# Flexibility of affine cones over complete intersection of 3 quadrics

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We work over the complex number field $\mathbb{C}$.

<u>problem</u>: studying the flexibility of affine var.

<u>Def</u>: • $\mathbb{G}_a = (\mathbb{C}, +)$.

• effective action $\mathbb{G}_a \times X \to X$ defines a one-parameter unipotent subgroup of $\mathrm{Aut}(X)$.

<u>Def</u>: $X$ : aff. var.

i) $p \in X$ is flexible if $T_p X$ is spanned by tangent vectors to the orbit $H.p$ of $\underset{\substack{\uparrow \\ \text{one-para. unipo. subgr.}}}{H} \subseteq \mathrm{Aut}(X)$

ii) $X$ is flexible if $\forall$ smooth point $p$ of $X$ is flexible.

⊛ The flexibility of aff. var. is closely related to the transitivity of properties.

<u>Def</u>: (i) An action $G \times A \to A$ is $m$-transitive if $\forall (a_1, \dots, a_m), (b_1, \dots, b_m)$, $a_i \neq a_j$, $b_i \neq b_j$ : $\exists g \in G : g.a_i = b_i$

ii) An action is $m$-transitive, $\forall m$ is ~~called~~ infinitely transitive.

[Arzhantsev et al, 2013] $X$: irre. aff. var. $\dim \geq 2$. TFAE.

(i) $\mathrm{SAut}(X)$ acts transitively on $\mathrm{reg}(X)$.

(ii) ——— infinitely transitive on $\mathrm{reg}(X)$.

(iii) $X$ is flexible.

$\mathrm{SAut}(X) = \langle \underset{\substack{\uparrow \\ \text{one-para. unipo. subgr.}}}{H} \rangle \subseteq \mathrm{Aut}(X)$.

<u>~~Def~~</u>:

<u>focus object</u>:

Def: $X$ : smooth proj. var.

$H$ : ample divisor on $X$

$$\mathrm{Affcone}_H X = \mathrm{Spec} \bigoplus_{m=0}^{\infty} H^0(X, \Gamma(\mathcal{O}_X(mH)))$$

⊛ The flexibility of affine cones is closely related to the cylindrocity of varieties.

<u>Def</u>: i) $\mathbb{A}^m$- cylinder in $X$ is a pair $(Z, \varphi)$

where $Z$ - affine var.

$\varphi : Z \times \mathbb{A}^m \to X$ ; open embedding s.t.

$\varphi(Z \times \mathbb{A}^m)$ is a principal open subset

ii) $(Z, \varphi)$ is $H$-polar if $\varphi(Z \times \mathbb{A}^m) = X \setminus \mathrm{Supp}\, D$,

$D \in |kH|$, $k > 0$.

iii) $Y \subseteq X$ is invariant w.r.t. $\mathbb{A}^m$- cylinder $(Z, \varphi)$ if

$Y \cap \varphi(Z \times \mathbb{A}^m) = \varphi(\pi_1^{-1}(\pi_1(\varphi^{-1}(Y))))$

$\pi_1 : Z \times \mathbb{A}^m \to Z$ : the first projection.

$$\begin{array}{ccc} Z \times \mathbb{A}^m & \xrightarrow{\varphi} & X \\ \downarrow \pi_1 & & \\ Z & & \end{array}$$

<u>iv</u> $X$ is transversally covered by $\mathbb{A}^m$- cylinder $U_i$ if

$X = \bigcup U_i$

$U_i$ is a $\mathbb{A}^m$- cylinder

$X$ do not admit proper invariant subset w.r.t. $U_i$, $\forall i$.

⊛ Two criteria for flexibility.

+) [Perepechko, 2013] $X$ : smooth proj. var.

$H$ : ample divisor on $X$

$$X \underset{\substack{\mathbb{A}^1\text{-cylinder} \\ H\text{-polar}}}{\overset{\text{trans.}}{=}} \bigcup U_i \Rightarrow \mathrm{Affcone}\, X \text{ is flexible}$$

+) [Hoff-Truong, 2022], [Prokhorov-Zaidenberg, 2023]

If $X \underset{\mathbb{A}^2\text{- cylinder}}{\overset{\text{trans.}}{=}} \bigcup U_i \Rightarrow X \underset{\mathbb{A}^1\text{-cylinder}}{\overset{\text{trans.}}{=}} \bigcup V_i$

$H$: very ample divisor.

If $X \underset{\mathbb{A}^2\text{- cylinder}, H\text{-polar}}{\overset{\text{trans.}}{=\!=\!=}} \bigcup U_i \Rightarrow$ Aff cone $X$ is flexible.

Aim: Explore the flexibility of affine cones over Fano var. using the above criteria.

*) Fano var. of dim. 2 is called del pezzo surface.

*) $X$: Fano var. dim $n \geq 3$, $\rho(X) = 1$, $\mathrm{Pic}(X) = \mathbb{Z}$

number $i$ s.t. $-K_X \sim iH$ is called the index of $X$.

$1 \leq i \leq n+1$.

If $i = n-1$ : $X$ is called a del pezzo variety.

if $i = n-2$, $X$ is called a Fano–Mukai variety.

$$g = \frac{1}{2} H^n + 1 \ : \text{genus of } X$$

$$2 \le g \le 12.$$

(*) know flexible affine cone over Fano var.

+) dim 2: → The affine cones over del Pezzo surfaces of degree $\geq 6$ [Arzhantsev et al, 2012)

The affine cones over del Pezzo surface of deg 5 [Perepechko, 2013].

4 [ .... ].

+) dim 3: Affine cones over certain Fano 3folds (Michalek et.al, 2018)

+) dim 4: → genus 10 (Prokhorov–Zaidenberg, 2023]

general 7, 8, 9 (Hoff–Truong, 2022]

Affine cone over every Fano–Mukai 4fold of genus 7 [Hang–Truong, 2024]

Affcone $X$ is flexible ⟹ $X$ is a rational Fano ~~4~~fold?

?? Which rational Fano var. (4folds) admit flexible affine cones?

$X$: Smooth complete intersection of 3 quadrics in $\mathbb{P}^7$

⟹ $X$ is a Fano–Mukai 4 fold of genus 5.

[Hassett et al, 2018] $X$ is either rational or irrational and the rational ones are dense in ~~the~~ moduli.

If $X$ contain a plane $P$ ⟹ the projection from $P$ induces a birational map

$$\phi : X \dashrightarrow \mathbb{P}^4$$

**Problem** Are the affine cones over rational complete intersection of three quadrics flexible?

<u>Thm</u>: let $X$ be general smooth complete intersection of 3 quadrics in $\mathbb{P}^7$ containing 5 planes. Then the affine cones over $X$ is flexible.

⊛ <u>Sketch of proof</u>: → Step 1: Study birational map from $X$ to $\mathbb{P}^4$

→ Step 2: Construct a transversal covering of $X$ by H-polar $\mathbb{A}^2$-cylinder.

⊛ Step 1:

[Prokhorov, 1993] $X$: complete intersection of 3 quadrics in $\mathbb{P}^7$ containing a plane $P$.

$\varphi: \tilde{X} \to X$: blow up of $X$ along $P$, exceptional divisor $D$

$L$: hyperplane section of $X$

$\varphi^*L$: full inverse image of $L$.

i) $|\varphi^*L - D|$ determines $\rho: \tilde{X} \to \mathbb{P}^4$ contracting irre. divisor $E \sim 3\varphi^*L - 3D$ to a surface $F \subseteq \mathbb{P}^4$

ii) If $X$ contains no other planes intersecting $P$ along a straight line $\Rightarrow$ $\rho$ is blow up of $\mathbb{P}^4$ along $F$. $F$ is a surface of general type with $K_F^2 = 2$, $\deg F = 9$.

$\Rightarrow$

$$\begin{tikzcd}
D \arrow[r, hook] & \tilde{X} \arrow[dl, "\varphi"'] \arrow[dr, "\rho"] & E \arrow[l, hook'] \\
P \arrow[r, hook] & X \arrow[r, dashed, "\Phi"] & \mathbb{P}^4 & F \arrow[l, hook']
\end{tikzcd}$$

Proposition 3.3: $\rho(D)$ is a singular cubic in $\mathbb{P}^4$ containing $F$.

$\varphi(E)$: cubic hypersurface section of $X$ singular along $P$

$\mathrm{Sing}\,\rho(D)$ = set of 6 ordinary double points.

proposition 3.4 : $X \setminus \varphi(E) \cong \mathbb{P}^4 \setminus p(D)$.

$\Rightarrow \exists$ singular cubic $W \subseteq \mathbb{P}^4$: $X \setminus \varphi(E) \cong \mathbb{P}^4 \setminus W$.

⊛ Step 2:

[Hoff – Truong, 2022] $W \subseteq \mathbb{P}^4$: singular cubic whose singular locus contains an ordinary double point. Then

$$\mathbb{P}^4 \setminus W \underset{A^2\text{-cylinders}}{\overset{\text{trans.}}{=}} \bigcup U_{Q,P} \quad \text{where } U_{Q,P} = \mathbb{P}^4 \setminus (P \cup Q \cup W)$$

$P$: hyperplane

$Q$: quadric

$$\Rightarrow X \setminus \varphi(E) \underset{A^2\text{-cylinders}}{\overset{\text{trans.}}{=}} \bigcup V_{Q,P} \quad \text{where } V_{Q,P} = \phi^{-1}(U_{Q,P})$$

Claim : $X$: general smooth complete intersection of 3 quadrics in $\mathbb{P}^7$ containing 5 planes.

$\Rightarrow \exists$ open covering $\{U_p \mid p \in \mathcal{I}\}$ of $X$ s.t.

- $U_p = X \setminus C_p$ – $C_p$ cubic hypersurface section.
- $\exists$ singular cubic $W_p \subseteq \mathbb{P}^4$, birational map $\phi_p : X \dashrightarrow \mathbb{P}^4$ s.t.

$$X \setminus C_p \cong \mathbb{P}^4 \setminus W_p.$$

$\Rightarrow \mathcal{I}$ = set of planes containg in $X$.

$\Rightarrow \forall p \in \mathcal{I} \Rightarrow \exists C_p, W_p, \phi_p : X \dashrightarrow \mathbb{P}^4$ s.t.

$$X \setminus C_p \cong \mathbb{P}^4 \setminus W_p.$$

$$\bigcap_{p \in \mathcal{I}} C_p = \phi \qquad (\text{Macaulay } 2) \qquad \square$$

Thm $X$: general ~~smooth~~ complete intersection of three quadrics in $\mathbb{P}^7$ containg 5 planes.

$\Rightarrow$ The affine cones over $X$ are flexible.