore, a lexsafe strategy is minimal. Indeed, x is safer than x' whenever $g(x) \subset g(x')$ and containment is strict.

Note also that Alice's lexsafe strategies are defined by g and $\succ_A$, while Bob's preference $\succ_B$ is irrelevant. Alice may be even unaware of it, which is important for applications.

Similarly, using $y \in Y$ and $\succ_B$ instead of x and $\succ_A$, we define Bob's lexsafe strategies. Respectively, they depend only on g and $\succ_B$, while $\succ_A$ is irrelevant.

The concept of a lexsafe strategy can be viewed as a refinement of the classical concept of a safe (maxmin) strategy. The latter optimizes the worst case scenario outcome, while lexsafe strategies optimize the whole set of outcomes in the lexicographical order defined above.

Thus, lexsafe strategies are safest, but sometimes may be not rational. For example, let $g(x) = \{\omega\}$, $g(x') = \{\omega, \omega'\}$ and $\omega \prec_A \omega'$. Then $x \succ_A x'$, although strategy x' is better for Alice than x. Indeed, x' gives her a chance to obtain the better outcome ω', while x excludes ω' and ensures ω; see Remark 5.

Consider a preference over Ω such that outcomes $\omega, \omega' \in \Omega$ are, respectively, the worst and the best outcomes for both Alice and Bob simultaneously. Consider a game form $g : X \times Y \to \Omega$ having two strategies $x^* \in X$ and $y^* \in Y$ such that $g(x, y) = \omega$ if and only if $x = x^*$ or $y = y^*$ and $g(x, y) = \omega'$ otherwise. Then, x^* and y^* are the only lexsafe strategies of Alice and Bob; furthermore, situation (x^*, y^*) is a unique lexsafe NE, but outcome $g(x^*, y^*) = \omega$ is worse than ω' for both players. One can view this as a price of stability. For comparison, recall that NE may be not Pareto optimal.

6.3. Lexsafe Nash equilibria in games with tight game forms

Recall that game form $g : X \times Y \to \Omega$ is tight if and only if its hypergraphs $\mathcal{A} = \mathcal{A}(g) = \{g(x) \mid x \in X\}$ and $\mathcal{B} = \mathcal{B}(g) = \{g(y) \mid y \in Y\}$ are dual.

Given a game $(g; \succ_A, \succ_B)$ with a tight game form g, choose any lexsafe strategy x^L of Alice. By Theorem 2 it is minimal. Let us show that there exists a strategy y^M of Bob such that (x^L, y^M) is a NE. By definition, y^M is a best response to x^L, that is, $g(x^L, y^M) \succeq_B g(x^L, y)$ for any $y \in Y$. (Note, however, that the preference is not strict, because for some y two outcomes may coincide: $g(x^L, y^M) = g(x^L, y)$.) Let us apply Theorem 2 setting

$$g(x^L) = A^L, \;\; g(y^M) = B^M, g(x^L, y^M) = \omega^*,$$

and conclude that there exists a (minimal) strategy y^M such that x^L, in its turn, is a best response to y^M. Thus, (x^L, y^M) is a NE. Theorem 1 is proven.

Moreover, we can strengthen it summarizing remarkable properties of the obtained NE. Recall that in Theorem 2 both edges A^L and B^M are minimal and $A^L \cap B^M = \{\omega^*\}$. Hence, for the obtained NE (x^L, y^M) both strategies x^L and y^M are minimal and situation (x^L, y^M) is simple, that is, $g(x^L) \cap g(y^M) = \{\omega^*\}$; see [23]. More precisely, X^L must be minimal, while Y^L can be chosen minimal.

Denote by X^L the set of all lexmax strategies of Alice. By definition, they all have the same support. Let us fix $x^L \in X^L$ and denote by $Y^M(x^L)$ the set of all Bob's best responses to x^L. In fact, $Y^M(x^L)$ does not depend on x^L provided $x^L \in X^L$. Indeed, set $g(x^L)$ is unique, that is, the same for all $x^L \in X_L$ and $g(x^L) \cap g(y^M) = \omega^*$ for all $y^M \in Y^M(x'^L)$ and for all $x'^L \in X^L$. Hence, $y^M(x^L)$ is a best response of Bob to each Alice's lexmax strategy. Denote by y^M the set of all such best responses.

Thus, we obtain $X^L \subseteq X$ and $Y^M \subseteq Y$ such that for any pair $x^L \in X^L$ and $y^M \in Y^M$ situation (x^L, y^M) is simple, $g(x^L, y^M) = \{\omega^*\}$, and (x^L, y^M) is a NE, because X^L is a best response to Y^M and vice versa.

In other words, the direct product NE-A $= (x^L \times y^M) \subseteq X \times Y$ consists of simple NE situations corresponding to the same outcome $\omega^* \in \Omega$. Furthermore, all strategy of X^L are lexsafe and, hence, minimal, while Y^M contains minimal strategies. We will call NE-A the *box of Alice's lexsafe equilibria*.

By construction, X^L depends only on Alice's preference $\succ_A$, while Bob's preference $\succ_B$ is irrelevant and Alice may be just unaware of it, which is important for applications. In contrast, Y^M is a set of some (special) Bob's best responses to X^L, which are the same for all $x^L \in X^L$.

Swapping the players, we obtain the *box of Bob's lexsafe equilibria* NE-B $= (x^M \times y^L) \subseteq X \times Y$ with similar properties. Thus, we can strengthen Theorem 1 as follows:

Theorem 3. *Every game $(g; \succ_A, \succ_B)$ with a tight game form g has two non-empty boxes of lexmax equilibria NE-A = $X^L \times Y^M$ and NE-B = $X^M \times Y^L$ of Alice and Bob satisfying the above properties.* □

Boxes NE-A and NE-B may intersect or even coincide. For example, this always happens in the zero-sum case. In this case X^L and X^M are maxmin strategies of Alice, while Y^M and Y^L are minmax strategies of Bob. More details can be found in the first arXiv version of this paper [31]. NE-A and NE-B may be equal in the non-zero-sum case too. For example a game may have a unique NE.

6.4. A pair of lexsafe strategies of Alice and Bob may be not a NE

For example, consider tight game form g_1 in Fig. 1. Define preferences $\succ_A$ and $\succ_B$ such that $\omega_2 \succ_A \omega_1 \succ_A \omega_3$ and $\omega_2 \succ_B \omega_3$. It is easy to verify that x_1 and y_1 are lexsafe strategies of Alice and Bob, respectively. Yet, situation (x_1, y_1) is not an NE. Alice can improve her result $g_1(x_1, y_1) = \omega_1$ by switching to x_2 and getting $g(x_2, y_1) = \omega_2$. Thus, two lexsafe strategies, of Alice and Bob, do not form an NE. However, sets NE-A and NE-B are not empty, in accordance with Theorem 3: NE-A $= \{(x_2, y_1)\}$ and NE-B $= \{(x_1, y_2)\}$. The corresponding NE outcomes are ω_1 and ω_2, respectively.

Note that ω_2 is the best outcome for both players if $\omega_2 \succ_B \omega_1$. In this case NE-B is not Pareto-optimal.

Remark 8. One could conjecture that each player prefers lexsafe NE of the opponent to his/her own. Such result would be similar to the analogous one from the matching theory; see, for example, [33]. There are two types of stable matchings given by the Gale–Shapley algorithm [16], depending on men propose to women or vice versa. Yet, this conjecture is disproved by the above example.

7. Computing lexsafe NE in polynomial time

If game form $g : X \times Y \to \Omega$ is given explicitly then to find all its NE is simple: one can just consider all situations $(x, y) \in X \times Y$ one by one verifying Nash's definition for each of them. Yet, in applications g is frequently given by an oracle $\mathcal{O}$ such that size of g is exponential in size $|\mathcal{O}|$ of this oracle. Then, the straightforward search for NE suggested above becomes not efficient. Four such oracles will be considered in the next section. The following three properties of oracle $\mathcal{O}$ will allow us to construct an algorithm computing two lexsafe NE (from NE-A and NE-B, respectively) for a given game $(g; \succ_A, \succ_B)$ with tight game form $g = g(\mathcal{O})$ realized by $\mathcal{O}$, in time polynomial in $|\mathcal{O}|$.

(I) Oracle $\mathcal{O}$ contains explicitly all outcomes Ω of g.
(Yet, strategies $x \in X$ and $y \in Y$ are implicit in $\mathcal{O}$; moreover, $|X|$ and $|Y|$ may be exponential in $|\mathcal{O}|$.)
(II) The game form $g = g(\mathcal{O})$ defined by $\mathcal{O}$ is tight.
(III) Every ± 1 game $(g(\mathcal{O}); \Omega_A, \Omega_B)$ can be solved in time polynomial in $|\mathcal{O}|$.

Requirement (III) needs a discussion. By tightness of g, exactly one of the following two options holds:

(a) there exists $x \in X$ with $g(x) \subseteq \Omega_A$ (Alice wins);
(b) there exists $y \in Y$ with $g(y) \subseteq \Omega_B$ (Bob wins).

Note that (a) (respectively, (b)) holds if and only if the monotone Boolean function corresponding to multi-hypergraph $\mathcal{A}(g)$ (respectively, $\mathcal{B}(g)$ takes value 1; see Remark 6.

To solve a ± 1 game we determine which option, (a) or (b), holds and output a winning strategy, x or y, respectively.

Note that it is possible to output a *minimal* winning strategy whenever (III) holds. Indeed, suppose Alice wins and we output her winning strategy x, with $g(x) \subseteq \Omega_A$. Reduce Ω_A by one outcome ω by moving it to Ω_B, solve the obtained ± 1 game, and repeat the procedure for all $\omega \in \Omega_A$. If Bob wins in all obtained games then x is already minimal. Otherwise we can move an outcome ω from Ω_A to Ω_B and Alice still wins. Repeating, we obtain a minimal winning strategy of Alice (in the original game) in at most $|\Omega_A|$ steps. We can speed up the above procedure using dichotomy. The same works for Bob.

Theorems 1–3 immediately imply the following statement.

Theorem 4. *Given an oracle $\mathcal{O}$ satisfying requirements (I,II,III), a lexsafe NE of Alice (of Bob) exists and can be computed in time polynomial in $|\mathcal{O}|$.* □

8. Examples of oracles

8.1. Summary

Here we consider four types of oracles known in the literature and verify that all four satisfy requirements (I, II, III).

In Section 8.3 we consider game forms corresponding to positional (graphical) game structures with perfect information, due to which Nash-solvability holds even in the n-person case. Yet, we substantially extend this class of game forms by modifying the set of outcomes. The standard approach assumes that the set of outcomes Ω is formed by the terminal vertices of the input directed graph Γ. Yet, Nash-solvability still holds if we extend Ω by redefining it as the set of all strongly connected components of Γ. Yet, in this case Nash-solvability holds only if we restrict ourselves to 2-person games.

In Section 8.4 we introduce so-called Jordan game forms in which Alice and Bob connect two pairs of opposite sides of the square. In Section 8.5 we consider monotone bargaining and in Section 8.6 veto voting schemes. In these three examples perfect information is not assumed, nevertheless requirements (I,II,III), tightness among them, hold.

Sections 8.3–8.6 can be read in an arbitrary order.

8.2. Game forms and game correspondences

A *game correspondence* is defined as an arbitrary mapping $G : X \times Y \to 2^{\Omega} \setminus \{\emptyset\}$, that is, G assigns a non-empty subset of outcomes to each situation.

Given G, define a game form $g \in G$, choosing an arbitrary outcome $g(x, y) \in G(x, y)$ for each situation (x, y). Conversely, given a game form $g : X \times Y \to \Omega$, define a game correspondence G setting $G(x, y) = g(x) \cap g(y)$. Then, obviously, $g \in G$.

By property (jj) of Section 4, if at least one $g^* \in G$ is tight then all $g \in G$ are tight. In this case G is called *tight* too. Moreover, all $g \in G$ have the same Sperner reduced dual hypergraphs $\mathcal{A}^0(g)$ and $\mathcal{B}^0(g)$, same simple situations, and for any u and w, the same sets of simple situations in NE-A and NE-B.

8.3. Deterministic graphical multi-stage game structures

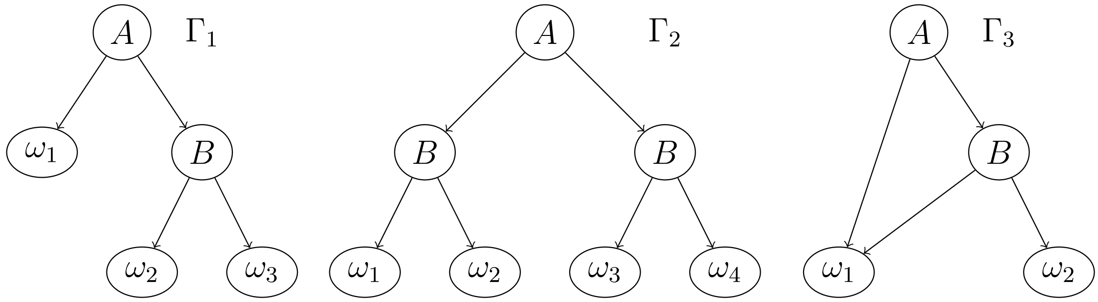
Let $\Gamma = (V, E)$ be a directed graph (digraph) whose vertices and arcs are interpreted as positions and moves, respectively. Denote by V_T the set of terminal positions (of out-degree zero) and by V_A, V_B the sets of positions of positive out-degree controlled by Alice and Bob, respectively. We assume that $V = V_A \cup V_B \cup V_T$ is a partition of V.

A strategy $x \in X$ of Alice (respectively, $y \in Y$ of Bob) is a mapping that assigns to each position $v \in V_A$ (respectively, $v \in V_B$) a move from this position. An initial position $v_0 \in V_A \cup V_B$ is fixed. Each situation (x, y) defines a unique walk in Γ that begins in v_0 and then follows the decisions made by strategies x and y. This walk $P(x, y)$ is called a *play*. Each play either terminates in V_T or is infinite. In the latter case, it forms a "lasso": first, an initial path, which may be empty, and then, a directed cycle (dicycle) repeated infinitely. (Indeed, since both players are restricted to their stationary strategies, a move may depend only on the current position but not on previous positions and/or moves. Hence, if a play visits a position twice then all further moves will be repeated as well.)

The (positional structure defined above can also be represented in normal form. We introduce a game form $g : X \times Y \to \Omega$, where, as before, Ω denotes a set of outcomes. Yet, there are several ways to define this set. One is to "merge" all infinite plays (lassos) and consider them as a single outcome c, thus, setting $\Omega = V_T \cup \{c\}$. This model was introduced by Washburn [47] and called *deterministic graphical game structure* (DGGS).

The following generalization was suggested in [27]. Digraph Γ is called *strongly connected* if for any $v, v' \in V$ there is a directed path from v to v' (and, hence, from v' to v, as well). By this definition, the union of two strongly connected digraphs with a common vertex is strongly connected. A vertex-inclusion-maximal strongly connected induced subgraph of Γ is called its *strongly connected component* (SCC). In particular, each terminal position $v \in V_T$ is an SCC. It is both obvious and well-known that any digraph $\Gamma = (V, E)$ admits a unique decomposition into SCCs: $\Gamma^{\omega} = \Gamma[V^{\omega}] = (V^{\omega}, E^{\omega})$ for $\omega \in \Omega$, where Ω is a set of indices. Furthermore, partition $V = \cup_{\omega \in \Omega} V^{\omega}$ can be constructed in time linear in the size of Γ, that is, in $(|V| + |E|)$.

Partitioning into SCCs has numerous applications; see [43,46] for more details. One more application was suggested in [27]. For each $\omega \in \Omega$, contract the SCC Γ^{ω} into a single vertex v^{ω}. Then, all edges of E^{ω} (including loops) disappear and we obtain an acyclic digraph $\Gamma^* = (\Omega, E^*)$. Set Ω can be treated as the set of outcomes. Each situation (x, y) uniquely defines a play $P = P(x, y)$. This play either comes to a terminal $v \in V_T$ or forms a lasso. The cycle of this lasso is contained



ω_1	ω_1
ω_2	ω_3

g_1

ω_1	ω_1	ω_2	ω_2
ω_3	ω_4	ω_3	ω_4

g_2

ω_1	ω_1
ω_1	ω_2

g_6

Fig. 2. Acyclic deterministic graphical game structures and corresponding game forms.

in an SCC ω of Γ. Each terminal is an SCC as well. In both cases an SCC $\omega \in \Omega$ is assigned to the play $P(x, y)$. Thus, we obtain a game form $g : X \times Y \to \Omega$, which is the normal form of the *multi-stage DGGS* (MSDGGS) defined by Γ.

An SCC is called *transient* if it is not a terminal and contains no dicycles. Obviously, a transient SCC consists of a single vertex and no play can result in it. Thus, it is not an outcome. For example, $\Omega = V_T$ in any acyclic digraph, while each remaining SCC is transient.

Proposition 1. *In both cases, DGGS and MSDGGS, the corresponding oracles satisfy requirements (I, II, III).*

Proof. Indeed, (I) holds since the SCCs, of a given digraph Γ can be generated in time linear in the size of Γ.

Both requirements, (II) and (III), for both oracles, DGGS and MSDGGS, can be verified simultaneously. Consider the corresponding game forms g' and g and note that g' is obtained from g by merging some outcomes. Namely, all outcomes corresponding to the non-terminal SCCs are replaced by a single outcome c. It is both obvious and well-known that merging outcomes respects tightness. Hence, it is enough to verify (II) and (III) for MSDGGSs,

By Theorem 1, to verify (II) it is sufficient to prove ± 1 solvability. For DGGS it was done in [47]; see also [5, Section 3], [2], [8, Section 12]. This result was extended to MSDGGS in [27]. Furthermore, all proofs in [27] were constructive, the corresponding ± 1 games were solved in time polynomial in the size of Γ, which implies (III).

For reader's convenience, we briefly sketch here the proof of (II,III) from [27]. Consider a ± 1 game $(g; \Omega_A, \Omega_B)$ with game form $g = g(\mathcal{O})$ generated by a MSDGGS oracle $\mathcal{O}$. We would like to apply Backward Induction, yet, digraph Γ may have dicycles. So we modify Backward Induction to make it work in presence of dicycles. Recall that Ω is the set of SCCs of Γ and $\Gamma^* = (\Omega, E^*)$ is acyclic. Consider an SCC $\Gamma' = (V', E')$ in Γ that is not terminal, but each move (v', v) from a position $v' \in V'$ either ends in a terminal $v \in V_T$, or stays in Γ', that is, $v, v' \in V'$. Obviously, such a SCC exists. Note that it may be transient. In this case the standard Backward Induction is applicable.

Suppose that Γ' is not transient, in other words, it contains a dicycle. Wlog we can assume that $\omega \in \Omega_A$, that is, Alice wins if the play cycles in Γ'. Then, Bob wins in a position $v' \in V'$ if and only if he can force the play to terminate in Ω_B, while Alice wins in all other positions of V'. Note that it is not necessary for Alice to force the play to come to a terminal from Ω_A, if the play cycles in Γ' Alice wins as well. Thus, every position of Γ' belongs either to Ω_A or to Ω_B. We make all these positions terminal, by eliminating all edges E' of Γ', and repeat until the initial position v_0 of Γ is evaluated.

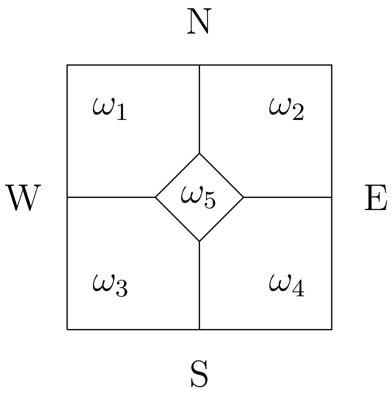
This procedure proves solvability of game form $g = g(\mathcal{O})$ (which is equivalent to its tightness (II), by Theorem 1), moreover, a ± 1 game $(g; \Omega_A, \Omega_B)$ is solved in time linear in the size of $\mathcal{O} = \Gamma$ (which is (III)). $\square$

Acyclic deterministic graphical game structures

A game form is called *rectangular* if all its situations are simple. It is shown in [22] that a game form g is generated by a DGGS whose graph is a tree if and only if g is tight and rectangular. Two examples, Γ_1 and Γ_2 generating game tight rectangular game forms g_1 and g_2 are given in Fig. 2; see also Fig. 1. More examples can be found in [25, Section 3], where the above characterization is extended to the n-person case.

Acyclic DGGS Γ_1 in Fig. 2 generates game form g_1. Recall game $(g_1; u, w)$ from Section 6.4 with $u(\omega_2) > u(\omega_1) > u(\omega_3)$ and $w(\omega_2) > w(\omega_3)$. Note that the Backward Induction NE (see [15,35] and also [26]) is NE-A and is not Pareto-optimal. In general, this NE may differ from both, NE-A and NE-B.

Acyclic DGGS Γ_3 in Fig. 2 generates game form g_6; see also Fig. 1.



ω_1	ω_2	ω_1	ω_2
ω_3	ω_4	ω_4	ω_3
ω_1	ω_4	$\omega_1\omega_4\omega_5$	ω_5
ω_3	ω_2	ω_5	$\omega_2\omega_3\omega_5$

Fig. 3. The Jordan game correspondence of the map of Wonderland.

Clearly, in absence of dicycles in Γ, the concepts of DGGS and MSDGGS coincide. It is also clear that an acyclic DGGS is a special case of MSDGGS. Thus, properties (I, II, III) required from an oracle hold for both.

Cyclic deterministic graphical game structures

The outcomes of MSDGGS are all its non-transient SCCs. In particular, each terminal position is an outcome. Let us now assume that every simple dicycle is a separate outcome (and each terminal remains an outcome as well). Such DGGSs, called *cyclic*, were studied in [8]; some special cases were considered earlier [17–19]. Cyclic DGGS can also serve as oracles generating game forms; see examples in [8, Figures 1 and 2]; compare examples 3 and 4 in [8, Figure 2] with game forms g_4 and g_5 in Fig. 1.

Game forms generated by the cyclic DGGS may be not tight; see Figure 1 in [8]. In other words, property (II) fails for the corresponding oracles, in general. Yet, it holds in some important special cases.

A digraph $G = (V, E)$ is called *symmetric* if $(v, v') \in E$ whenever $(v', v) \in E$. Cyclic DGGS on symmetric digraphs are called *symmetric*. Symmetric Cyclic DGGSs satisfying (II) are called *solvable* and explicitly characterized in [8, sections 1–3 and Theorems 1–3]. It follows from results of [8] that (III) also holds for solvable cyclic symmetric DGGS. Hence, Theorem 4 is applicable.

8.4. Jordan oracle; choosing battlefields in Wonderland

Wonderland is a subset of the plane homeomorphic to the closed disc. Wlog, we can consider a square Q with the sides N, E, S, W. Let us partition Q into areas $\Omega = \{\omega_1, \dots, \omega_p\}$ each of which is homeomorphic to the closed disc, too. Every two distinct areas $\omega_i, \omega_j \in \Omega$ are either disjoint or intersect in a set homeomorphic to a closed interval that contains more than one point. Equivalently, we can require that the borders of the areas in Q form a regular graph of degree 3. (Note that four vertices of the square are not vertices of this graph.) Two examples are given in Figs. 3 and 4.

Remark 9. Consider game form g_5 in Fig. 1 and merge outcomes ω_5 and ω_6 in it getting g_5'. (This operation respects tightness). Note that $g_5' \in G$, where G is the game correspondence given in Fig. 3. See also [8, Figure 4], where g_5 also appears as the normal form of a cyclic game form.

The following interpretation was suggested in [30]. Two players, Alice Tweedledee and Bob Tweedledum, agreed to have a battle. The next thing to do is to agree on a battlefield, which should be an area $\omega \in \Omega$. The strategies $x \in X$ of Alice are all (inclusion-minimal) subsets $x \subseteq \Omega$ connecting W and E, Respectively, the strategies $y \in Y$ of Bob are all (inclusion-minimal) subsets $y \subseteq \Omega$ connecting N and S.

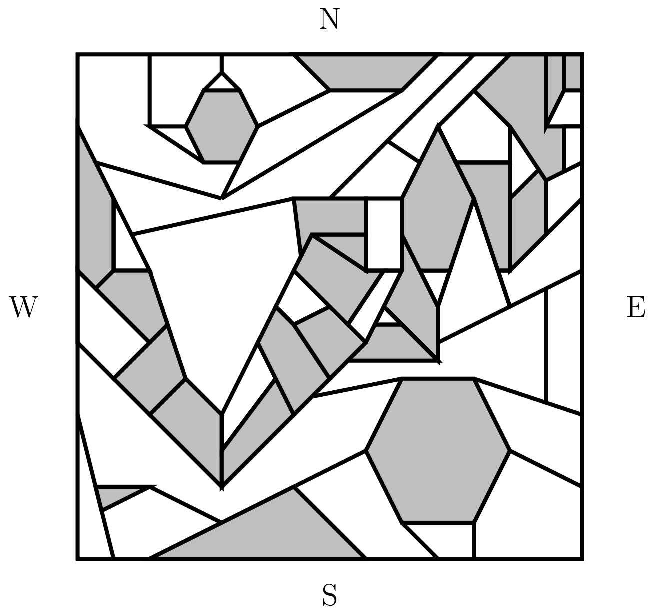


Fig. 4. Gray and white areas are in Ω_A and Ω_B, respectively. Alice wins.

Proposition 2. *Any two such subsets x and y intersect.*

Proof. It follows the Jordan curve theorem and the fact that all vertices in the square are of degree 3 (except its four corners, which are of degree 2). Note that x and y might be disjoint if we allow vertices of degree 4 or more. □

Intersection $x \cap y$ may contain several areas of Ω. Thus, a game correspondence $G : X \times Y \to 2^{\Omega} \setminus \{\emptyset\}$ is defined.

Proposition 3. *Game correspondence G is tight.*

Proof. Again, it follows from the Jordan curve theorem and the assumption that all vertices in the square are of degree 3. Choose an arbitrary $g \in G$ and consider a ± 1 game $(g; \Omega_A, \Omega_B)$ determined by a partition $\Omega = \Omega_A \cup \Omega_B$. Then, from the following two options exactly one holds:

(a) areas from Ω_A connect W and E; (b) areas from Ω_B connect N and S. □

The above observations imply that Jordan oracle $\mathcal{O}$ satisfies requirements (I) and (II). It remains to verify (III),

Proposition 4. *By using oracle $\mathcal{O}$, one can decide whether (a) or (b) holds and find corresponding x or y, respectively, in time linear in $|\mathcal{O}|$.*

Proof. Consider all areas from Ω_B boarding N, then add all areas from Ω_B boarding these areas, etc. Such iterations will stop in time linear in $|\mathcal{O}|$ either reaching S (then, obviously, (b) holds) or not (then (a) holds, again by the Jordan curve theorem). Moreover, in the first case we obtain a set of areas y' from Ω_B connecting N and S; in the second case — a set of areas x' from Ω_A connecting W and E. The former strategy y' is obtained explicitly; the latter one, x', is easy to construct. To do so, denote by Ω'_B the set of areas obtained in the course of iterations. It does not reach S. Hence, the areas from Ω_A that border Ω'_B connect W and E, by the Jordan curve theorem once more. □

This case is realized in Fig. 4; Alice wins.

Remark 10. It is not necessary to restrict ourselves by minimal strategies. In linear time we can reduce arbitrary strategy (set) x' of Alice to an inclusion-minimal set x connecting W and E, thus, getting minimal strategies of Alice. To do so, we eliminate areas from x' one by one until (a) still holds. We require inclusion-minimality of subsets $x \in \Omega$ just to reduce the number of strategies (which may still remain exponential in $|\mathcal{O}|$). Of course, the same is true for Bob's strategies.

8.5. Monotone bargaining schemes

The following oracle was introduced in [30]. Two players, Alice and Bob, possess items $A = \{a_1, \ldots, a_m\}$ and $B = \{b_1, \ldots, b_n\}$, respectively. Both sets are ordered: $a_1 \prec \cdots \prec a_m$ and $b_1 \prec \cdots \prec b_n$. Both players know both orders.

The direct product $\Omega = A \times B = \{(a, b) \mid a \in A, b \in B\}$ is the set of *outcomes*.

Alice's strategies are monotone non-decreasing mappings $x : A \to B$ (that is, $x(a) \geq x(a')$ whenever $a > a'$) showing that she is ready to exchange a for $x(a)$ for any $a \in A$. Similarly, Bob's strategies are monotone non-decreasing mappings $y : B \to A$ (that is, $y(b) \geq y(b')$ whenever $b > b'$) showing that he is ready to exchange b for $y(b)$ for any $b \in B$.

It is not difficult to compute the numbers of strategies and outcomes:

$$|X| = \binom{m+n-1}{m}, \quad |Y| = \binom{m+n-1}{n}; \quad |\Omega| = |A \times B| = mn. \tag{2}$$

Given a situation (x, y), an outcome $(a, b) \in \Omega$ is called a *deal* (in this situation) if $x(a) = b$ and $y(b) = a$. Denote by $G(x, y) \subseteq \Omega$ the set of all deals in the situation (x, y). We will show that $G(x, y) \neq \emptyset$. Yet, $G(x, y)$ may contain several deals.

This construction is called a *monotone bargaining (MB) scheme*. It can be viewed as an oracle $\mathcal{O}$ generating game correspondence $G : X \times Y \to 2^{\Omega} \setminus \{\emptyset\}$. By (2), requirement (I) holds for $\mathcal{O}$.

Note that $G = G_{m,n}$ is uniquely defined by m and n. A game form $g \in G$ is called an *MB game form*. For example, if $m = n = 3$ then $|X| = |Y| = 3$ and we obtain game form g_4 in Fig. 1; game correspondence $G(x, y)$ is given in [30, Figure 1a].

The following interpretation was suggested in [30]. Alice and Bob are dealers possessing the sets of objects A and B, respectively, and a deal $(a, b) \in A \times B$ means that they exchange a and b. They may be art-dealers, car dealers; or one of them may be just a buyer with a discrete budget. For example, $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_n\}$ may be paintings or sculptures ordered in accordance with their age (not price or value).

To any pair of mappings $x : A \to B$ and $y : B \to A$ (not necessarily monotone non-decreasing) let us assign a bipartite digraph $\Gamma = \Gamma(x, y)$ on the vertex-set $A \cup B$ as follows: $[a, b)$ (respectively, $[b, a)$) is an arc of $\Gamma(x, y)$ whenever $x(a) = b$ (respectively, $y(b) = a$).

Some visualization helps. Embed $\Gamma(x, y)$ into a plane; putting ordered A and B in two parallel columns. Two arcs corresponding to x may have a common head, but not tail. Furthermore, they cannot cross if mapping x is monotone non-decreasing. Similarly for y. By construction, digraph Γ is bipartite, with parts A and B. Hence, every dicycle in Γ is even. There is an obvious one-to-one correspondence between the dicycles of length 2 in $\Gamma(x, y)$ and the deals of $G(x, y)$.

Proposition 5. *For each situation (x, y) its digraph $\Gamma(x, y)$ contains at least one dicycle of length 2 (a deal) and cannot contain longer dicycles.*

Proof. For any initial vertex $v \in A \cup B$, strategies x and y uniquely define an infinite walk from v, which is called a play. Since sets A and B are finite and there are no terminals, this play is a *lasso*: it consists of an initial directed path, which may be empty, and a dicycle C repeated infinitely. Furthermore, C must be a dicycle of length 2 whenever mappings x and y are monotone non-decreasing. Indeed, if C is longer than 2 then crossing arcs appear and, hence, either x, or y, or both are not monotone, □

Consider a ± 1 MB game $(g; \Omega_A, \Omega_B)$, where $g = g(\mathcal{O})$ is an MB game form generated by an MB scheme $\mathcal{O}$. As we already mentioned, requirement (I) holds for $\mathcal{O}$. The following statement shows that (II) and (III) hold as well.

Proposition 6. *Game form $g = g(\mathcal{O})$ is tight and each ± 1 MB game $(g; \Omega_A, \Omega_B)$ can be solved in time polynomial in $|\mathcal{O}| = mn$.*

The first part was already proven in [30]. Yet, here we provide a much shorter proof.

Proof. For the sake of simplicity, we will slightly abuse notation writing that both directed edges $[a, b)$ and $[b, a)$ are in Ω_A or in Ω_B whenever the corresponding deal (a, b) is in Ω_A or in Ω_B, respectively.

Consider complete bipartite symmetric digraph Γ on $m + n$ vertices $A = \{a_1, \dots, a_m\}$, $B = \{b_1, \dots, b_n\}$, and with $2mn$ directed edges $\{[a_i, b_j), [b_j, a_i) \mid i = 1, \dots, m; j = 1, \dots, n\}$. The following two statements are obvious:

(a) Alice wins if she has a monotone non-decreasing strategy $x^* : A \to B$ such that $[a, x^*(a)) \in \Omega_A$ for all $a \in A$.

(b) Bob wins if he has a monotone non-decreasing strategy $y^* : B \to A$ such that $[b, y^*(b)) \in \Omega_B$ for all $b \in B$.

Indeed, it is easily seen that x^* and y^* are the winning strategies of Alice and Bob, respectively. It is enough to show that $g(x^*, y) \in \Omega_A$ for any $y \in Y$. Recall the proof of Proposition 5: Fix x^*, choose an arbitrary $y \in Y$, and consider the play $P = P(x^*, y)$ beginning from an arbitrary initial position $v \in A \cup B$. By Proposition 5, P is a lasso resulting in a 2-cycle (a, b).

The corresponding deal $(a, b) \in \Omega_A$, in case (a), for any y, by the choice of x^*, and hence, Alice wins. Similarly, $g(x, y^*) \in \Omega_B$ in case (b) for any $x \in X$, by the choice of y^*, and Bob wins.

Obviously, (a) and (b) cannot hold simultaneously, since otherwise $(a, b) \in \Omega_A \cap \Omega_B$, which is a contradiction, since $\Omega = \Omega_A \cup \Omega_B$ is a partition.

Let us show that either (a) or (b) holds (in other words, g is tight, which implies (II)). The proof will be constructive: we obtain either x^* satisfying (a) or y^* satisfying (b) in time polynomial in mn (which in its turn, implies (III)).

We will construct a play P by the following greedy iterative algorithm. Let $a^1 = a_1$ be an initial position of P. (We use superscripts to number iterations.)

If $[a^1, b) \in \Omega_B$ for all $b \in B$ then Bob wins. (His winning strategy y^* is defined by: $y^*(b) = a^1$ for all $b \in B$. Then $[y^*(b), b) \in \Omega_B$ for all $b \in B$ and (b) holds.) Otherwise, denote by b^1 the (unique) minimal $b \in B$ such that $[a^1, b) \in \Omega_A$. Then, by definition, $[b^1, a^1) \in \Omega_A$ too. Furthermore, by this choice of b^1, we have: $[b, a^1) \in \Omega_B$ for all $b \prec b^1$, while $[b^1, a^1) \in \Omega_A$.

If $[b^1, a) \in \Omega_A$ for all $a \succeq a^1$ then Alice wins. (Her winning strategy x^* is defined by: $x^*(a) = b^1$ for all $a \in A$. Then $[a, x^*(a)) \in \Omega_A$ for all $a \in A$.) Otherwise, denote by a^2 the (unique) minimal $a \in A$ such that $[b^1, a) \in \Omega_B$. Then, by definition, $[a^2, b^1) \in \Omega_B$ too. Furthermore, by the choice of a^2, we have: $[a, b^1) \in \Omega_A$ for all $a \prec a^2$, while $[a^2, b^1) \in \Omega_B$.

The general kth step of this greedy recursion is as follows.

If $[a^k, b) \in \Omega_B$ for all $b \succeq b^{k-1}$ then Bob wins. (His winning strategy y^* is defined by: $y^*(b) = a^i$ for each b such that $b^i \succ b \succeq b^{i-1}$, for $i = 1, \ldots, k$, assuming conventionally that $b \succ b^0$ holds for all $b \in B$).

Otherwise, denote by b^k the (unique) minimal $b \in B$ such that $b \succ b^{k-1}$ and $[a^k, b) \in \Omega_A$. Then $[b^k, a^k) \in \Omega_A$ too.

Furthermore, by the choice of b^k, we have: $[b, a^k) \in \Omega_B$ for all b such that $b^k \succ b \succeq b^{k-1}$, while $[b^k, a^k) \in \Omega_A$.

If $[b^k, a) \in \Omega_A$ for all $a \succeq a^k$ then Alice wins. (Her winning strategy x^* is defined by: $x^*(a) = b^j$ for each a such that $a^{j+1} \succ a \succeq a^j$, for $j = 1, \ldots, k$, assuming conventionally that $a^{k+1} \succ a$ holds for all $a \in A$.)

Otherwise, denote by a^{k+1} the (unique) minimal $a \in A$ such that $[b^k, a) \in \Omega_B$. Then $[a^{k+1}, b^k) \in \Omega_B$, too.

Furthermore, by the choice of a^{k+1}, we have: $[a, b^k) \in \Omega_A$ for all a such that $a^{k+1} \succ a \succeq a^k$, while $[a^{k+1}, b^k) \in \Omega_B$. After each iteration a^k (respectively, b^k) both Alice and Bob have winning moves in all positions $a \prec a^k$ and $b \preceq b^{k-1}$ (respectively, $a \preceq a^k$ and $b \prec b^k$). Since sets A and B are finite, the procedure will stop on some iteration either $a^{k^*} \prec a_m$ or $b^{k^*} \prec b_n$, indicating that Bob or, respectively, Alice wins.

Furthermore, we obtain his or her winning strategy in time linear in mn.

The following slightly different procedure can be applied too. First, we start looking for a winning strategy x^* for Alice. Consider successively $a_1, a_2, \ldots$ and construct (again recursively and greedily) her monotone non-decreasing strategy x^* as follows: $x^*(a_i) = b^i$ such that $[a_i, b^i) \in \Omega_A$, $b^i \succeq b^{i-1}$, and b^i is the minimal element of B satisfying these two properties. If this will work for all $i = 1, \ldots, m$ then Alice wins and we obtain her winning strategy x^* satisfying (a). Otherwise, if the procedure stops on some $i < m$ (no required b^i exists for a_i) then Bob wins. His winning strategy y^* satisfying (b) is defined as follows: $y^*(b) = a^i$ for all b such that $b^{i-1} \preceq b \prec b^i$, where a_i is the smallest a such that $x^*(a) = b^i$, for $i = 1, 2, \ldots$ By convention, $b^0 \prec b$ for all $b \in B$. $\square$

Thus, requirements (I,II,III) hold for the MB schemes and, hence, Theorem 4 is applicable.

8.6. Veto voting schemes

Two voters (players), Alice and Bob choose among candidates (options, outcomes) $\Omega = \{\omega_1, \ldots, \omega_p\}$. They are assigned some positive integer veto powers and given μ_A and μ_B veto cards, respectively. Each candidate $\omega \in \Omega$ is assigned an integer positive veto resistance λ_ω. We assume that

$$\mu_A + \mu_B + 1 = \lambda_{\omega_1} + \cdots + \lambda_{\omega_p}. \tag{3}$$

A strategy of a voter is an arbitrary distribution of her/his veto cards among the candidates. Given a pair of strategies x and y, a candidate $\omega \in \Omega$ who got at least λ_ω veto cards (from Alice and Bob together) is vetoed. From the set $G(x, y)$ of all not vetoed candidates one $g(x, y) \in G(x, y)$ is elected. By (3), $G(x, y) \neq \emptyset$. Thus, we obtain a *veto voting* (VV) scheme $\mathcal{O}$, VV game form $g = g(\mathcal{O})$, and VV game correspondence $G = G(\mathcal{O})$; see, for example, [24],[38, Chapter 6],[41, Chapter 5] for more details.

By construction, VV schemes are oracles satisfying (I). For example, game form g_3 in Fig. 1 corresponds to the VV scheme defined by

$$\mu_A = \mu_B = \lambda_{\omega_1} = \lambda_{\omega_2} = \lambda_{\omega_3} = 1.$$

Let us show that requirements (II) and (III) also hold for VV schemes.

Proposition 7. *Each game form g defined by a VV scheme satisfying (3) is tight. Furthermore, every ± 1 game $(g; \Omega_A, \Omega_B)$ can be solved in time linear in $|\mathcal{O}| = \log(\mu_A \mu_B \prod_{\omega \in \Omega} \lambda_\omega)$.*

Proof. To see this, consider a ± 1 game $(g; \Omega_A, \Omega_B)$. By (3), from two options, (a) Alice can veto Ω_B and (b) Bob can veto Ω_A, exactly one holds. Alice or Bob wins in case of (a) or (b), respectively. Given numbers μ_A, μ_B, and λ_ω, $\omega \in \Omega$, one can decide in linear time whether (a) or (b) holds. In each case the winning strategy of Alice or Bob is straightforward: just veto all opponent's candidates, Ω_B or Ω_A, respectively. $\square$

Thus, the VV oracles satisfy (I,II,III) and Theorem 4 is applicable.

8.7. Tight game correspondences and forms of arbitrary monotone properties

The most general setting is defined as follows. Given a finite ground set Ω, consider a family of its subsets $\mathcal{P} \subseteq 2^{\Omega}$. Standardly, we call $\mathcal{P}$ a *property* and say that a subset $\Omega' \subseteq \Omega$ satisfies $\mathcal{P}$ or not if $\Omega' \in \mathcal{P}$ or $\Omega' \notin \mathcal{P}$, respectively. Property $\mathcal{P}$ is called *inclusion monotone non-decreasing* (or simply *monotone*, for short) if $\Omega'' \in \mathcal{P}$ implies $\Omega' \in \mathcal{P}$ whenever $\Omega'' \subseteq \Omega' \subseteq \Omega$. We restrict ourselves to monotone properties.

Define the sets of strategies X of Alice and Y of Bob as follows:

$x \in X$ is any (inclusion minimal) subset $\Omega_A \subseteq \Omega$ such that $\Omega_A \in \mathcal{P}$;

$y \in Y$ is any (inclusion minimal) subset $\Omega_B \subseteq \Omega$ such that $\Omega \setminus \Omega_B \notin \mathcal{P}$.

The restriction in parenthesis does not matter, it can be waved or kept. In the latter case, sets X and Y are significantly reduced.

Define a game correspondence $G = G(\mathcal{P})$ by setting $G(x, y) = x \cap y$. It is both obvious and well-known that $G(x, y) \neq \emptyset$ for any $x \in X, y \in Y$ and, moreover, G is tight. Hence, any game form $g \in G$ is tight too.

Thus, (I) and (II) hold automatically whenever a monotone property $\mathcal{P}$ is given by an oracle $\mathcal{O}(\mathcal{P})$. Yet, (III) must be required in addition. In other words, $\mathcal{O}(\mathcal{P})$ must be a polynomial membership oracle, which for a given subset $\Omega' \subseteq \Omega$, decides if $\Omega' \in \mathcal{P}$ in time polynomial in $|\Omega| + |\mathcal{O}(\mathcal{P})|$.

It is easily seen that this general setting includes in particular all four examples of oracles given in this section before; see more examples in [6,28].

Data availability

No data was used for the research described in the article.

Acknowledgments

This research was prepared within the framework of the HSE University Basic Research Program. The authors are thankful to Endre Boros and to three anonymous reviewers for helpful remarks and suggestions.

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