Flexibility of affine cones over del Pezzo surfaces

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Notation

- The ground field $\mathbb{K}$ is algebraically closed of characteristic zero.
- We denote by $\mathbb{G}_a$ the additive group $\mathbb{G}_a(\mathbb{K})$ of $\mathbb{K}$.
- All the $\mathbb{G}_a$-actions on an affine variety X over $\mathbb{K}$ generate the *special automorphism group* $\mathrm{SAut}(X)$.
- A point $x \in X$ is *flexible* if T_xX is generated by the tangent vectors to the orbits of $\mathbb{G}_a$-actions.

Theorem (Arzhantsev–Flenner–Kaliman–Kutzschebauch–Zaidenberg 2010)

Let X be an affine algebraic variety of dimension ≥ 2 and X_{reg} be its smooth locus. Then the following conditions are equivalent:

1. *all points in X_{reg} are flexible (and X is called flexible);*
2. *the group $\mathrm{SAut}(X)$ acts transitively on X_{reg};*
3. *the group $\mathrm{SAut}(X)$ acts infinitely transitively on X_{reg}.*

Theorem (Arzhantsev–Flenner–Kaliman–Kutzschebauch–Zaidenberg 2010)

Let X be an affine algebraic variety of dimension ≥ 2 and U be an open $\mathrm{SAut}(X)$-invariant subset in X. Then the following conditions are equivalent:

1. *all points in U are flexible (and X is called generically flexible);*
2. *the group $\mathrm{SAut}(X)$ acts transitively on U;*
3. *the group $\mathrm{SAut}(X)$ acts infinitely transitively on U.*

Let $Y \subset \mathbb{P}^n$ and a hyperplane section $H \subset Y$ be given by an equation, say, $x_0 = 0$.
The projection $\pi \colon \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ induces the affine cone $X = \pi^{-1}(Y) \cup \{0\}$.

Cylinders: motivation

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Assume that $U := Y \setminus H$ is a *cylinder*, i.e., $U \cong \mathbb{A}^1 \times Z = X \cap \{x_0 \neq 0\}$ for some affine variety Z with $\operatorname{Pic} Z = 0$.

Then $\pi^{-1}(U)$ is a cylinder, hence it possesses a $\mathbb{G}_a$-action G. On the other hand, we have $\pi^{-1}(U) = X \cap \{x_0 \neq 0\}$, so x_0 is a G-invariant, and we can extend G to X.
(see Kishimoto–Prokhorov–Zaidenberg'13)

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Important: G-orbits on X are projected into fibers of U on Y.

In sequel we denote by Y a normal projective variety, by H a very ample divisor on Y, and by

$$X = \operatorname{AffCone}_H Y = \operatorname{Spec} \bigoplus_{k \in \mathbb{Z}_{\geq 0}} H^0(Y, kH)$$

the affine cone over Y polarized by H.

Definition

A cylinder U in Y is an open subset isomorphic to $Z \times \mathbb{A}^1$ for an affine Z. An open subset $U \subset Y$ is H-polar if $Y \setminus U = \operatorname{supp} D$ for some effective $\mathbb{Q}$-divisor $D \sim H$.

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A principal cylinder in an affine variety X corresponds to a $\mathbb{G}_a$-action on X. An H-polar cylinder in a projective variety Y corresponds to a $\mathbb{G}_a$-action on $\operatorname{AffCone}_H Y$.

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A principal cylinder in an affine variety X corresponds to a $\mathbb{G}_a$-action on X. An H-polar cylinder in a projective variety Y corresponds to a $\mathbb{G}_a$-action on $\operatorname{AffCone}_H Y$. (see the survey Cheltsov–Park–Prokhorov–Zaidenberg)

SAut(X)-ORBITS

Let $\mathcal{U} = \{U\}$ be a set of H-polar cylinders on Y.

Definition

A subset Z of Y is $\mathcal{U}$-invariant if for any $U \in \mathcal{U}$ the intersection $U \cap Z$ is SAut(U)-invariant.

For any SAut(X)-orbit $S \subset X$ its image $\pi(S) \subset Y$ is $\mathcal{U}$-invariant. In particular, each fiber F of a cylinder $U \in \mathcal{U}$ either is contained in $\pi(S)$ or does not intersect it.

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Definition

$\mathcal{U}$ is *transversal* if $\bigcup_{U \in \mathcal{U}} U$ does not admit nontrivial $\mathcal{U}$-invariant subsets;

In this case $\pi(S)$, where S is a general G-orbit on X, contains $\operatorname{supp} \mathcal{U} := \bigcup_{U \in \mathcal{U}} U$.

Definition

A subset $Z \subset Y$ is *H-complete* if $(\operatorname{supp} D) \cap Z \neq \emptyset$ for any $\mathbb{Q}$-divisor $D \sim H$. A $\mathcal{U}$ is called *H-complete* if $\operatorname{supp} \mathcal{U}$ is.

If $\pi(S) \subset Y$ is open, then $\operatorname{codim}_X S$ equals either 0 or 1. If $\pi(S)$ is H-complete, then $S \subset X$ is open.

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Theorem (P.'20)

Let $\mathcal{U}$ be a transversal collection of H-polar open affine subsets of Y. Then there exists an $\mathrm{SAut}(X)$-orbit S on $X = \mathrm{AffCone}_H Y$ whose image contains $\bigcup_{U \in \mathcal{U}} U$. If, moreover, $\mathcal{U}$ is H-complete, then S is open in X.

Cylinders on $\mathbb{P}^2$

Let $p_1, p_2, p_3 \in \mathbb{P}^2$ be non-collinear. Then lines $L_{ij} := \overline{p_i p_j}$ induce three cylinders: $U_1 = \mathbb{P}^2 \setminus (L_{12} \cup L_{13})$ etc.
Fibers of U_1 are curves in the pencil $\langle L_{12}, L_{13} \rangle$ minus the base point.
Cylinders U_i are lifted isomorphically to the blowup of $\mathbb{P}^2$ at p_1, p_2, p_3 and correspond to the complete linear systems $|L - E_i|$.

We study cylinders on (weak) del Pezzo surfaces, which are blowups of $\mathbb{P}^2$, simultaneously.

Bubble Picard group

We fix a smooth projective surface X and consider the category $\mathcal{B}_X$ of iterative blowups

$$\phi_Y : Y \xrightarrow{\sigma_1} Y_1 \xrightarrow{\sigma_2} \dots \xrightarrow{\sigma_n} Y_n = X,$$

where each σ_i is a contraction of a (-1)-curve into a smooth point.

Definition

The *bubble Picard group* of X is the direct limit

$$\operatorname{Pic} \mathcal{B}_X = \varinjlim_{Y \in \mathcal{B}_X} \operatorname{Pic} Y \tag{1}$$

under injective maps $\sigma^* : \operatorname{Pic} Y \to \operatorname{Pic} Y'$ for $Y' \xrightarrow{\sigma} Y \to X$. Since maps σ^* preserve intersection forms, $\operatorname{Pic} \mathcal{B}_X$ is also endowed with the intersection form.

Bubble cycles

Definition (Bubble space)

The bubble space is $X^{bb} = \sqcup_{\mathcal{B}_X} Y / \{p_1 \sim p_2\}$,
where $p_1 \sim p_2$ if there exist neighbourhoods $p_i \in U_i \subset Y_i$ such that $\sigma_{Y_1}^{-1} \circ \sigma_{Y_1} : U_1 \cong U_2$.

A point q is infinitely near p (denoted $q \succ p$) if $q \neq p$ and $\sigma(q) = p$ for some morphism σ in $\mathcal{B}_X$.

Definition

A bubble cycle on X is a finite formal sum

$$\eta = \sum_{p \in X^{bb}} \eta(p) p \in \mathcal{Z}(Y^{bb}) = \bigoplus_{p \in X^{bb}} \mathbb{Z} p.$$

We have a correspondence $\eta \to \{\mathrm{Bl}_\eta X \to X\}$ by blowing up points in $\operatorname{supp} \eta := \{p \mid \eta(p) \neq 0\}$.

(cf. I.Dolgachev "Classical Algebraic Geometry: a Modern View")

Bubble linear systems

Bubble cycles can be viewed as elements of $\operatorname{Pic}\mathcal{B}_X$ by sending $p \in Y \subset X^{bb}$ to the class of the corresponding exceptional curve $[E_p] \in \operatorname{Pic}\mathrm{Bl}_p Y$. This defines an injective map $\mathcal{Z}(X^{bb}) \hookrightarrow \operatorname{Pic}\mathcal{B}_X$ and the factorization

$$\operatorname{Pic}\mathcal{B}_X = \operatorname{Pic} Y \oplus \mathcal{Z}(Y^{bb})$$

for any $Y \in \mathcal{B}_X$.

Definition

A bubble class (D, η) on $Y \in \mathcal{B}_X$ is a pair of a divisor class $D \in \operatorname{Pic} Y$ and a bubble cycle η on $Y^{bb} \subset X^{bb}$. We let $D + \eta \in \operatorname{Pic}\mathcal{B}_X$ be the corresponding divisor class.

Definition

A bubble linear system corresponding to the bubble class $(D, -\eta)$ on $Y \in \mathcal{B}_X$, where η is effective, is the linear system $|D - \eta|$ with the base scheme η, cf. notation of [CAG, Section 7.3.2].
That is, $|D - \eta|$ consists of divisors $D' \in |D|$ that (up to taking a strict transform) have multiplicity at least $\eta(p)$ at p for each $p \in \operatorname{supp}\eta$.

Definition

A del Pezzo surface Y of degree d distinct from $\mathbb{P}^1 \times \mathbb{P}^1$ is obtained from $\mathbb{P}^2$ by blowing up $k := 9 - d$ points $p_1, \ldots, p_k \in \mathbb{P}^2$ in general position:

- No three points on a line;
- No 5 points on a conic;
- No 8 points on a cuspidal cubic, one of them is a node.

Equivalently, the only negative curves are (-1)-curves, namely:

- exceptional curves $E_1, \ldots, E_k$;
- $L_{ij} \in [L - E_i - E_j]$, which is the strict transform of the line passing through p_i and p_j, for $i \neq j$;
- Strict transforms of conics passing through 5 points, e.g., $Q \in [2L - E_1 - \ldots - E_5]$.

Definition

A weak del Pezzo surface Y of degree d distinct from $\mathbb{P}^1 \times \mathbb{P}^1$ is obtained from $\mathbb{P}^2$ by blowing up $k := 9 - d$ points $p_1, \ldots, p_k \in \mathbb{P}^2$ in almost general position:

- Diagram of infinitely near points (Enriques diagram) is a union of chains;
- No four points on a line;
- No seven points on a conic.

Negative curves are (-1)- and (-2)-curves, the latter ones are:

- The strict transform of E_i if there is p_j infinitely near p_i, its class $|E_i - E_j|$;
- The strict transform of a line passing through 3 points, its class $|L - E_i - E_j - E_k|$;
- The strict transform of a conic passing through 6 points,
- The strict transform of a cuspidal curve passing through 8 points, one of them is a node.

Types of weak surfaces

Definition

We say that $Y, Y' \in \mathcal{B}_{\mathbb{P}^2}$ are equivalent if there is an isomorphism $\operatorname{Pic} Y \cong \operatorname{Pic} Y'$ respecting the intersection form that induces a bijection on classes of negative curves.
The equivalence class of surfaces in $\mathcal{B}_{\mathbb{P}^2}$ is called the type of surfaces.
We define the type of a blowup cycle η on $\mathbb{P}^2$ as one of $\mathrm{Bl}_\eta \mathbb{P}^2$.

The type of a weak del Pezzo surface is determined by the same data:

- p_j infinitely near p_i;
- a line passing through 3 points;
- a conic passing through 6 points,
- a cuspidal curve passing through 8 points, one of them is a node.

Cylinders from $\mathbb{P}^2$

A cylinder $U \subset \mathrm{Bl}_\eta \mathbb{P}^2$ is described by the corresponding pencil of curves, which is a one-dimensional linear system $S \subset |(kL, -\eta')|$, and reducible fibers. The conditions on $(kL, -\eta')$ are:

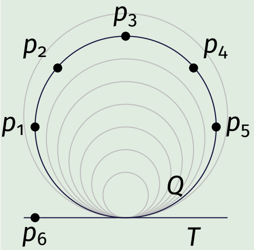
- $0 \leq \eta' \leq \eta$,
- $\dim |kL - \eta'| \geq 1$,
- a general element of $|kL - \eta'|$ is irreducible and contains $\mathbb{A}^1$ outside of the base locus.

So, U can be expressed in terms of the combinatorial type of η' and similar data for components of reducible fibers and their intersections.

The conic and a tangent

Example

Consider a cubic del Pezzo surface Y, which is the blowup of $\mathbb{P}^2$ at points $p_1, \ldots, p_6$, and choose a point $p \in \mathbb{P}^2$ such that $p, p_1, \ldots, p_5$ lie on a conic. Consider a conic $Q \in |2L - p_1 - \ldots - p_5|$ and a line $T \in |L - p - p_6|$ tangent to Q.



The cylinder U is the isomorphic preimage of $\mathbb{P}^2 \setminus T \cup Q$. Fibration is given by the pencil $\langle Q, 2T \rangle$, the corresponding bubble class is $(2L, -p)$.

Example

Consider a del Pezzo surface Y of degree 5, which is the blowup of $\mathbb{P}^2$ at points $p_1, \ldots, p_4$.
Choose another point $p \in Y$ and consider a cuspidal curve C, a conic Q and lines $L_1, \ldots, L_4$ as follows.

$$\begin{aligned} C \in &(3L, -2p - p_1 - \ldots - p_4), \\ Q \in &(2L, -p - p_1 - \ldots - p_4), \\ L_i \in &(L, -p - p_i). \end{aligned}$$

Then $U = Y \setminus (C \cup Q \cup L_1 \cup \ldots \cup L_4)$ is a cylinder corresponding to the pencil $\langle 2C, Q + L_1 + \ldots + L_4 \rangle$ and the bubble class $(6L, -2p - p_1 - \ldots - p_4)$ on $\mathbb{P}^2$, which is $(6L - 2E_1 - \ldots - 2E_4, -2p)$ on Y.

On $\mathrm{Bl}_p Y$ we can take the contraction to $\sigma' : \mathrm{Bl}_p Y \to \mathbb{P}^2$ of $\{E'_1, \ldots, E'_5\} := \{Q', L'_1, \ldots, L'_4\}$. Then $C \in |L'|$, $E_p \in |2L - p'_1 - \ldots - p'_5|$, and $\sigma'(C)$ is a tangent line to the conic $\sigma'(E_p)$.