Calabi-Yau threefolds in $\mathbb{P}^7$ and extensions of curves of genus 5

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Extensions

Extendibility problem

Given a projective (irreducible) variety $X \subset \mathbb{P}^n$, when does there exist a projective variety $Y \subset \mathbb{P}^{n+1}$, not a cone, of which X is a hyperplane section?

$$X = Y \cap \mathbb{P}^n.$$

Definition

- Given a positive integer r, an *r-extension* of $X \subset \mathbb{P}^n$ is a variety $Y \subset \mathbb{P}^{n+r}$ having X as a section by a linear space.
- The variety X is r-extendable if it has an r-extension that is not a cone,
- The variety X is extendable if it is at least 1-extendable.

Motivations

Theorem (C. Segre)

Let $C \subseteq \mathbb{P}^{n-1}$ be a smooth linearly normal and non-degenerate curve of genus $g > 0$. Assume there exists a scroll $\Sigma \subseteq \mathbb{P}^n$ such that C is a hyperplane section of Σ. Then the scroll Σ is necessarily a cone.

Definition

- Surface S is a ruled surface if S is surface equipped with a locally trivial morphism $S \to C$ onto a smooth curve whose fibres are $\mathbb{P}^1$.
- Surface S is scrolls if it is a ruled surface embedded in some projective space in such a way that the fibres are lines.

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[1] C. Segre, *Ricerche sul le rigate el littiche di qualunque ordine*, Atti Accad. Sci. Torino **21** (1885–86), 628–651.

Curves of degree $d \geq 4g - 4$

Theorem (R. Hartshorne)

Let C be a smooth curve of genus g, sitting in a smooth surface S. If $C^2 > 4g + 5$, then S is a ruled surface having C as a section.
If $C^2 = 4g + 5$, the only other possibility is that C is a cubic curve in $S = \mathbb{P}^2$.

Theorem (C. Ciliberto and T. Dedieu)

Let $S \subseteq \mathbb{P}^{r+1}$ be a nondegenerate irreducible, projective surface of degree $d \geqslant 4g - 4$, and whose general hyperplane section C is smooth, of genus $g \geqslant 2$, and linearly normal. If S is not a cone, then one of the following holds:

- *S is the image by the Veronese map v_2 of a cone over an elliptic normal curve of degree $g - 1$, and the hyperplane sections of S are bielliptic bicanonical curves;*
- *S is a rational surface represented by a linear system of plane δ-ics, $4 \leqslant \delta \leqslant 6$;*
- *S is the image by the Veronese map v_2 of a Del Pezzo surface;*
- *S is a rational surface with hyperelliptic sections, represented by a linear subsystem of $|2H + (g + 1 - e)F|$ on $\mathbb{F}_e$;*
- *S is a rational surface with trigonal sections, represented by a linear subsystem of $\left|3H + \frac{1}{2}(g - 3e + 2)F\right|$ on $\mathbb{F}_e$.*

[2] [3]

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[2] R. Hartshorne, *Curves with high self-intersection on algebraic surfaces*, Inst. Hautes Etudes Sci. Publ. Math. (1969), no. 36, 111-125.

[3] C. Ciliberto and T. Dedieu *Extensions of curves with high degree with respect to the genus*, Épijournal de Géométrie Algébrique, special volume in honour of C. Voisin, article no. 16 (2024)

Curves of degree $d < 4g - 4$

Open problem (C. Ciliberto and T. Dedieu)

Let $S \subseteq \mathbb{P}^{r+1}$ be a nondegenerate irreducible, projective surface of degree $d < 4g - 4$, and whose general hyperplane section C is smooth, of genus $g \geqslant 2$, and linearly normal. When is S not a cone?

$g = 3$ C. Ciliberto and T. Dedieu [4]

$g = 4$??

$g = 5$ Main topic in this talk

Open Problem

How many ways can a smooth curve (C, η) of degree d genus 5 extend to a surface S such that $C \in |H|$ and $\omega_S \otimes \mathcal{O}_C = \eta$?

[4] C. Ciliberto and T. Dedieu *Extensions of curves with high degree with respect to the genus*, Épijournal de Géométrie Algébrique, special volume in honour of C. Voisin, article no. 16 (2024)

General curves

Theorem ($d \geq 12$)

Let $C \subseteq \mathbb{P}^{d-5}$ be a smooth irreducible nondegenerate linearly normal general curve of genus 5 and degree d. Then C is not extendable if $d \geq 12$.

Theorem ($d = 11$)

1) *A smooth general curve C of degree 11 and genus 5 in $\mathbb{P}^6$, with a divisor $\eta \in \mathrm{Pic}^0(C)$, can be extended at most 3 steps.*
2) *Let (C, η) be a smooth general curve in $\mathbb{P}^6$ of degree 11, genus 5 with a divisor $\eta \in \mathrm{Pic}^0(C)$. Then there exists a unique family of 1-extensions S of (C, η) with $K_S^2 = -1$.*
3) *Let (S, H) be a general, non-degenerate projective surface in $\mathbb{P}^7$ of degree 11, sectional genus 5, and $K_S^2 = -1$. Then the surface S possesses only a two-step extension pathway.*

General curves

Theorem ($d = 10$)

1) *General curves C of degree 10 and genus 5 in $\mathbb{P}^5$ exhibit two distinct extension pathways. One of these pathways involves three steps, while the other comprises just two steps.*
2) *Let (C, η) be a smooth general curve in $\mathbb{P}^5$ of degree 10, genus 5 with a divisor $\eta \in \mathrm{Pic}^0(C)$. Then there exists a unique family of 1-extensions S of (C, η) with $K_S^2 = -2$.*
3) *Let (S, H) be a general, non-degenerate projective surface in $\mathbb{P}^6$ of degree 10, sectional genus 5 and $K_S^2 = -2$. Then the surface S exhibits two distinct extension pathways. One of these pathways involves two steps, while the other comprises just one step.*

General curves

Theorem ($d = 9$)

1) *General curves C of degree 9 and genus 5 in $\mathbb{P}^4$ exhibit three distinct extension pathways. Two of these pathways involve three steps, while the other comprises just one step.*

2) *Let (C, η) be a smooth general curve in $\mathbb{P}^4$ of degree 9, genus 5 with a divisor $\eta \in \mathrm{Pic}^0(C)$. Then*

   a) *there is a unique family of an 1-extension S of (C, η) with $K_S^2 = -3$,*
   b) *there is a unique family of an 1-extension S of (C, η) with $K_S^2 = -1$,*

   and no other.

3) *Let (S, H) be a general, non-degenerate projective surface in $\mathbb{P}^5$ of degree 9 sectional genus 5. Then where $K_S^2 = -1$, the surface S is not extendable. However, if $K_S^2 = -3$, the surface exhibits two distinct extension pathways, each comprising two steps.*

Trigonal Curves

Theorem ($d \geq 10$)

1) *Trigonal curves C of degree d and genus 5 in $\mathbb{P}^4$ have only a $17-d$-step extension pathway, for $10 \leq d \leq 15$.*
2) *Let C be trigonal curve of degree d and genus 5 in $\mathbb{P}^{d-g}$ for $10 \leq d \leq 15$. Then there is a variety Σ in $\mathbb{P}^{12}$ of dimension $18-d$, degree d and sectional genus 5, not a cone, having C as a linear section, and satisfying the following property: the surface linear sections of Σ containing C are in one-to-one correspondence with the surface extensions of C in $\mathbb{P}^{d-g+1}$ that are not cones.*
3) *Let (S, H) be a general non-degenerate projective surface of degree $d \geq 10$ sectional genus 5 and $K_S^2 = d-13$. Then the surface S has only a $16-d$-step extension pathway. In particular, there exists a variety $\Sigma \subset \mathbb{P}^{12}$ of dimension $18-d$ with S as a section through a linear space and satisfying the following property: the threefold linear sections of Σ containing S are in one-to-one correspondence with the threefold extensions of S in $\mathbb{P}^{d-g+2}$ that are not cones.*

Trigonal curves

Theorem (Trigonal curves of degree 9)

1) *Trigonal curves C of degree 9 and genus 5 in $\mathbb{P}^4$ exhibit two distinct extension pathways. One involves a single step, while the other comprises seven steps.*

2) *Let (C, η) be a smooth trigonal curve in $\mathbb{P}^4$ of degree 9, genus 5 with a divisor $\eta \in \mathrm{Pic}^0(C)$. Then*

   a) *there is a unique family of an 1-extension S of (C, η) with $K_S^2 = -4$,*
   b) *there is a unique family of an 1-extension S of (C, η) with $K_S^2 = -1$,*

   and no other.

3) *Let (S, H) be a smooth trigonal non-degenerate projective surface in $\mathbb{P}^5$ of degree 9 sectional genus 5 and $K_S^2 = -1$. Then the surface S are not extendable.*

4) *Let (S, H) be a smooth trigonal non-degenerate projective surface in $\mathbb{P}^5$ of degree 9 sectional genus 5 and $K_S^2 = -4$. Then the surface S only has a 7-step extension pathway. Furthermore, there exists a variety $Y_4 \subset \mathbb{P}^{12}$ with S as a section through a linear space and satisfying the following property: the threefold linear sections of Σ containing S are in one-to-one correspondence with the threefold extensions of S in $\mathbb{P}^6$ that are not cones.*

Calabi-Yau varieties

- X : a closed subscheme in $\mathbb{P}^n$ of codimension c.
- R : the polynomial ring $k[x_0, \ldots, x_n]$ over an algebraically closed field k.
- $A(X) = R/I_X$: the coordinate ring of X.
- $\omega_{A(X)}$: the canonical module of X.
- $\mathcal{E}$: coherent sheaf on X.
- $H^i_\bullet(\mathcal{E}) = \bigoplus_{m \in \mathbb{Z}} H^i(X, \mathcal{E}(m))$.

Definition 1

A smooth threefold X is Calabi-Yau if

$$K_X \simeq \mathcal{O}_X \text{ and } H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0.$$

Examples

Definition 1

A smooth threefold X is Calabi-Yau if

$$K_X \simeq \mathcal{O}_X \text{ and } H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0.$$

- A quintic hypersurface in $\mathbb{P}^4$.
- When X is a complete intersection (CI) of type $\{d_1, \ldots, d_{n-3}\} \subseteq \mathbb{P}^n$ we have

$$K_X \simeq \mathcal{O}_X\left(-n - 1 + \sum d_i\right)$$

So a complete intersection Calabi-Yau (CICY) threefold in $\mathbb{P}^n$ must have $\{d_1, \ldots, d_{n-3}\}$ satisfying

$$\begin{array}{ccc} \{5\} & \text{in} & \mathbb{P}^4 \\ \{2,4\} & \text{in} & \mathbb{P}^5 \\ \{3,3\} & \text{in} & \mathbb{P}^5 \\ \{2,2,3\} & \text{in} & \mathbb{P}^6 \\ \{2,2,2,2\} & \text{in} & \mathbb{P}^7 \end{array}$$

Problems

- Calabi-Yau threefolds $X \subseteq \mathbb{P}^6$ giving the complete classification of the examples of degree $d \leq 14$ (Kapustka G., Kapustka M. 2013)
- If X is Calabi-Yau threefolds in $\mathbb{P}^6$ then $11 \leq d(X) \leq 41$.

Problem 1 (Okonek)

Classify the Calabi-Yau threefolds in $\mathbb{P}^6$

- All smooth threefolds (so also all Calabi-Yau threefolds) can be smoothly projected to $\mathbb{P}^7$.

Problem 2 (Coughlan-Golebiowski-Kapustka-Kapustka (CGKK) 2016)

Classify the Calabi-Yau threefolds X in $\mathbb{P}^7$ such that X is ACM.

- A closed subscheme $X \subset \mathbb{P}^n_k$ is *arithmetically Cohen-Macaulay* (ACM), if $R_X = R/I_X$ is a Cohen-Macaulay ring.
- A closed subscheme $X \subset \mathbb{P}^n_k$ is *arithmetically Gorenstein*, if X is ACM and

$$\exists\, a \in \mathbb{Z} : \omega_{R_X} \cong R_X(a).$$

arithmetically Gorenstein

Theorem (Goto-Watanabe, 1978)

Let X be a projective variety and D an ample divisor.
Set $R = R(X, D) = \bigoplus_{m\geq 0} H^0(X, mD)$, the corresponding graded ring, so that $X = \operatorname{Proj} R$.
If R is Cohen-Macaulay then:

(i) $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$;

(ii) R is Gorenstein if and only if $K_X = kD$ for some integer k.

Problem 2' (Coughlan-Golebiowski-Kapustka-Kapustka (CGKK) 2016)

Classify the Gorenstein Calabi-Yau threefolds X in $\mathbb{P}^7$.

Remark

- $14 \leq \deg X \leq 20$.
- The Hodge numbers of X :

$$h^{i,j}(X) := \dim_{\mathbb{C}} H^j\left(X, \Omega_X^i\right), 0 \leq i, j \leq d.$$

Then we may characterize a Calabi-Yau variety of dimension d in terms of its Hodge numbers.

List of Coughlan-Golebiowski-Kapustka-Kapustka

No.	Deg.	$h^{1,1}$	$h^{1,2}$	Description	cite
1	14	2	86	$(2,4)$ divisor in $\mathbb{P}^1 \times \mathbb{P}^3$	Kapustka G.
2	15	1	76	$G(2,5) \cap X_3 \cap \mathbb{P}^7$	Bertin
3	16	1	65	$X_{2,2,2,2}$	
4	17	1	55	bilinked on $Y_{2,2,2}$ to $\mathbb{P}^3$	Bertin
5	17	2	58	2×2 minors of a 3×3 matrix with degrees $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$	Bertin
6	17	2	54	rolling factors, codim 2 in cubic scroll	CGKK
7	18	1	46	bilinked on $Y_{2,2,3} \subset \mathbb{P}^7$ to F_1	Kapustka G.
8	18	1	45	bilinked on $Y_{2,2,3} \subset \mathbb{P}^7$ to F_2	Kapustka G.
9	19	2	37	bilinked on special P_{13} to F_1	CGKK
10	19	2	36	bilinked on special P_{13} to F_2	CGKK
11	20	2	34	3×3 minors of 4×4 matrix with linear forms in $\mathbb{P}^7$	Bertin

Problem (Coughlan-Golebiowski-Kapustka-Kapustka, 2016)

Does Table give the complete classification of a Gorenstein Calabi-Yau smooth threefolds in $\mathbb{P}^7$?

Main theorems

Theorem 1

There are a new family of smooth Gorenstein Calabi-Yau threefolds X in $\mathbb{P}^7$ with

$$\deg(X) = 19 \text{ and } h^{1,1}(X) = 1 \text{ and } h^{1,2} = 35.$$

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Deformations of Calabi-Yau threefolds

	0	1	2	3	4
0	1	·	·	·	·
1	·	1	·	·	·
2	·	12	24	12	·
3	·	·	·	1	·
4	·	·	·	·	1

	0	1	2	3	4
0	1	·	·	·	·
1	·	1	·	·	·
2	·	4	·	·	·
3	·	·	4	·	·
4	·	·	15	24	9

- Let X be Calabi-Yau threefold with $h^{1,1}(X) = 1$ and $h^{2,1}(X) = 38$. Then

$$(A + \varphi) \circ (B + \psi) = 0.$$

- $\mathbb{P}^7 \cup \mathbb{P}^{14} = V(AB) \subset \mathbb{P}^{16}$.
- Writing I/I^2 for both the graded S-module and the sheaf, by local duality

$$h^1\left(I/I^2\right) = \dim \operatorname{Ext}^{n-1}\left(I/I^2, S\right)_{-n-1} = 0 \text{ and } h^2\left(I/I^2\right) = \dim \operatorname{Ext}^{n-2}\left(I/I^2, S\right)_{-n-1} = 0$$

- $$h^{1,1}(X) = h^2\left(I/I^2\right) - h^1\left(I/I^2\right) + 1 = 1.$$

- Euler characteristic X is 34. Thus $h^{2,1}(X) = 35$.
- X is smooth.

Thank you for your attention