Computing lexicographically safe Nash equilibria in finite two-person games with tight game forms given by oracles

Lecture series

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Examples of oracles

Summary

Here we consider four types of oracles known in the literature
and verify that all four satisfy requirements (I, II, III).

J. Game forms corresponding to positional (graphical) game structures with perfect information, for which Nash-solvabilty holds even in the n-person case.

We extend this class of such game forms by modifying the set of outcomes. The standard approach assumes that the set of outcomes Ω is formed by the terminal vertices of the input directed graph Γ.

Nash-solvability still holds if we extend Ω by redefining it as the set of all strongly connected components of Γ.

But only if we restrict ourselves to 2-person games.

JJ. So-called Jordan game forms in which Alice and Bob connect two pairs of opposite sides of the square.

JJJ. Monotone bargaining Schemes

JV.Veto voting schemes.

In the last 3 examples perfect information is not assumed, nevertheless requirements (I,II,III), tightness among them, hold.

Game forms and game correspondences

A *game correspondence* is defined as an arbitrary mapping

$G : X \times Y \to 2^\Omega \setminus \{\emptyset\}$,

that is, G assigns a non-empty subset of outcomes to each situation.

Given G, define a game form $g \in G$, choosing an arbitrary outcome $g(x, y) \in G(x, y)$ for each situation (x, y).

Conversely, given a game form $g : X \times Y \to \Omega$, define a game correspondence G setting $G(x, y) = g(x) \cap g(y)$. Then, obviously, $g \in G$.

If at least one $g^* \in G$ is tight then all $g \in G$ are tight.

In this case G is called *tight* too.

Moreover, all $g \in G$ have the same Sperner reduced dual hypergraphs $\mathcal{A}^0(g)$ and $\mathcal{B}^0(g)$, same simple situations, and for any u and w, the same sets of simple situations in NE-A and NE-B.

Deterministic graphical multi-stage game structures

Let $\Gamma = (V, E)$ be a directed graph (digraph) whose vertices and arcs are interpreted as positions and moves, respectively. Denote by V_T the set of terminal positions (of out-degree zero) and by V_A, V_B the sets of positions of positive out-degree controlled by Alice and Bob, respectively. We assume that $V = V_A \cup V_B \cup V_T$ is a partition of V.

A strategy $x \in X$ of Alice (respectively, $y \in Y$ of Bob) is a mapping that assigns to each position $v \in V_A$ (respectively, $v \in V_B$) a move from this position. An initial position $v_0 \in V_A \cup V_B$ is fixed. Each situation (x, y) defines a unique walk in Γ that begins in v_0 and then follows the decisions made by strategies x and y. This walk $P(x, y)$ is called a *play*. Each play either terminates in V_T or is infinite. In the latter case, it forms a “lasso": first, an initial path, which may be empty, and then, a directed cycle (dicycle) repeated infinitely. (Indeed, since both players are restricted to their stationary strategies, a move may depend only on the current position but not on previous positions and/or moves. Hence, if a play visits a position twice then all further moves will be repeated as well.)

The (positional structure defined above can also be represented in normal form. We introduce a game form $g : X \times Y \to \Omega$, where, as before, Ω denotes a set of outcomes. Yet, there are several ways to define this set. One is to “merge" all infinite plays (lassos) and consider them as a single outcome c, thus, setting $\Omega = V_T \cup \{c\}$. This model was introduced by Washburn [**Was90**] and called *deterministic graphical game structure* (DGGS).

The following generalization was suggested in [**Gur18**]. Digraph Γ is called *strongly connected* if for any $v, v' \in V$ there is a directed path from v to v' (and, hence, from v' to v, as well). By this definition, the union of two strongly connected digraphs with a common vertex is strongly connected. A vertex-inclusion-maximal strongly connected induced subgraph of Γ is called its *strongly connected component* (SCC). In particular, each terminal position $v \in V_T$ is an SCC. It is both obvious and well-known that any digraph $\Gamma = (V, E)$ admits a unique decomposition into SCCs: $\Gamma^\omega = \Gamma[V^\omega] = (V^\omega, E^\omega)$ for $\omega \in \Omega$, where Ω is a set of indices. Furthermore, partition $V = \cup_{\omega \in \Omega} V^\omega$ can be constructed in time linear in the size of Γ, that is, in $(|V| + |E|)$.

Partitioning into SCCs has numerous applications;
see [**Sha81**, **Tar72**] for more details.

One more application was suggested in [**Gur18**].

For each $\omega \in \Omega$, contract the SCC Γ^ω into a single vertex v^ω.

Then, all edges of E^ω (including loops) disappear and we obtain an acyclic digraph $\Gamma^* = (\Omega, E^*)$.

Set Ω can be treated as the set of outcomes.

Each situation (x, y) uniquely defines a play $P = P(x, y)$.

This play either comes to a terminal $v \in V_T$ or forms a lasso.

The cycle of this lasso is contained in an SCC ω of Γ.

Each terminal is an SCC as well. In both cases an SCC $\omega \in \Omega$ is assigned to the play $P(x, y)$.

Thus, we obtain a game form $g : X \times Y \to \Omega$,
which is the normal form of the *multi-stage DGGS* (MSDGGS) defined by Γ.

An SCC is called *transient* if it is not a terminal and contains no dicycles.

Obviously, a transient SCC consists of a single vertex and no play results in it.

Thus, it is not an outcome. For example, $\Omega = V_T$ in any acyclic digraph, while each remaining SCC is transient.

Proposition

In both cases, DGGS and MSDGGS, the corresponding oracles satisfy requirements (I, II, III).

Proof

Indeed, (I) holds since the SCCs, of a given digraph Γ
can be generated in time linear in the size of Γ.

Both requirements, (II) and (III), for both oracles, DGGS and MSDGGS, can be verified simultaneously.

Consider the corresponding game forms g' and g and note
that g' is obtained from g by merging some outcomes.

Namely, all outcomes corresponding to the non-terminal SCCs are replaced by a single outcome c.

It is both obvious and well-known that merging outcomes respects tightness.

Hence, it is enough to verify (II) and (III) for MSDGGSs.

By our main Theorem, to verify (II) it is sufficient to prove ± 1 solvability.

For DGGS it was done in [**Was90**]; see also [**BG03**], [**AHMS08**], [**BGMS07**].

This result was extended to MSDGGS in [**Gur18**].

Furthermore, all proofs in [**Gur18**] were constructive,
the corresponding ± 1 games were solved in time
polynomial in the size of Γ, which implies (III).

For your convenience, we briefly sketch here the proof of (II,III) from [**Gur18**].

Consider a ± 1 game $(g; \Omega_A, \Omega_B)$ with game form $g = g(\mathcal{O})$ generated by a MSDGGS oracle $\mathcal{O}$.

We would like to apply Backward Induction, yet, digraph Γ may have dicycles.

So we modify Backward Induction to make it work in presence of dicycles.

Recall that Ω is the set of SCCs of Γ and $\Gamma^* = (\Omega, E^*)$ is acyclic.

Consider an SCC $\Gamma' = (V', E')$ in Γ that is not terminal,
but each move (v', v) from a position $v' \in V'$ either ends in a terminal $v \in V_T$, or stays in Γ', that is, $v, v' \in V'$.

Obviously, such a SCC exists. Note that it may be transient.

In this case the standard Backward Induction is applicable.

Suppose that Γ' is not transient, in other words, it contains a dicycle.

Wlog we can assume that $\omega \in \Omega_A$, that is, Alice wins if the play cycles in Γ'.

Then, Bob wins in a position $v' \in V'$ if and only if he can force the play to terminate in Ω_B, while Alice wins in all other positions of V'.

Note that it is not necessary for Alice to force the play to come to a terminal from Ω_A, if the play cycles in Γ' Alice wins as well.

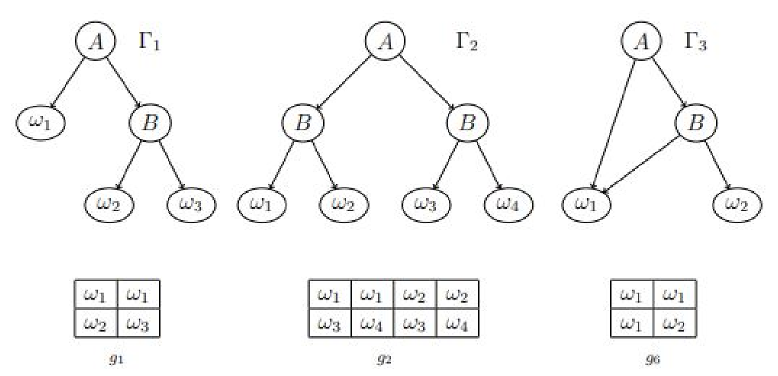
Thus, every position of Γ' belongs either to Ω_A or to Ω_B.

We make all these positions terminal, by eliminating all edges E' of Γ', and repeat until the initial position v_0 of Γ is evaluated.

This procedure proves solvability of game form $g = g(\mathcal{O})$
(which is equivalent to its tightness (II), by Theorem 1),
moreover, a ± 1 game $(g; \Omega_A, \Omega_B)$ is solved in time
linear in the size of $\mathcal{O} = \Gamma$ (which is (III)). $\square$

Acyclic deterministic graphical game structures

A game form is called *rectangular* if all its situations are simple. It is shown in [**Gur82**] that a game form g is generated by a DGGS whose graph is a tree if and only if g is tight and rectangular. Two examples, Γ_1 and Γ_2 generating game tight rectangular game forms g_1 and g_2 are given in Figure 5; see also Figure **??**. More examples can be found in [**Gur09**], where the above characterization is extended to the n-person case.



Acyclic DGGS Γ_1 in the above Figure generates game form g_1.

Recall game $(g_1; u, w)$ from Introduction with

$u(\omega_2) > u(\omega_1) > u(\omega_3)$ and $w(\omega_2) > w(\omega_3)$.

Note that the Backward Induction NE (see [**Gal53**, **Kuh53**] and also [**Gur17**]) is NE-A and is not Pareto-optimal.

In general, this NE may differ from both, NE-A and NE-B.

In absence of dicycles in Γ, the concepts of DGGS and MSDGGS coincide.

It is also clear that an acyclic DGGS is a special case of MSDGGS.

Thus, properties (I, II, III) required from an oracle hold for both.

Cyclic deterministic graphical game structures

The outcomes of MSDGGS are all its non-transient SCCs.

In particular, each terminal position is an outcome.

Let us now assume that every simple dicycle is a separate outcome
(and each terminal remains an outcome as well).

Such DGGSs, called *cyclic*, were studied in [**BGMS07**];
some special cases were considered earlier [**GG91**, **GG91a**, **GG92**].

Cyclic DGGS can also serve as oracles generating game forms; [**BGMS07**]; compare examples 3 and 4 in [**BGMS07**] with game forms g_4 and g_5 from our 9 game forms given in Introduction.

Game forms generated by the cyclic DGGS may be not tight; see Figure 1 in [**BGMS07**]. In other words, property (II) fails for the corresponding oracles, in general. Yet, it holds in some important special cases.

A digraph $G = (V, E)$ is called *symmetric* if $(v, v') \in E$ whenever $(v', v) \in E$.

Cyclic DGGS on symmetric digraphs are called *symmetric*.

Symmetric Cyclic DGGSs satisfying (II) are called *solvable* and explicitly characterized in [**BGMS07**].

It follows from results of [**BGMS07**] that (III) also holds for solvable cyclic symmetric DGGS. Hence, main Theorem is applicable.

Jordan oracle; choosing Battlefields in Wonderland

Wonderland is a subset of the plane homeomorphic to the closed disc.

Wlog, we can consider a square Q with the sides N, E, S, W.

Let us partition Q into areas $\Omega = \{\omega_1, \dots, \omega_p\}$ each of which is homeomorphic to the closed disc, too.

Every two distinct areas $\omega_i, \omega_j \in \Omega$ are either disjoint or intersect in a set homeomorphic to a closed interval that contains more than one point.

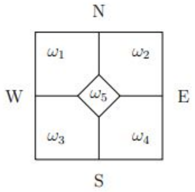
Equivalently, we can require that the borders of the areas in Q form a regular graph of degree 3.

(Note that four vertices of the square are not vertices of this graph.)

Remark

*Consider game form g_5 from Introduction and merge outcomes ω_5 and ω_6 in it getting g_5'. (This operation respects tightness). Note that $g_5' \in G$, where G is the game correspondence given in Figure 1. See also [***BGMS07***], where g_5 also appears as the normal form of a cyclic game form.*

Two examples are given in the Figures and 3.



ω_1	ω_2	ω_1	ω_2
ω_3	ω_4	ω_4	ω_3
ω_1	ω_4	$\omega_1\omega_4\omega_5$	ω_5
ω_3	ω_2	ω_5	$\omega_2\omega_3\omega_5$

Figure 1. The Jordan game correspondence of the map of Wonderland.

The following interpretation was suggested in [**GK18**]. Two players, Alice Tweedledee and Bob Tweedledum, agreed to have a battle. The next thing to do is to agree on a battlefield, which should be an area $\omega \in \Omega$. The strategies $x \in X$ of Alice are all (inclusion-minimal) subsets $x \subseteq \Omega$ connecting W and E, Respectively, the strategies $y \in Y$ of Bob are all (inclusion-minimal) subsets $y \subseteq \Omega$ connecting N and S.

Proposition

Any two such subsets x and y intersect.

Proof It follows the Jordan curve theorem and the fact that all vertices in the square are of degree 3
(except its four corners, which are of degree 2).
Note that x and y might be disjoint if we allow vertices of degree 4 or more. □

Camille Jordan

Intersection $x \cap y$ may contain several areas of Ω. Thus, a game correspondence $G : X \times Y \to 2^{\Omega} \setminus \{\emptyset\}$ is defined.

Proposition

Game correspondence G is tight.

Proof

Again, it follows from the Jordan curve theorem and the assumption that all vertices in the square are of degree 3. Choose an arbitrary $g \in G$ and consider a ± 1 game $(g; \Omega_A, \Omega_B)$ determined by a partition $\Omega = \Omega_A \cup \Omega_B$. Then, from the following two options exactly one holds:

(a) areas from Ω_A connect W and E; (b) areas from Ω_B connect N and S. $\square$

The above observations imply that Jordan oracle $\mathcal{O}$ satisfies requirements (I) and (II). It remains to verify (III),

Proposition

By using oracle $\mathcal{O}$, one can decide whether (a) or (b) holds and find corresponding x or y, respectively, in time linear in $|\mathcal{O}|$.

Proof Consider all areas from Ω_B boarding N, then add all areas from Ω_B boarding these areas, etc. Such iterations will stop in time linear in $|\mathcal{O}|$ either reaching S (then, obviously, (b) holds) or not (then (a) holds, again by the Jordan curve theorem). Moreover, in the first case we obtain a set of areas y' from Ω_B connecting N and S; in the second case - a set of areas x' from Ω_A connecting W and E. The former strategy y' is obtained explicitly; the latter one, x', is easy to construct. To do so, denote by Ω'_B the set of areas obtained in the course of iterations. It does not reach S. Hence, the areas from Ω_A that border Ω'_B connect W and E, by the Jordan curve theorem once more. □

This case is realized in Figure 3; Alice wins.

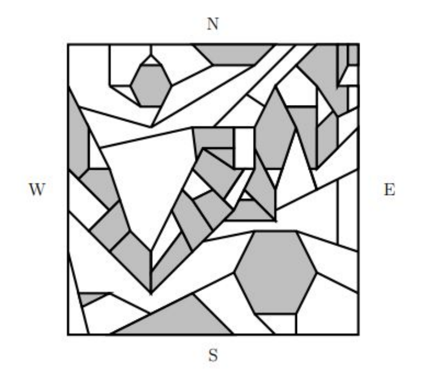


Figure 2. Gray and white areas are in Ω_A and Ω_B, respectively. Alice wins.

Remark

It is not necessary to restrict ourselves by minimal strategies. In linear time we can reduce arbitrary strategy (set) x' of Alice to an inclusion-minimal set x connecting W and E, thus, getting minimal strategies of Alice.
To do so, we eliminate areas from x' one by one until (a) still holds.
We require inclusion-minimality of subsets $x \in \Omega$ just to reduce the number of strategies (which may still remain exponential in $|\mathcal{O}|$).
Of course, the same is true for Bob's strategies.

Monotone bargaining schemes

The following oracle was introduced in [**GK18**].
Two players, Alice and Bob, possess items

$A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_n\}$, respectively. Both sets are ordered: $a_1 \prec \cdots \prec a_m$ and $b_1 \prec \cdots \prec b_n$. Both players know both orders.

The direct product $\Omega = A \times B = \{(a, b) \mid a \in A, b \in B\}$ is the set of *outcomes*.

Alice's strategies are monotone non-decreasing mappings $x : A \to B$
(that is, $x(a) \geq x(a')$ whenever $a > a'$)
showing that she is ready to exchange a for $x(a)$ for any $a \in A$.

Similarly, Bob's strategies are monotone non-decreasing mappings $y : B \to A$
(that is, $y(b) \geq y(b')$ whenever $b > b'$)
showing that he is ready to exchange b for $y(b)$ for any $b \in B$.

It is not difficult to compute the numbers of strategies and outcomes:

$$|X| = \binom{m+n-1}{m}, \; |Y| = \binom{m+n-1}{n}; \;\; |\Omega| = |A \times B| = mn. \qquad (1)$$

Given a situation (x, y), an outcome $(a, b) \in \Omega$ is called a *deal*
(in this situation) if $x(a) = b$ and $y(b) = a$.
Denote by $G(x, y) \subseteq \Omega$ the set of all deals in the situation (x, y).

We will show that $G(x, y) \neq \emptyset$. Yet, $G(x, y)$ may contain several deals.

This construction is called a *monotone bargaining (MB) scheme*.

It can be viewed as an oracle $\mathcal{O}$ generating game correspondence
$G : X \times Y \to 2^{\Omega} \setminus \{\emptyset\}$. By (1), requirement (I) holds for $\mathcal{O}$.

Note that $G = G_{m,n}$ is uniquely defined by m and n.

A game form $g \in G$ is called an *MB game form*.

For example, if $m = n = 3$ then $|X| = |Y| = 3$
and we obtain game form g_4 in Figure 1;
game correspondence $G(x, y)$ is given in [**GK18**].

The following interpretation was suggested in [**GK18**].

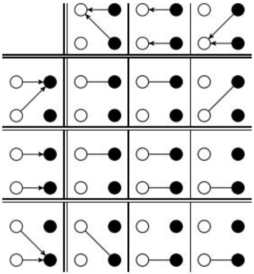
Alice and Bob are dealers possessing the sets of objects A and B, respectively, and a deal $(a, b) \in A \times B$ means that they exchange a and b.

They may be art-dealers, car dealers; or one of them may be just a buyer with a discrete budget. For example, $A = \{a_1, \dots, a_m\}$ and $B = \{b_1, \dots, b_n\}$ may be paintings or sculptures ordered in accordance with their age (not price or value).

To any pair of mappings $x : A \to B$ and $y : B \to A$
(not necessarily monotone non-decreasing) let us assign a bipartite digraph $\Gamma = \Gamma(x, y)$ on the vertex-set $A \cup B$ as follows:

$[a, b)$ (resp., $[b, a)$) is an arc of $\Gamma(x, y)$ whenever $x(a) = b$ (resp., $y(b) = a$).

Some visualization helps.
Embed $\Gamma(x, y)$ into a plane; putting ordered A and B in two parallel columns.
Two arcs corresponding to x may have a common head, but not tail.
Furthermore, they cannot cross if mapping x is monotone non-decreasing.
Similarly for y. By construction, digraph Γ is bipartite, with parts A and B.
Hence, every dicycle in Γ is even.
There is an obvious one-to-one correspondence between the dicycles of length 2 in $\Gamma(x, y)$ and the deals of $G(x, y)$.



	1 1 1 2	1 1 2 2	2 1 1 2
1 1 2 1	1 1	1 1	2 1
1 1 2 2	1 1	1 1 2 2	2 2
1 2 2 2	1 2	2 2	2 2

Figure 3. Monotone bargaining game correspondence $G(2,2)$.

Proposition

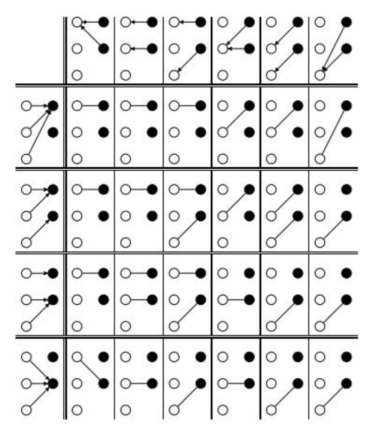
For each situation (x, y) its digraph $\Gamma(x, y)$ contains at least one dicycle of length 2 (a deal) and cannot contain longer dicycles.

Proof For any initial vertex $v \in A \cup B$, strategies x and y uniquely define an infinite walk from v, which is called a play.

Since sets A and B are finite and there are no terminals, this play is a *lasso*:

it consists of an initial directed path, which may be empty, and a dicycle C repeated infinitely.

Furthermore, C must be a dicycle of length 2 whenever mappings x and y are monotone non-decreasing. Indeed, if C is longer than 2 then crossing arcs appear and, hence, either x, or y, or both are not monotone, $\square$



	1 1 1 2	1 1 2 2	1 1 3 2	2 1 2 2	2 1 3 2	3 1 3 2
1 1 2 1 3 2	1 1	1 1	1 1	2 1	2 1	3 1
1 1 2 2 3 2	1 1	1 1	1 1 3 2	2 1	2 1 3 2	3 2
1 1 2 2 3 2	1 1	1 1 2 2	1 1 3 2	2 2	3 2	2 2
1 2 2 2 3 2	1 2	2 2	3 2	2 2	3 2	3 2

Figure 4. Monotone bargaining game correspondence $G(2,3)$.

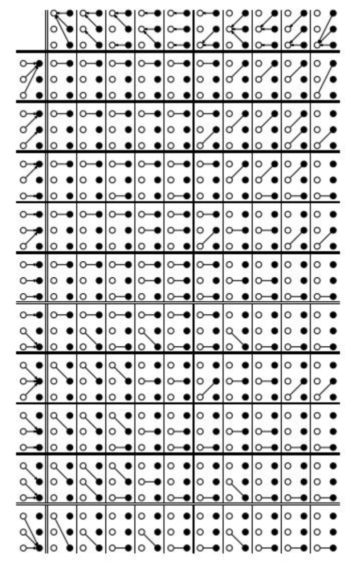


Figure 5. Monotone bargaining game correspondence $G(3,3)$.

Consider a ± 1 MB game $(g; \Omega_A, \Omega_B)$, where $g = g(\mathcal{O})$ is an MB game form generated by an MB scheme $\mathcal{O}$. As we already mentioned, requirement (I) holds for $\mathcal{O}$. The following statement shows that (II) and (III) hold as well.

Proposition

Game form $g = g(\mathcal{O})$ is tight and each ± 1 MB game $(g; \Omega_A, \Omega_B)$ can be solved in time polynomial in $|\mathcal{O}| = mn$.

The first part was already proven in [**GK18**]. Yet, here we provide a much shorter proof.

Proof

For the sake of simplicity, we will slightly abuse notation writing that both directed edges $[a, b)$ and $[b, a)$ are in Ω_A or in Ω_B whenever the corresponding deal (a, b) is in Ω_A or in Ω_B, respectively.

Consider complete bipartite symmetric digraph Γ on $m + n$ vertices $A = \{a_1, \ldots, a_m\}, B = \{b_1, \ldots, b_n\}$, and with $2mn$ directed edges $\{[a_i, b_j), [b_j, a_i) \mid i = 1, \ldots, m; j = 1, \ldots, n\}$.

The following two statements are obvious:

(a) Alice wins if she has a monotone non-decreasing strategy $x^* : A \to B$ such that $[a, x^*(a)) \in \Omega_A$ for all $a \in A$.

(b) Bob wins if he has a monotone non-decreasing strategy $y^* : B \to A$ such that $[b, y^*(b)) \in \Omega_B$ for all $b \in B$.

Indeed, it is easily seen that x^* and y^* are the winning strategies of Alice and Bob, respectively. It is enough to show that $g(x^*, y) \in \Omega_A$ for any $y \in Y$.

Recall the proof of Proposition 5: Fix x^*, choose an arbitrary $y \in Y$, and consider the play $P = P(x^*, y)$ beginning from an arbitrary initial position $v \in A \cup B$. By Proposition 5, P is a lasso resulting in a 2-cycle (a, b).

The corresponding deal $(a, b) \in \Omega_A$, in case (a), for any y, by the choice of x^*, and hence, Alice wins. Similarly, $g(x, y^*) \in \Omega_B$ in case (b) for any $x \in X$, by the choice of y^*, and Bob wins.

Obviously, (a) and (b) cannot hold simultaneously, since otherwise $(a, b) \in \Omega_A \cap \Omega_B$, which is a contradiction, since $\Omega = \Omega_A \cup \Omega_B$ is a partition.

Let us show that either (a) or (b) holds (in other words, g is tight, which implies (II)).

The proof will be constructive: we obtain either x^* satisfying (a) or y^* satisfying (b) in time polynomial in mn
(which in its turn, implies (III)).

We will construct a play P by the following greedy iterative algorithm.
Let $a^1 = a_1$ be an initial position of P.
(We use superscripts to number iterations.)

If $[a^1, b) \in \Omega_B$ for all $b \in B$ then Bob wins.
(His winning strategy y^* is defined by: $y^*(b) = a^1$ for all $b \in B$.

Then $[y^*(b), b) \in \Omega_B$ for all $b \in B$ and (b) holds.)

Otherwise, denote by b^1 the (unique) minimal $b \in B$ such that $[a^1, b) \in \Omega_A$.

Then, by definition, $[b^1, a^1) \in \Omega_A$ too.

Furthermore, by this choice of b^1, we have:

$[b, a^1) \in \Omega_B$ for all $b \prec b^1$, while $[b^1, a^1) \in \Omega_A$.

If $[b^1, a) \in \Omega_A$ for all $a \succeq a^1$ then Alice wins.
(Her winning strategy x^* is defined by: $x^*(a) = b^1$ for all $a \in A$.
Then $[a, x^*(a)) \in \Omega_A$ for all $a \in A$.)

Otherwise, denote by a^2 the (unique) minimal $a \in A$ such that $[b^1, a) \in \Omega_B$.

Then, by definition, $[a^2, b^1) \in \Omega_B$ too. Furthermore, by the choice of a^2, we have: $[a, b^1) \in \Omega_A$ for all $a \prec a^2$, while $[a^2, b^1) \in \Omega_B$.

The general k-th step of this greedy recursion is as follows.

If $[a^k, b) \in \Omega_B$ for all $b \succeq b^{k-1}$ then Bob wins.
(His winning strategy y^* is defined by:
$y^*(b) = a^i$ for each b such that $b^i \succ b \succeq b^{i-1}$,
for $i = 1, \ldots, k$, assuming conventionally that $b \succ b^0$ holds for all $b \in B$).

Otherwise, denote by b^k the (unique) minimal $b \in B$ such that
$b \succ b^{k-1}$ and $[a^k, b) \in \Omega_A$. Then $[b^k, a^k) \in \Omega_A$ too.

Furthermore, by the choice of b^k, we have:
$[b, a^k) \in \Omega_B$ for all b such that $b^k \succ b \succeq b^{k-1}$, while $[b^k, a^k) \in \Omega_A$.

If $[b^k, a) \in \Omega_A$ for all $a \succeq a^k$ then Alice wins.
(Her winning strategy x^* is defined by:
$x^*(a) = b^j$ for each a such that $a^{j+1} \succ a \succeq a^j$,
for $j = 1, \ldots, k$, assuming conventionally that $a^{k+1} \succ a$ holds for all $a \in A$.)

Otherwise, denote by a^{k+1} the (unique) minimal $a \in A$ such that $[b^k, a) \in \Omega_B$.
Then $[a^{k+1}, b^k) \in \Omega_B$, too.

Furthermore, by the choice of a^{k+1}, we have: $[a, b^k) \in \Omega_A$ for all a such that $a^{k+1} \succ a \succeq a^k$, while $[a^{k+1}, b^k) \in \Omega_B$.

After each iteration a^k (respectively, b^k) both Alice and Bob have winning moves in all positions $a \prec a^k$ and $b \preceq b^{k-1}$ (respectively, $a \preceq a^k$ and $b \prec b^k$). Since sets A and B are finite, the procedure will stop on some iteration either $a^{k^*} \prec a_m$ or $b^{k^*} \prec b_n$, indicating that Bob or, respectively, Alice wins.

Furthermore, we obtain his or her winning strategy in time linear in mn.

The following slightly different procedure can be applied too.

First, we start looking for a winning strategy x^* for Alice.

Consider successively $a_1, a_2, \dots$ and construct (again recursively and greedily) her monotone non-decreasing strategy x^* as follows:

$x^*(a_i) = b^i$ such that $[a_i, b^i) \in \Omega_A$, $b^i \succeq b^{i-1}$, and

b^i is the minimal element of B satisfying these two properties.

If this will work for all $i = 1, \dots, m$ then Alice wins and
we obtain her winning strategy x^* satisfying (a).

Otherwise, if the procedure stops on some $i < m$
(no required b^i exists for a_i) then Bob wins.
His winning strategy y^* satisfying (b) is defined as follows:
$y^*(b) = a^i$ for all b such that $b^{i-1} \preceq b \prec b^i$, where a_i is the smallest a such that $x^*(a) = b^i$, for $i = 1, 2, \dots$

By convention, $b^0 \prec b$ for all $b \in B$. □

Thus, requirements (I,II,III) hold for the MB schemes and, hence, our main Theorem is applicable.

Veto voting schemes

Two voters (players), Alice and Bob choose among candidates (options, outcomes) $\Omega = \{\omega_1, \ldots, \omega_p\}$.

They are assigned some positive integer veto powers and given μ_A and μ_B veto cards, respectively.

Each candidate $\omega \in \Omega$ is assigned an integer positive veto resistance λ_ω. We assume that

$$\mu_A + \mu_B + 1 = \lambda_{\omega_1} + \cdots + \lambda_{\omega_p}. \qquad (2)$$

A strategy of a voter is an arbitrary distribution of her/his veto cards among the candidates.

Given a pair of strategies x and y, a candidate $\omega \in \Omega$ who got at least λ_ω veto cards (from Alice and Bob together) is vetoed.

From the set $G(x, y)$ of all not vetoed candidates one $g(x, y) \in G(x, y)$ is elected. By (2), $G(x, y) \neq \emptyset$.

Thus, we obtain a *veto voting* (VV) scheme $\mathcal{O}$, VV game form $g = g(\mathcal{O})$, and VV game correspondence $G = G(\mathcal{O})$; see, for example, [**Gur08**],[**Mou83**],[**Pel84**] for more details.

By construction, VV schemes are oracles satisfying (I). For example, game form g_3 in Figure 1 corresponds to the VV scheme defined by

$$\mu_A = \mu_B = \lambda_{\omega_\mathbf{1}} = \lambda_{\omega_\mathbf{2}} = \lambda_{\omega_\mathbf{3}} = 1.$$

Let us show that requirements (II) and (III) also hold for VV schemes.

Proposition

Each game form g defined by a VV scheme satisfying (2) is tight.

Furthermore, every ± 1 game $(g; \Omega_A, \Omega_B)$ can be solved in time linear in $|\mathcal{O}| = \log(\mu_A \mu_B \prod_{\omega \in \Omega} \lambda_\omega)$.

Proof

To see this, consider a ± 1 game $(g; \Omega_A, \Omega_B)$.

By (2), from two options,
(a) Alice can veto Ω_B and (b) Bob can veto Ω_A, exactly one holds.

Alice or Bob wins in case of (a) or (b), respectively. Given numbers μ_A, μ_B, and λ_ω, $\omega \in \Omega$, one can decide in linear time whether (a) or (b) holds.

In each case the winning strategy of Alice or Bob is straightforward:
just veto all opponent's candidates, Ω_B or Ω_A, respectively. □

Thus, the VV oracles satisfy (I,II,III) and main Theorem is applicable.

Tight game correspondences and forms of arbitrary monotone properties

The most general setting is defined as follows. Given a finite ground set Ω, consider a family of its subsets $\mathcal{P} \subseteq 2^{\Omega}$. Standardly, we call $\mathcal{P}$ a *property* and say that a subset $\Omega' \subseteq \Omega$ satisfies $\mathcal{P}$ or not if $\Omega' \in \mathcal{P}$ or $\Omega' \notin \mathcal{P}$, respectively. Property $\mathcal{P}$ is called *inclusion monotone non-decreasing* (or simply *monotone*, for short) if $\Omega'' \in \mathcal{P}$ implies $\Omega' \in \mathcal{P}$ whenever $\Omega'' \subseteq \Omega' \subseteq \Omega$. We restrict ourselves to monotone properties.

Define the sets of strategies X of Alice and Y of Bob as follows:

$x \in X$ is any (inclusion minimal) subset $\Omega_A \subseteq \Omega$ such that $\Omega_A \in \mathcal{P}$;

$y \in Y$ is any (inclusion minimal) subset $\Omega_B \subseteq \Omega$ such that $\Omega \setminus \Omega_B \notin \mathcal{P}$.

The restriction in parenthesis does not matter, it can be waved or kept. In the latter case, sets X and Y are significantly reduced.

Define a game correspondence $G = G(\mathcal{P})$ by setting $G(x, y) = x \cap y$. It is both obvious and well-known that $G(x, y) \neq \emptyset$ for any $x \in X, y \in Y$ and, moreover, G is tight. Hence, any game form $g \in G$ is tight too.

Thus, (I) and (II) hold automatically whenever a monotone property $\mathcal{P}$ is given by an oracle $\mathcal{O}(\mathcal{P})$. Yet, (III) must be required in addition. In other words, $\mathcal{O}(\mathcal{P})$ must be a polynomial membership oracle, which for a given subset $\Omega' \subseteq \Omega$, decides if $\Omega' \in \mathcal{P}$ in time polynomial in $|\Omega| + |\mathcal{O}(\mathcal{P})|$.

It is easily seen that this general setting includes in particular all four examples of oracles given in this section before; see more examples in [**BGEK02**, **Gur18a**].

References

AFPT10 N. Alon, M. Feldman, A. D. Procaccia, and M. Tennenholtz, A note on competitive diffusion through social networks, Information Processing Letters, 110:6 (2010) 221–225.

AHMS08 D. Andersson, K. Hansen, P. Miltersen, and T. Sorensen, Deterministic graphical games, revisited, J. Logic and Computation. 22:2 (2012) 165-178. Preliminary version in Fourth Conference on Computability in Europe (CiE-08), Lecture Notes in Computer Science. 5028 (2008) 1–10.

BBM90 A.B. Barabas, R.E. Basko, and I.S. Menshikov, On an approach to analysis of conflict situations, Proc. Moscow Comput. Center 2 (1990) 13–20 (in Russian).

BCGM20 E. Boros, O. Čepek, V. Gurvich, and K. Makino, Recognizing distributed approval voting forms and correspondences, http://arxiv.org/abs/2010.15730 (2020) 1–16, Annals of Operations Research, to appear.

BG03 E. Boros and V. Gurvich, On Nash-solvability in pure strategies of finite games with perfect information which may have cycles, Math. Soc. Sciences 46 (2003) 207–241.

BGEK02 E. Boros, V. Gurvich, K. Elbassioni, and L. Khachiyan, Generating dual-bounded hypergraphs, Optimization Methods and Software 17:5 (2002) 749 – 781.

BGMP10 E. Boros, V. Gurvich, K. Makino, and D. Papp, Acyclic, or totally tight, two-person game forms; a characterization and main properties; Discrete Math. 310:6-7 (2010) 1135–1151.

BGMS07 E. Boros, V. Gurvich, K. Makino, and Wei Shao, Nash-solvabile two-person symmetric cycle game forms, Discrete Appl. Math. 159:15 (2011) 1461–1487.

BFT17 L. Bulteau, V. Froese, and N. Talmon, Multi-player diffusion games on graph classes, https://arxiv.org/abs/1412.2544 (2017) 1–21.

CH11 Y. Crama and P. L. Hammer, Boolean functions: Theory, algorithms, and applications, Cambridge University Press, 2011.

DS91 V.I. Danilov and A.I. Sotskov, Social Choice Mechanisms, Moscow Nauka 1991 (in Russian); English translation in Studies of Economic Design, Springer, Berlin-Heidelberg, 2002.

EF70 J. Edmonds and D.R. Fulkerson, Bottleneck extrema, J. of Combinatorial Theory 8 (1970) 299–306.

FK96 M. Fredman and L. Khachiyan, On the complexity of dualization of Monotone Disjunctive Normal Forms, J. Algorithms 21 (1996) 618–628.

FHKOY20 N. Fukuzono, T. Hanaka, H. Kiya, H. Ono, and R. Yamaguchi, Two-player competitive diffusion game: graph classes and the existence of a Nash equilibrium, 46th International Conference on Current Trends in Theory and Practice of Informatics, Limassol, Cyprus, Jan. 20–24, 2020, Proceedings (2020) 627–635,

Gal53 D. Gale, A theory of N-person games with perfect information, Proc. Natl. Acad. Sci. 39 (1953) 496–501.

GS62 D. Gale and L.S. Shapley, College admissions and the stability of marriage, The American Mathematical Monthly 69:1 (1962) 9–15.

GG91 A.I. Gol'berg and V.A. Gurvich, Tight cyclic game forms, Russian Math. Surveys, 46:2 (1991) 241–242.

GG91a A.I. Gol'berg and V.A. Gurvich. Some properties of tight cyclic game forms, Soviet. Math Dokl., 43:3 (1991) 898–903.

GG92 A.I. Gol'berg and V.A. Gurvich, A tightness criterion for reciprocal bipartite cyclic game forms, Russian Acad. Sci. Dokl. Math. 45:2 (1992) 348–354.

Gur73 V. Gurvich, To theory of multi-step games, USSR Comput. Math. and Math. Phys. 13:6 (1973) 143–161.

Gur75 V. Gurvich, Solution of positional games in pure strategies, USSR Comput. Math. and Math. Phys. 15:2 (1975) 74–87.

Gur82 V. Gurvich, On the normal form of positional games, Soviet Math Dokl. 25:3 (1982) 572–575.

Gur89 V. Gurvich, Equilibrium in pure strategies, Soviet Math. Dokl. 38:3 (1989) 597–602.

Gur08 V. Gurvich, War and Peace in veto voting, European J. of Operational Research 185 (2008) 438–443.

Gur09 V. Gurvich, Decomposing complete edge-chromatic graphs and hypergraphs, revisited, Discrete Appl. Math. 157 (2009) 3069–3085.

Gur17 V. Gurvich, Generalizing Gale's theorem on backward induction and domination of strategies, arXiv http://arxiv.org/abs/1711.11353 (2017) 12 pp.

Gur18 V. Gurvich, Backward induction in presence of cycles, Oxford Journal of Logic and Computation 28:7 (2018) 1635–1646.

Gur18a V. Gurvich, Complexity of generation, Computer Science in Russia, XIII-th International Computer Science Symposium in Russia (CSR-13) Moscow, June 6–10, 2018; Lecture Notes in Computer Science LNCS 10846 (2018) 1–14.

GK99 V. Gurvich and L. Khachiyan, On generating the irredundant conjunctive and disjunctive normal forms of monotone boolean functions, Discrete Appl. Math. 96-97 (1999) 363–373.

GK18 V. Gurvich and G. Koshevoy, Monotone bargaining is Nash-solvable, Discrete Appl. Math. 250 (2018) 1–15.

GN21 V. Gurvich and M. Naumova, Polynomial algorithms computing two lexicographically safe Nash equilibria in finite two-person games with tight game forms given by oracles; https://arxiv.org/abs/2108.05469; 2021.

GN22 V. Gurvich and M. Naumova, Lexicographically maximal edges of dual hypergraphs and Nash-solvability of tight game forms; https://arxiv.org/abs/2204.10213; Published online: 19 October 2022, Annals of Math. and Artificial Intelligence.

GI89 D. Gusfield and R. W. Irving, The stable marriage problem: Structure and Algorithms, MIT Press, 1989.

HKM21 T. Harks, M. Klimm, and J. Matuschke, Pure Nash equilibria in resource graph games, Journal of Artificial Intelligence Research (2021).

Kuh53 H. Kuhn, Extensive games and the problem of information, in Contributions to the theory of games, Volume 2, Princeton (1953) 193–216.

Kuk11 N.S. Kukushkin, Acyclicity of improvements in finite game forms, Int. J. Game Theory 40 (2011) 147–177.

MS96 D. Monderer and L. S. Shapley, Potential games, Games and Economic behavior 14 (1996) 124–143.

Mou83 H. Moulin, The Strategy of social choice, Advanced textbooks in Economics, 18, North Holland Publishing Company, Amsterdam, New York, Oxford, 1983.

Nas50 J. Nash, Equilibrium points in *n*-person games, Proceedings of the National Academy of Sciences 36:1 (1950) 48–49.

Nas51 J. Nash, Non-cooperative games, Annals of Math. 54:2 (1951) 286–295.

Pel84 B. Peleg, Game theoretic analysis of voting in committees, Cambridge University Press, Cambridge, London, New York, New Rochelle, Melbourne, Sydney, 1984.

Ros73 R. Rosenthal. A class of games possessing pure-strategy Nash equilibria: Internat. J. Game Theory, 2:1 (1973) 65–67.

Sha81 M. Sharir, A strong-connectivity algorithm and its application in data flow analysis, Comput. Math. Appl. 7 (1981) 67–72.

SM13 L. Small and O. Mason, Nash equilibria for competitive information diffusion on trees, Information Processing Letters 113:7 (2013) 217–219.

Tar72 R.E. Tarjan, Depth-first search and linear graph algorithms, SIAM J. Computing 1:2 (1972) 146–160.

THS12 R. Takehara, M. Hachimori, and M. Shigeno, A comment on pure-strategy Nash equilibria in competitive diffusion games, Information Processing Letters 112:3 (2012) 59—60.

Was90 A. R. Washburn, Deterministic graphical games, J. of Math. Analysis and Appl. 153 (1990) 84–96.