Ivan Arzhantsev (HSE University, Moscow)

Flexibility and Surjective Morphisms from Affine Spaces

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Algebraically Generated Automorphism Group

Basic Facts on Linear Algebraic Groups - I

In this talk we work over an algebraically closed field $\mathbb{K}$ of characteristic zero

Any connected linear algebraic group G has two decompositions:

$G = G^{ss} \leftthreetimes \mathrm{Rad}(G)$, where G^{ss} is a maximal semisimple subgroup and $\mathrm{Rad}(G)$ is the radical;

$G = G^{red} \leftthreetimes \mathrm{Rad}^{u}(G)$, where G^{red} is a maximal reductive subgroup and $\mathrm{Rad}^{u}(G)$ is the unipotent radical

Basic Facts on Linear Algebraic Groups - II

Any connected linear algebraic group G is generated by one-parameter subgroups, i.e., $\mathbb{G}_m$- and $\mathbb{G}_a$-subgroups

Popov'2011:

G is generated by $\mathbb{G}_a$-subgroups $\Leftrightarrow$ $G = G^{ss} \rightthreetimes \mathrm{Rad}^u(G)$

G is generated by $\mathbb{G}_m$-subgroups $\Leftrightarrow$ A'2024 (next slide)

Any connected linear algebraic group G is generated by a maximal torus T and root subgroups, i.e., T-normalized $\mathbb{G}_a$-subgroups

Algebraic Groups Generated by Seimisimple Elements - I

Proposition (A.'2024)

Let G be a connected linear algebraic group. Then the following conditions are equivalent:

(a) *G is generated by $\mathbb{G}_m$-subgroups;*

(b) *G is generated by semisimple elements;*

(c) *any homomorpism $G \to \mathbb{G}_a$ is trivial;*

(d) *G has no proper normal subgroup containing G^{red};*

(e) *the derived subgroup $[G, G]$ equals $G^{ss} \ltimes Rad^u(G)$.*

Algebraic Groups Generated by Seimisimple Elements - II

Let $\mathfrak{g} = \mathrm{Lie}(G)$. The nilpotent ideal $\mathfrak{n} = \mathrm{Lie}(Rad^{\mathrm{u}}(G))$ of $\mathfrak{g}$ is a G^{red}-module with respect to the adjoint action. Let $\mathfrak{n}_1$ be the sum of all non-trivial simple G^{red}-submodules in $\mathfrak{n}$ and $\mathfrak{s}$ be the subalgebra in $\mathfrak{n}$ generated by $\mathfrak{n}_1$. Set $U = \exp(\mathfrak{s}) \subseteq Rad^{\mathrm{u}}(G)$

Theorem (A.'2024)

Let G be a connected linear algebraic group and G^{sem} be the subgroup of G generated by all semisimple elements. Then

$$G^{\mathrm{sem}} = G^{\mathrm{red}} \rightthreetimes U.$$

Corollary

Let G be a regular subgroup of a reductive group. Then G is generated by semisimple elements. In particular, any parabolic subgroup is generated by semisimple elements.

Automorphism Groups of Complete Varieties

Let X be a complete (projective) algebraic variety and $\mathrm{Aut}(X)$ be the automorphism group

Then $\mathrm{Aut}(X)^0$ is an algebraic group (not necassarily linear)

and $\mathrm{Aut}(X)/\mathrm{Aut}(X)^0$ need not be finitely generated

Theorem (Brion'2012)

Any connected algebraic group over a perfect field is the neutral component of the automorphism group scheme of some normal projective variety.

Automorphism Groups of Affine Varieties - I

Theorem (A.'2018, Kraft'2018)

Let X be affine. Assume that $\mathrm{Aut}(X)$ has a structure of a linear algebraic group such that the action map $\mathrm{Aut}(X) \times X \to X$ is a morphism. Then either $X = \mathbb{A}^1$, or $\mathrm{Aut}(X)$ is finite, or $\mathrm{Aut}(X)$ is a finite extension of a torus.

Theorem (Jelonek'2015)

Let X be quasi-affine. Assume that $\mathrm{Aut}(X)$ is infinite. Then X is uniruled, i.e., X is covered by rational curves.

Theorem (Jelonek'2014)

For every $k \geqslant 1$ and every finite group Γ there is a k-dimensional smooth affine non-uniruled variety X such that $\mathrm{Aut}(X) \cong \Gamma$.

Automorphism Groups of Affine Varieties - II

Let X be an affine algebraic variety. Then $\mathrm{Aut}(X)$ can be infinite dimensional, i.e., $\mathrm{Aut}(X)$ contains an algebraic subgroup G of arbitrary dimension

For example, take $X = \mathbb{A}^2$ and $(x + f(y), y)$, $\deg(f) < n$

More generally, if there is a non-trivial action $\mathbb{G}_a \times X \to X$ and $\dim X \geqslant 2$, then $\mathrm{Aut}(X)$ is infinite dimensional: we consider replicas of the $\mathbb{G}_a$-action

It is known that $\mathrm{Aut}(X)$ is an ind-group, i.e., $\mathrm{Aut}(X) = \bigcup_i Z_i$ and $Z_i Z_j \subseteq Z_{f(i,j)}$, where each Z_i has a structure of an affine variety

Algebraically Generated Groups

Let X be an algebraic variety. A subgroup $H \subseteq \mathrm{Aut}(X)$ is *algebraic* if H admits a structure of an algebraic group such that the action map $H \times X \to X$ is a morphism

A subgroup $G \subseteq \mathrm{Aut}(X)$ is *algebraically generated* if G is generated by a family of connected algebraic subgroups in $\mathrm{Aut}(X)$

The group of special automorphisms $\mathrm{SAut}(X)$ is the subgroup of $\mathrm{Aut}(X)$ generated by all $\mathbb{G}_a$-subgroups

It is not known whether the group $\mathrm{Aut}(\mathbb{A}^n)$ is algebraically generated and whether $\mathrm{SAut}(\mathbb{A}^n)$ coincides with the group of all automorphisms with Jacobian 1

Basic Properties of Algebraically Generated Groups

Let X be an algebraic variety and G be an algebraically generated subgroup in $\mathrm{Aut}(X)$. Then

1) any G-orbit on X is locally closed;

2) there are finitely many rational G-invariants on X that separate generic G-orbits (Rosenlicht's Theorem)

3) Kleinman's Transversality Theorem holds for actions with an open orbit.

Open Problems Related to Automorphism Groups

The Jacobian Conjecture

The Cancellation Problem

The Rectification Problem

The Linearization Problem

The structure of $\mathrm{Aut}(\mathbb{A}^n)$, tame and wild automorphisms

Flexibility

and

Infinite Transitivity

Multiple and Infinite Transitivity

Definition

Let G be a group, X a set, and m a positive integer. An action $G \times X \to X$ is *m-transitive* if for any two tuples $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_m)$ of pairwise distinct points on X there is $g \in G$ such that $(ga_1, \ldots, ga_m) = (b_1, \ldots, b_m)$.

Definition

An action $G \times X \to X$ is *infinitely transitive* if it is m-transitive for any positive integer m.

Example (of infinite transitivity)

1) Let X be an infinite set and G the group of all permutations of X
2) Let X be an infinite set and G the group of all permutations with finite support of X

The Case of Finite Groups

Let X be a finite set witn n elements.

1) The group S_n of all permutations of X is n-**transitive**

2) The group A_n of all even permutations of X is $(n-2)$-**transitive**

3) All other finite permutation groups are at most **5-transitive**

4) **5-transitive** finite groups are precisely the Mathieu groups M_{12} and M_{24} and all **4-transitive** finite groups are precisely the Mathieu groups M_{11} and M_{23} (1861-1873)

5) There are infinitely many **3-transitive** finite permutation groups, for example, $\mathrm{PGL}_2(\mathbb{F}_q)$ acting on $\mathbb{P}^1(\mathbb{F}_q)$

The Case of Algebraic Groups

Let G be a connected algebraic group over an algebraically closed field $\mathbb{K}$ acting on an algebraic variety X.

1) Such an action is at most **3-transitive**

2) The only **3-transitive** action is $\mathrm{PGL}_2(\mathbb{K})$-action on $\mathbb{P}^1(\mathbb{K})$

3) The action of $\mathrm{PGL}_{n+1}(\mathbb{K})$ on $\mathbb{P}^n(\mathbb{K})$ for $n \geqslant 2$ is **2-transitive**

4) There is a classification of **2-transitive actions**: Borel, Knop, Kramer,....

The Case of Affine Spaces

Theorem

Over an infinite ground field K, the group $\mathrm{Aut}(\mathbb{A}^n)$ is infinitely transitive on $\mathbb{A}^n$ for any $n \geqslant 2$.

Idea ($n = 2$): use parallel translations $(x_1 + a, x_2)$, $(x_1, x_2 + b)$ and their replicas $(x_1 + af_1(x_2), x_2)$, $(x_1, x_2 + bf_2(x_1))$, where $a, b \in K$.

Example

The group $\mathrm{Aut}(\mathbb{A}^1)$ is isomorphic to $K^\times \rightthreetimes K$. It is 2-transitive, but not 3-transitive on $\mathbb{A}^1$.

General Problems

Let X be an affine algebraic variety over an algebraically closed field $\mathbb{K}$ of characteristic zero.

When the group $\mathrm{Aut}(X)$ is infinitely transitive on X?

If X is singular, we ask this question for the smooth locus X^{reg}

Idea: use $\mathbb{G}_a$-subgroups in the group $\mathrm{Aut}(X)$ and their replicas

Recall: $\mathrm{SAut}(X)$ is the subgroup of $\mathrm{Aut}(X)$ generated by all $\mathbb{G}_a$-subgroups

Locally Nilpotent Derivations

Definition

A derivation $D\colon A \to A$ of an algebra A is *locally nilpotent* if for any $a \in A$ there is a positive integer k such that $D^k(a) = 0$.

Locally nilpotent derivations on $\mathbb{K}[X] \Leftrightarrow \mathbb{G}_a$-subgroups in $\mathrm{Aut}(X)$:

$$D \in \mathrm{LND}(\mathbb{K}[X]) \iff \exp(\mathbb{K}D) \subseteq \mathrm{Aut}(X)$$

If $D \in \mathrm{LND}(A)$ and $f \in \mathrm{Ker}(D)$, then $fD \in \mathrm{LND}(A)$

$\mathbb{G}_a$-subgroups corresponding to LNDs of the form fD are *replicas* of the $\mathbb{G}_a$-subgroup corresponding to D

Flexibility vs Infinite Transitivity

Definition

An affine variety X is *flexible* if the tangent space $T_x(X)$ at any smooth point $x \in X^{\mathrm{reg}}$ is generated by velocity vectors to orbits of $\mathbb{G}_a$-subgroups passing through x

Theorem (A.-Flenner-Kaliman-Kutzschebauch-Zaidenberg'2013)

Let X be an irreducible affine variety of dimension $\geqslant 2$. The following conditions are equivalent:

(a) the group $\mathrm{SAut}(X)$ *acts transitively on* X^{reg}*;*

(b) the group $\mathrm{SAut}(X)$ *acts infinitely transitively on* X^{reg}*;*

(c) the variety X is flexible

Examples of Flexible Varieties

- Suspensions $\mathrm{Susp}(X, f)$ given by $\{uv = f(x)\}$, $f \in \mathbb{K}[X] \setminus \mathbb{K}$, in $\mathbb{A}^2 \times X$ over a flexible variety X;
- Non-degenerate ($\mathbb{K}[X]^\times = \mathbb{K}^\times$) affine toric varieties;
- Non-degenerate horospherical varieties of reductive groups;
- Homogeneous spaces G/F, where G is semisimple and F is reductive;
- Normal affine SL(2)-embeddings;
- Affine cones over flag varieties and over del Pezzo surfaces of degree $\geqslant 4$

The Gromov-Winkellmann Theorem

Theorem (Flenner-Kaliman-Zaidenberg'2016)

Let X be a flexible quasiaffine variety and $Z \subseteq X$ be a closed subvariety with $\mathrm{codim}_X Z \geqslant 2$. Then $X \setminus Z$ is flexible.

Infinite Transitivity

and

Finite Generation

Finite Generation

Conjecture A. Any flexible affine variety X admits a finite collection $H_1, \ldots, H_k$ of $\mathbb{G}_a$-subgroups in $\mathrm{Aut}(X)$ such that the group $G = \langle H_1, \ldots, H_k \rangle$ acts infinitely transitively on X^{reg}.

Plan of a possible proof:

Step 1. Find $G = \langle H_1, \ldots, H_s \rangle$ that acts transitively on X^{reg}

Step 2. Prove that the closure $\overline{G}$ of the subgroup G in $\mathrm{Aut}(X)$ in the ind-topology contains 'many other' $\mathbb{G}_a$-subgroups

Step 3. Prove that $\overline{G}$ acts infinitely transitively on X^{reg}

Step 4. Prove that G acts infinitely transitively on X^{reg}

Implication Step 3 $\Rightarrow$ Step 4 turns out to be true in general.

A Conjecture on Locally Nilpotent Derivations

To Step 2:

Conjecture B. Let X be an affine variety and $A = \mathbb{K}[X]$. Consider the group $G = \langle H_1, \ldots, H_k \rangle$ generated by a finite collection of $\mathbb{G}_a$-subgroups $H_i = \exp(\mathbb{K}D_i) \subseteq \mathrm{SAut}(X)$, where $D_i \in \mathrm{LND}(A)$. Then the $\mathbb{G}_a$-subgroup

$$H = \exp(\mathbb{K}D) \subseteq \mathrm{SAut}(X),$$

where $D \in \mathrm{LND}(A)$, is contained in $\overline{G} \quad \Leftrightarrow \quad D \in \mathrm{Lie}\,\langle D_1, \ldots, D_k \rangle$.

Kraft-Zaidenberg's Theorem

Theorem (Kraft-Zaidenberg'2024)

Let X be an affine variety. A subgroup $G \subseteq \mathrm{Aut}(X)$ generated by a family of connected algebraic subgroups G_i is algebraic if and only if the Lie algebras $\mathrm{Lie}(G_i) \subseteq \mathrm{Vec}(X)$ generate a finite-dimensional Lie subalgebra in $\mathrm{Vec}(X)$.
In particular, $G = \langle H_1, \ldots, H_k \rangle$ is a linear algebraic group if and only if $\mathrm{Lie}\,\langle D_1, \ldots, D_k \rangle$ is finite-dimensional.

Root Subgroups and Demazure Roots

Let X be a variety with an action of a torus T. A $\mathbb{G}_a$-subgroup H in $\mathrm{Aut}(X)$ is called a *root subgroup* if H is normalized in $\mathrm{Aut}(X)$ by the torus T. In this case T acts on H by some character e. Such a character is called a *root* of the T-variety X.

Assume X is toric with acting torus T. What are the roots of X?

Let $p_1, \ldots, p_s$ be the primitive lattice vectors on rays of the fan Σ_X.

Definition

A *Demazure root* of the fan Σ_X in a character $e \in M$ such that there exists $1 \leqslant i \leqslant s$ with $\langle e, p_i \rangle = -1$ and $\langle e, p_j \rangle \geqslant 0$ for $j \neq i$.

Theorem (Demazure'1970)

Let X be a complete or affine toric variety. Then root subgroups on X are in bijection with Demazure roots of the fan Σ_X.

The Toric Case

Theorem (A.-Kuyumzhiyan-Zaidenberg'2019)

For any non-degererate affine toric variety X of dimension at least 2, which is smooth in codimention 2, one can find root subgroups $H_1, \ldots, H_k$ such that the group $G = \langle H_1, \ldots, H_k \rangle$ acts infinitely transitively on the smooth locus X^{reg}.

In the proof we use Cox rings and the quotient presentation $\pi \colon \mathbb{A}^s \to X$ by an action of a quasitorus.

Finite Generation for Affine Spaces - I

Theorem (Bodnarchuk'2001)

For any $n \geqslant 3$ and any triangular $h \in \mathrm{Aut}(\mathbb{A}^n) \setminus \mathit{Aff}_n$ we have $\langle \mathit{Aff}_n, h \rangle = \mathit{Tame}_n$.

Corollary

For any $n \geqslant 3$ and any non-affine root subgroup H in $\mathrm{Aut}(\mathbb{A}^n)$ the group $\langle \mathit{Aff}_n, H \rangle$ acts on $\mathbb{A}^n$ infinitely transitively. In particular, one can find $n+2$ root subgroups which generate a subgroup acting infinitely transitively on $\mathbb{A}^n$.

Theorem (Andrist'2019, A.-Kuyumzhiyan-Zaidenberg'2019)

For any $n \geqslant 2$ one can find $\mathbb{G}_a$-subgroups H_1, H_2, H_3 in $\mathrm{Aut}(\mathbb{A}^n)$ such that $G = \langle H_1, H_2, H_3 \rangle$ acts infinitely transitively on $\mathbb{A}^n$.

Finite Generation for Affine Spaces - II

Let H be the $\mathbb{G}_a$-subgroup of $\mathrm{Aut}(\mathbb{A}^n)$ given by

$$(x_1 + ax_2^2, x_2, \ldots, x_n).$$

Theorem (A.-Kuyumzhiyan-Zaidenberg'2019)

Consider the action of the symmetric group S_n on $\mathbb{A}^n$ by permutations. Then for any $n \geqslant 3$ the subgroup

$$G = \langle H, S_n \rangle \subset \mathrm{Aut}(\mathbb{A}^n)$$

acts infinitely transitively in $\mathbb{A}^n \setminus \{0\}$.

Finite Generation for Affine Plane - I

Let L_k and R_s be the $\mathbb{G}_a$-subgroups of $\mathrm{Aut}(\mathbb{A}^2)$ given by

$$(x_1 + ax_2^k, x_2) \text{ and } (x_1, x_2 + bx_1^s), \text{ respectively.}$$

Let $G_{k,s} = \langle L_k, R_s \rangle$. We claim that if $ks \neq 2$ then $G_{k,s}$ can not be 2-transitive. Indeed,

if $k = 0$ or $s = 0$, then there are only parallel translations along one coordinate;

if $k = s = 1$, then $G_{1,1}$ is the group SL(2) preserving colinearity;

if $ks > 2$, we take a root of unity ω of degree $ks - 1$ and consider

$$S = \{(P, Q) \in \mathbb{A}^2 \times \mathbb{A}^2 \mid P = (x_1, x_2),\ Q = (\omega x_1,\ \omega^s x_2)\}$$

$$P' = (x_1 + ax_2^k, x_2),\ Q' = (\omega x_1 + a(\omega^s x_2)^k,\ \omega^s x_2) = (\omega(x_1 + ax_2^k), \omega^s x_2)$$

$$P'' = (x_1, x_2 + bx_1^s),\ Q'' = (\omega x_1,\ \omega^s x_2 + b(\omega x_1)^s) = (\omega x_1, \omega^s(x_2 + bx_1^s))$$

Finite Generation for Affine Plane - II

Theorem (Lewis-Perry-Straub'2019)

The group $G_{1,2}$ generated by two subgroups

$$(x_1 + ax_2, x_2) \quad and \quad (x_1, x_2 + bx_1^2)$$

acts infinitely transitively on $\mathbb{A}^2 \setminus \{0\}$.

The proof is based on a detailed study of the Polydegree Conjecture for plane polynomial automorphisms.

Chistopolskaya-Taroyan: arxiv.org/abs/2202.02214, a simpler proof

Finite Generation for Other Affine Varieties

Theorem (Andrist'2023)

There are $\mathbb{G}_a$-subgroups H_1, H_2, H_3 such that $G = \langle H_1, H_2, H_3 \rangle$ is infinitely transitive on the regular locus of the cone $xy = z^2$.

Theorem (Andrist'2024)

There are $\mathbb{G}_a$-subgroups H_1, H_2, H_3, H_4 such that the group $G = \langle H_1, H_2, H_3, H_4 \rangle$ is infinitely transitive on the smooth Danielewski surface $xy = p(z)$. A generalization to higher dimensions is also available.

Theorem (Andrist'2024)

If for X and Y the infinite transitivity is achieved on finite collections of $\mathbb{G}_a$-subgroups, then the same holds for $X \times Y$.

Tits Type Alternative - I

Conjecture C. Let X be an affine variety and $G = \langle H_1, \ldots, H_k \rangle$, where $H_1, \ldots, H_k$ are $\mathbb{G}_a$-subgroups in $\mathrm{Aut}(X)$. Then the group G is either a unipotent linear algebraic group, or contains the free group F_2.

Corollary

If G is 2-transitive then G contains F_2 and is of exponential growth.

Tits Type Alternative - II

Theorem (A.-Zaidenberg'2022)

Let X be an affine toric variety and $G = \langle H_1, \ldots, H_k \rangle$, where $H_1, \ldots, H_k$ are root $\mathbb{G}_a$-subgroups in $\mathrm{Aut}(X)$. Then the group G is either a unipotent linear algebraic group, or contains F_2.

Theorem (A.-Zaidenberg'2023)

Let X be an affine surface and $G = \langle H_1, \ldots, H_k \rangle$, where $H_1, \ldots, H_k$ are $\mathbb{G}_a$-subgroups in $\mathrm{Aut}(X)$. Then the group G is either a metabelian unipotent linear algebraic group, or contains F_2.

Unirationality

and

Images of Affine Spaces

A Key Lemma

Lemma

Let X be a flexible variety. Then there are $\mathbb{G}_a$-subgroups $H_1, \ldots, H_m$ in $\mathrm{Aut}(X)$ such that

(a) $H_1 \cdot H_2 \cdot \ldots \cdot H_m \cdot x = X^{\mathrm{reg}}$ *for any* $x \in X^{\mathrm{reg}}$;

(b) $T_x(X) = \langle h_1, \ldots, h_m \rangle$, *where* h_i *is a tangent vector to the orbit* $H_i \cdot x$ *at* x.

Remark: properties (a) and (b) do not imply each other.

Rationality and Unirationality

An irreducible variety X is

- *rational* if $\mathbb{K}(X) = \mathbb{K}(x_1, \ldots, x_n)$ with algebraically independent $x_1, \ldots, x_n \Leftrightarrow$ there is an open $U \subseteq X$ such that U is isomorphic to an open subset of $\mathbb{K}^n \Leftrightarrow$ there is a birational rational map from $\mathbb{K}^n$ to X;
- *stably rational* if $X \times \mathbb{K}^d$ is rational for some $d \in \mathbb{Z}_{\geqslant 0}$;
- *unirational* if $\mathbb{K}(X) \subseteq \mathbb{K}(y_1, \ldots, y_m)$ with algebraically independent $y_1, \ldots, y_m \Leftrightarrow$ there is a dominant rational map from $\mathbb{K}^m$ to X

Unirationality vs Flexibility

Theorem (A.-Flenner-Kaliman-Kutzschebauch-Zaidenberg'2013)

Any flexible variety X is unirational.

Proof.

The morphism $\varphi \colon \mathbb{A}^m \to X$,

$$(a_1, a_2, \ldots, a_m) \mapsto H_1(a_1) \cdot H_2(a_2) \cdot \ldots \cdot H_m(a_m) \cdot x,$$

is dominant for any $x \in X^{\mathrm{reg}}$. □

Remark: there are homogeneous spaces SL_n/F, where F is a finite subgroup, that are not stably rational.

Bogomolov's Conjecture

Definition

An irreducible variety X is *stably birationally infinitely transitive* if for some $m > 0$ the variety $X \times \mathbb{A}^m$ is birational to an affine flexible variety.

Conjecture (Bogomolov'2013) An irreducible variety X is unirational if and only if it is stably birationally infinitely transitive.

In Bogomolov-Karzhemanov-Kuyumzhiyan'2013, this conjecture is proved for some classes of (unirational non-rational) varieties.

Unirationality vs Images of Affine Spaces

Observation. If a flexible variety X is smooth, then the morphism $\varphi\colon \mathbb{A}^m \to X$ is surjective.

Definition

An algebraic variety X is an *A-image*, if there is a surjective morphism $\mathbb{A}^m \to X$ for some $m \in \mathbb{Z}_{>0}$.

Corollary

Any smooth flexible variety X is an A-image.

Example

The morphism $\mathbb{A}^1 \to \mathbb{P}^1$, $x \mapsto [x : 1 + x^2]$ is surjective.

Three Necessary Conditions

If X is an A-image, then

(1) X is irreducible;

(2) $\mathbb{K}[X]^\times = \mathbb{K}^\times$;

(3) X is unirational

Quetsion. Do (1)-(3) imply that X is an A-image?

Three Results on A-Images

Definition

An algebraic variety X is *A-covered* if $X = \cup_i U_i$, and each U_i is isomorphic to $\mathbb{A}^n$.

Problem

Which homogeneous spaces G/H are A-covered?

Theorem (A.'2023)

1) *Any A-covered variety is an A-image.*
2) *A toric variety X is an A-image $\Leftrightarrow \mathbb{K}[X]^\times = \mathbb{K}^\times$.*
3) *A homogeneous space X of a linear algebraic group G is an A-image $\Leftrightarrow \mathbb{K}[X]^\times = \mathbb{K}^\times$.*

Infinite Transitivity for Endomorphisms

Definition

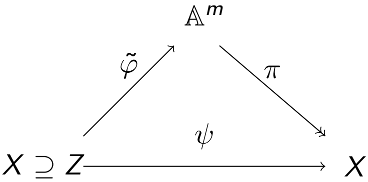
Let X be an algebraic variety. The monoid $\mathrm{End}(X)$ acts on X *infinitely transitively* if for any finite subset $Z \subseteq X$ and any map $\psi\colon Z \to X$ there is a morphism $\varphi\colon X \to X$ such that $\varphi|_Z = \psi$.

Theorem (Kaliman-Zaidenberg'2023)

Let X be an affine variety, which is an $\mathbb{A}$-image. Then the monoid $\mathrm{End}(X)$ acts on X infinitely transitively.

Proof

Let $Z = \{z_1, \ldots, z_k\}$. Given $\psi\colon Z \to X$, fix $a_1, \ldots, a_k \in \mathbb{A}^m$ with $\pi(a_i) = \psi(z_i)$.



Consider $h_j \in \mathbb{K}[Z]$, $j = 1, \ldots, m$, where $h_j(z_i)$ is the jth coordinate of a_i. There are $f_j \in \mathbb{K}[X]$ with $f_j|_Z = h_j$ and the functions $(f_1, \ldots, f_m)$ define the morphism $\tilde{\varphi}\colon X \to \mathbb{A}^m$ with $\tilde{\varphi}(z_i) = a_i$. Then with $\varphi := \pi \circ \tilde{\varphi}\colon X \to X$ we have $\varphi|_Z = \psi$.

The Main Result

Theorem (A.-Kaliman-Zaidenberg'2024)

Let X be a complete variety. Then X is an A-image if and only if X is unirational.

The proof is based on the ellipticity property.

Ellipticity after Mikhail Gromov

Elliptic Varieties: Gromov'1989

Definition

A *spray* of rank r over a smooth algebraic variety X is a triple (E, p, s), where $p \colon E \to X$ is a vector bundle of rank r with zero section Z, and $s \colon E \to X$ is a morphism such that $p|_Z = s|_Z$.

Definition

A spray (E, p, s) is *dominant* at $x \in X$ if the map $s_x := s|_{E_x} : E_x \to X$ is dominant at zero, i.e. $ds_x : E_x \to T_x X$ is surjective.

Definition

A smooth algebraic variety X is *elliptic* if there is a spray (E, p, s), which is dominant at every $x \in X$.

Ellipticity vs Flexibility

Remark: elliptic $\Longrightarrow$ unirational

Example

Any smooth flexible variety X is elliptic: take $E = \mathbb{A}^m \times X$ and $s_x \colon \mathbb{A}^m \to X$, $s_x(a_1, \ldots, a_m) = H_1(a_1) \cdot \ldots \cdot H_m(a_m) \cdot x$.

Locally Elliptic Varieties

Definition

A *local spray* on a smooth algebraic variety X is a tuple (U, E, p, s), where $U \subseteq X$ is open, $p \colon E \to U$ is a vector bundle with $s \colon E \to X$ and $p|_Z = s|_Z$.

Definition

A smooth algebraic variety X is *locally elliptic* if for every $x \in X$ there is a local spray (U, E, p, s) with $x \in U$ that is dominant at x.

Subelliptic Varieties

Definition

A smooth algebraic variety X is *subelliptic* if there is a finite collections (E_i, p_i, s_i) of sprays such that $T_x X = \sum_i (ds_i)_x((E_i)_x)$ for all $x \in X$.

Theorem (Kaliman-Zaidenberg'2023)

Let X be a smooth algebraic variety. Then

$$\text{elliptic} \Leftrightarrow \text{locally elliptic} \Leftrightarrow \text{subelliptic}.$$

Uniformly Rational Varieties

Definition

An irreducible variety X is *uniformly rational* if $X = \cup_i U_i$ and every U_i is isomorphic to an open subset of $\mathbb{A}^n$.

Theorem (Gromov,...)

Uniform rationality is preserved under blow ups of smooth subvarieties.

Question (Gromov):
Is any smooth rational variety X uniformly rational?

Uniform Rationality vs Ellipticity

Theorem (A.-Kaliman-Zaidenberg'2024)

Let X be a smooth complete uniformly rational variety. Then

(a) *X is elliptic;*

(b) *if X is projective and Y is an affine cone over X, then $Y \setminus \{0\}$ is elliptic.*

Ellipticity vs A-Image

Theorem (Kusakabe'2022)

Any elliptic variety X is an A-image. Moreover, if $\dim X = n$ then there is a surjective morphism $\mathbb{A}^{n+1} \to X$.

Proof.

Let (E, p, s) be a dominant spray of rank r on X. $\exists\, x_1, \ldots, x_k \in X$ with $s(E_{x_1}) \cup \ldots \cup s(E_{x_k}) = X$. Since X is unirational $\Rightarrow$ rationally connected $\Rightarrow \exists\, \gamma\colon \mathbb{A}^1 \to X$, $x_1, \ldots, x_k \in \gamma(\mathbb{A}^1)$. Lift E to $\mathbb{A}^1 \Rightarrow$ trivial vector bundle, so $\mathbb{A}^r \times \mathbb{A}^1 \to X$ is surjective. Take $V_i \subseteq E_{x_i}$, $\dim V_i = n$, such that $ds_{x_i}\colon V_i \to T_{x_i}(X)$ is surjective. We can assume that $s(V_1) \cup \ldots \cup s(V_k) = X$. Fix $y_i \in \mathbb{A}^1$, $\gamma(y_i) = x_i$. As $\gamma^*(E) \cong \mathbb{A}^r \times \mathbb{A}^1$, take linear operators L_i on $\gamma^*(E)_{y_i}$ that map a fixed subspace $\mathbb{A}^n$ in $\mathbb{A}^r$ to the preimage of V_i. Then $L\colon \mathbb{A}^n \times \mathbb{A}^1 \to \mathbb{A}^r \times \mathbb{A}^1$, $L(v, a) = (\sum_i \xi_i(a) L_i(v), a)$, where $\xi_i(y_j) = \delta_{ij}$. Then $s \circ \gamma_* \circ L\colon \mathbb{A}^{n+1} \to X$ is surjective. $\square$

The Field of Complex Numbers

Theorem (Forstnerič'2017)

Let $\mathbb{K} = \mathbb{C}$ and X be a compact algebraically (sub)elliptic manifold of dimension n. Then X admits a surjective strongly dominating algebraic map $\mathbb{C}^n \to X$.

A morphism $F\colon Y \to X$ is *strongly dominating* if for any $x \in X$ there is $y \in Y$ such that $F(y) = x$ and the tangent map

$$dF_y\colon T_yY \to T_xX$$

is surjective.

Return to the Result

Theorem (A.-Kaliman-Zaidenberg'2024)

Let X be a complete variety. Then X is an A-image if and only if X is unirational.

Let us prove that a complete unirational variety X is an A-image.

By Chow's Lemma, $\exists$ a birational surjection $X' \to X$ with X' projective $\Rightarrow$ we assume further that X is projective.

Proof

X is unirational $\Rightarrow \exists$ a dominant rational map h from $\mathbb{P}^n$ to X.

By Hironaka's Theorem on elimination of indeterminacy, we have

$$\begin{array}{ccc} & \tilde{X} & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ \mathbb{P}^n & \overset{h}{\dashrightarrow} & X, \end{array}$$

where f is a composition of blowups with smooth centers and g is a generically finite morphism, which is birational if h is.

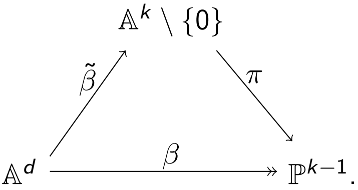
So $\tilde{X}$ is uniformly rational $\Rightarrow$ elliptic $\Rightarrow$ A-image

$\Rightarrow X$ is an A-image.

Affine Cones

Theorem (A.'2025)

Let X be an affine cone. Then X is an A-image if and only if X is unirational.



So we have $\beta\colon \mathbb{A}^d \to \mathrm{Proj}(X)$ and let the morphism $\tilde{\beta}\colon \mathbb{A}^d \to \mathbb{A}^k \setminus \{0\}$ be given by polynomials $h_1, \ldots, h_k \in \mathbb{K}[\mathbb{A}^d]$, which have no common zero. Then we have

$$\gamma\colon \mathbb{A}^{d+1} = \mathbb{A}^d \times \mathbb{A}^1 \to \mathbb{A}^k, \quad (x, z) \mapsto (h_1(x)z, \ldots, h_k(x)z).$$

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