Intersection numbers on Grassmannians via localization in equivariant cohomology

Dang Tuan Hiep

Da Lat University

Outline

Schubert Calculus

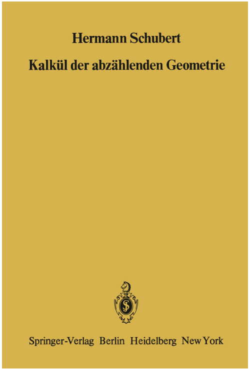
Intersection Theory on Grassmannians

Geometry of Lagrangian Grassmannians

Schubert Calculus

Schubert calculus is a classical field in algebraic geometry beginning from the 19th century.

- Hermann Schubert's book with many deep ideas.



Counting problems in Algebraic Geometry

Q: How many geometric objects satisfy given geometric conditions?

Objects:	curves, surfaces, ...
Conditions:	passing through given points, curves, ...
	tangent to given curves, surfaces, ...
	having given shape: genus, degree

The only requirement is that the conditions are chosen so that the answer is *finite* (usually general).

Example 0

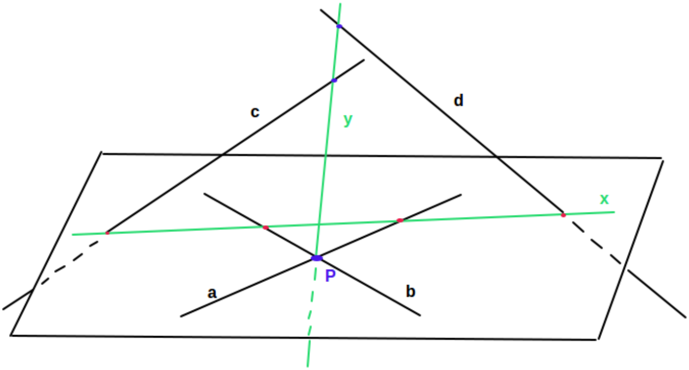
Q: How many lines pass through 2 general (distinct) points?

1

Example 1

How many lines pass through 4 general lines in $\mathbb{P}^3$?

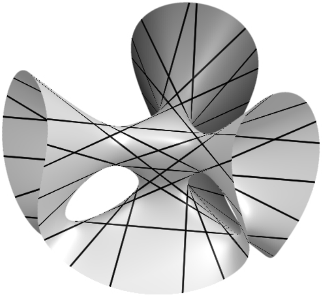
$\boxed{2}$



Example 2

How many lines lie on a general cubic surface in $\mathbb{P}^3$?

$\boxed{27}$



Example 3

How many lines lie on a general quintic hypersurface in $\mathbb{P}^4$? $\boxed{2875}$

More generally:

1. How many lines lie on a general hypersurface of degree $2n-3$ in $\mathbb{P}^n$?
2. How many k-planes lie on a general hypersurface of degree d in $\mathbb{P}^n$? If $k, d, n \in \mathbb{N}$ satisfy $d \geq 3$ and $\binom{d+k}{k} = (k+1)(n-k)$, then the answer is *finite*.

How to solve the counting problems?

1. Find suitable *parameter spaces*: Grassmannians (linear subspaces), projective bundles (conics), moduli spaces of stable maps (higher degree).
2. Using *intersection theory* on these parameter spaces, express the locus of geometric objects satisfying given geometric conditions as a certain 0-dimensional *charactersitic class* on the parameter space.
3. Compute the *degree* of the charactersitic class:
   - Localization in equivariant cohomology.

Hilbert's fifteenth problem

Question: *Construct the rigorous foundation of Schubert's enumerative calculus.*

- As the question would be understood today we can split it into *algebraic combinatorics* and *enumerative geometry*.
- Using the modern language of algebraic geometry, the question can be translated into *Gromov-Witten invariants* related to physics (string theory, mirror symmetry)
- The computation of Gromov-Witten invariants leads to the desired counting numbers.

Schubert calculus on Grassmannians

Schubert calculus on Grassmannians

Consider the *Grassmannian*

$$G(k,n) = \{W \subset \mathbb{C}^n \mid \dim(W) = k\}.$$

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathcal{P}_{k,n}$ be a partition, i.e. a sequence of integers such that

$$n-k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0.$$

The set

$$X_\lambda = \{W \in G(k,n) \mid \dim(W \cap \mathbb{C}^{n-k+i-\lambda_i}) \geq i, \forall i\}$$

is called a *Schubert variety*.

Schubert calculus on Grassmannians

The Poincaré class

$$\sigma_\lambda := [X_\lambda] \in H^{2|\lambda|}(G(k,n),\mathbb{Z})$$

is called a *Schubert class*.
The set

$$\{\sigma_\lambda \mid \lambda \in \mathcal{P}_{n,k}\}$$

forms a linear basis for $H^\star(G(k,n),\mathbb{Z})$

Example 1

How many lines are there in $\mathbb{P}^3$ meeting 4 general lines?

- The parameter space for lines in $\mathbb{P}^3$ is the Grassmannian $G(2,4)$.
- Given a line $L \subset \mathbb{P}^3$, the class of lines meeting L is the Schubert class
$$\sigma_{(1,0)} = c_1(\mathcal{S}^\vee).$$
- The number of lines meeting 4 general lines in $\mathbb{P}^3$ is equal to the degree of the class $\sigma_{(1,0)}^4$ on $G(2,4)$, that is
$$\int_{G(2,4)} c_1(\mathcal{S}^\vee)^4 = \int_{G(2,4)} \sigma_{(1,0)}^4 = 2.$$

Intersection numbers on Grassmannians

Suppose that $\Phi(\mathcal{S})$ is represented by a symmetric polynomial $P(x_1, \ldots, x_k)$ of degree not greater than $k(n-k)$ in k variables $x_1, \ldots, x_k$ which are the Chern roots of the tautological subbundle $\mathcal{S}$ on $G(k, n)$. Then the intersection number

$$\int_{G(k,n)} \Phi(\mathcal{S}) = (-1)^{k(n-k)} \frac{c(k, n)}{k!},$$

where $c(k, n)$ is the coefficient of the monomial $x_1^{n-1} \cdots x_k^{n-1}$ in the polynomial

$$P(x_1, \ldots, x_k) \prod_{i \neq j} (x_i - x_j).$$

Localization in equi. cohomology

Theorem (Atiyah-Bott-Berline-Vergne)

Suppose that X is a compact manifold endowed with a torus action and the fixed point locus X^T is finite. For $\alpha \in H^*(X)$, we have

$$\int_X \alpha = \sum_{p \in X^T} \frac{\alpha|_p}{e_p},$$

where e_p is the T-equivariant Euler class of the tangent bundle at the fixed point p, and $\alpha|_p$ is the restriction of α to the point p.

Localization in equi. cohomology

Consider the natural action of $T = (\mathbb{C}^*)^n$ on $\mathbb{C}^n$ given in coordinates by

$$(a_1, \ldots, a_n) \cdot (x_1, \ldots, x_n) = (a_1 x_1, \ldots, a_n x_n).$$

This induces a torus action on $G(k, n)$ with isolated fixed points p_I corresponding to coordinate k-planes in $\mathbb{C}^n$. Each fixed point p_I is indexed by a subset $I \subset [n]$ of size k.

Localization in equi. cohomology

By the Atiyah-Bott-Berline-Vergne formula, we have

$$\int_{G(k,n)} \Phi(\mathcal{S}) = \sum_{p_I} \frac{\Phi^T(\mathcal{S}|_{p_I})}{e_{p_I}}$$

$$= (-1)^{k(n-k)} \sum_{I\subseteq[n],|I|=k} \frac{P(\lambda_I)}{\prod_{i\in I, j\notin I} (\lambda_i - \lambda_j)}.$$

Identity on symmetric polynomials

Theorem (H., 2019)

Let $P(x_1, \ldots, x_k)$ be a symmetric polynomial of degree not greater than $k(n-k)$. Then the sum

$$\sum_{I \subseteq [n], |I| = k} \frac{P(\lambda_I)}{\prod_{i \in I, j \notin I} (\lambda_i - \lambda_j)} = \frac{c(k,n)}{k!},$$

where $c(k,n)$ is the coefficient of $x_1^{n-1} \ldots x_k^{n-1}$ in the polynomial

$$P(x_1, \ldots, x_n) \prod_{i \neq j} (x_i - x_j).$$

Geometry of Lagrangian Grassmannians

Lagrangian Grassmannians

Let V be a complex vector space of dimension $2n$ endowed with a symplectic form ω. A subspace W of V is said to be *maximal isotropic* if

$$\omega(x,y) = 0, \forall x,y \in W \text{ and } \dim(W) = n.$$

The Lagrangian Grassmannian

$$LG(n) = \{W \subset V \mid W \text{ is maximal isotropic}\}.$$

$LG(n)$ is a smooth subvariety of the ordinary Grassmannian $G(n,2n)$. Its dimension is

$$\dim(LG(n)) = \frac{n(n+1)}{2}.$$

Intersection numbers on $LG(n)$

Consider the following intersection number

$$\int_{LG(n)} \Phi(\mathcal{S}),$$

where $\Phi(\mathcal{S})$ is a characteristic class of the tautological sub-bundle $\mathcal{S}$ on $LG(n)$.
Suppose that $\Phi(\mathcal{S})$ is represented by a symmetric polynomial $P(x_1, \ldots, x_n)$ of degree not greater than $\frac{n(n+1)}{2}$ in n variables $x_1, \ldots, x_n$ which are the Chern roots of the tautological sub-bundle $\mathcal{S}$ on $LG(n)$.

Intersection numbers on $LG(n)$

Theorem (H.-Tu, 2021)

Then the intersection number

$$\int_{LG(n)} \Phi(\mathcal{S}) = (-1)^{\frac{n(n+1)}{2}} \frac{c(n)}{n!}, \tag{1}$$

where $c(n)$ is the coefficient of $x_1^{2n-1} \cdots x_n^{2n-1}$ in the polynomial

$$P(x_1, \dots, x_n) \prod_{i \neq j} (x_i - x_j) \prod_{i<j} (x_i + x_j).$$

Relation to the Grassmannian

Let $G(n, 2n)$ be the Grassmannian parametrizing n-dimensional subspaces of a $2n$-dimensional complex vector space. We denote by σ_{δ_n} the Schubert class on $G(n, 2n)$ with respect to the partition

$$\delta_n = (n-1, n-2, \ldots, 1).$$

It is easy to see that σ_{δ_n} is represented by the symmetric polynomial

$$(-1)^{\frac{n(n-1)}{2}} \prod_{i<j} (x_i + x_j).$$

Relation to the Grassmannian

Let $\Phi(\mathcal{S})$ be a characteristic class on $LG(n)$. Then we have

$$\int_{LG(n)} \Phi(\mathcal{S}) = \int_{G(n,2n)} \Phi(\mathcal{S})\sigma_{\delta_n},$$

where, by abuse of notation, the later class $\Phi(S)$ is on $G(n, 2n)$.

Degree formula for $LG(n)$

The degree of $LG(n)$, considered as a subvariety of a projective space thanks to the Plücker embedding, is given by the following formula

$$\deg(LG(n)) = \frac{\frac{n(n+1)}{2}!}{\prod_{i=1}^{n}(2i-1)!} \prod_{1 \leq i < j \leq n} (2j - 2i).$$

Schubert calculus on $LG(n)$

For each positive integer n, we denote by $\mathcal{D}_n$ the set of strict partitions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ such that $n \geq \alpha_1 > \alpha_2 > \cdots > \alpha_k > 0$.
Fix an isotropic flag of subspaces F_i of V:

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset V,$$

where $\dim(F_i) = i$ for all i and F_n is isotropic.

Schubert calculus on $LG(n)$

For each strict partition $\alpha \in \mathcal{D}_n$, define

$$X_\alpha = \{W \in LG(n) \mid \dim(W \cap F_{n+1-\alpha_i}) \geq i, \forall i\}$$

to be a *Schubert variety* in $LG(n)$.
The Poincaré class

$$\sigma_\alpha = [X_\alpha] \in H^\star(LG(n), \mathbb{Z})$$

does not depend on the choice of the flag, is called a *Schubert class* on $LG(n)$.

Schubert calculus on $LG(n)$

The set $\{\sigma_\alpha \mid \alpha \in D_n\}$ forms a $\mathbb{Z}$-basis for $H^\star(LG(n),\mathbb{Z})$, that is

$$H^\star(LG(n),\mathbb{Z}) = \bigoplus_{\alpha\in D_n} \mathbb{Z}\sigma_\alpha.$$

The ring structure

$$\sigma_\alpha \cdot \sigma_\beta = \sum_{\gamma\in D_n} N^\gamma_{\alpha,\beta}\sigma_\gamma,$$

where $N^\gamma_{\alpha,\beta}$ is the intersection number of three Schubert vareties X_α, X_β and $X_{\gamma'}$ with γ' is the complement of γ in the set $\{1,2,\dots,n\}$.

Quantum cohomology of Lagrangian Grassmannians

Quantum cohomology of $LG(n)$

We denote by $QH^*(LG(n), \mathbb{Z})$ the *(small) quantum cohomology ring* of $LG(n)$. This is a deformation of $H^*(LG(n), \mathbb{Z})$ which first appeared in the work of string theorists.
The ring $QH^*(LG(n), \mathbb{Z})$ is an algebra over $\mathbb{Z}[q]$, where q is a formal variable of degree $n+1$.
By taking $q = 0$, we recover the classical cohomology ring $H^*(LG(n), \mathbb{Z})$.

Quantum cohomology of $LG(n)$

The set $\{\sigma_\alpha \mid \alpha \in D_n\}$ forms a $\mathbb{Z}[q]$-basis for $QH^\star(LG(n), \mathbb{Z})$, that is

$$QH^*(LG(n), \mathbb{Z}) = \bigoplus_{\lambda \in D_n} \mathbb{Z}[q]\sigma_\lambda.$$

In $QH^*(LG(n), \mathbb{Z})$, the *quantum product* is given by the following formula:

$$\sigma_\alpha \star \sigma_\beta = \sum N_{\alpha,\beta}^{\gamma,d} q^d \sigma_\gamma,$$

where the sum runs over $d \in \mathbb{N}$ and $\gamma \in D_n$ with $|\gamma| + d(n+1) = |\alpha| + |\beta|$.

Quantum cohomology of $LG(n)$

The *quantum structure constant* $N_{\alpha,\beta}^{\gamma,d}$ is equal to the *three-point, genus* 0 *Gromov–Witten invariant*, which counts the number of rational curves of degree d meeting three Schubert varieties X_α, X_β and $X_{\gamma'}$ in general position.

Problem

Find a formula for $N_{\alpha,\beta}^{\gamma,d}$.

Quantum cohomology of $LG(n)$

For each $\alpha \in D_n$, we define the class

$$Q_\alpha \in H^*(G(n, 2n), \mathbb{Z})$$

as follows:

- $Q_i = \sigma_i$ for all $i = 1, \ldots, n$.
- For $i > j > 0$, we define

$$Q_{i,j} = Q_i Q_j + 2 \sum_{k=1}^{\min\{n-i,j\}} (-1)^k Q_{i+k} Q_{j-k}.$$

Quantum cohomology of $LG(n)$

- For any $\alpha \in D_n$ of length at least 3, we define

$$Q_\alpha = \text{Pfaffian}[Q_{\alpha_i,\alpha_j}]_{1 \le i < j \le r},$$

where r is the least even integer which is greater than or equal to the length of α. If the length α is odd then we set $\alpha_r = 0$ and $Q_{\alpha_i,0} = Q_{\alpha_i}$.

Quantum cohomology of $LG(n)$

We denote by σ_{δ_n} the Schubert class on $G(n, 2n)$ with respect to the partition

$$\delta_n = (n-1, n-2, \ldots, 1).$$

Theorem (H.-Tu)

Let $\alpha, \beta, \gamma \in D_n$ such that $|\gamma| = |\alpha| + |\beta|$. The Schubert structure constant $N_{\alpha,\beta}^{\gamma}$ can be expressed as an intersection number on the ordinary Grassmannian $G(n, 2n)$, that is

$$N_{\alpha,\beta}^{\gamma} = \int_{G(n,2n)} Q_\alpha Q_\beta Q_{\gamma'} \sigma_{\delta_n}.$$

Quantum cohomology of $LG(n)$

Theorem (H.-Tu, 2021)

Let $\alpha, \beta, \gamma \in D_n$ such that $|\gamma| + n + 1 = |\alpha| + |\beta|$. Then the Gromov–Witten invariant

$$N_{\alpha,\beta}^{\gamma,1} = \frac{1}{2} \int_{G(n+1,2n+2)} Q_\alpha Q_\beta Q_{\gamma'} \sigma_{\delta_{n+1}},$$

where Q_α, Q_β and $Q_{\gamma'}$ are determined as the classes on $G(n+1, 2n+2)$.

Thank you for your attention!