Varieties of minimal rational tangents and Lie algebras associated with embedded projective varieties

Dmitry A. Timashev

Faculty of Mechanics and Mathematics
Lomonosov Moscow State University

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Fano manifolds

Ground field is $\mathbb{C}$.

Definition

A *Fano manifold* is a smooth projective algebraic variety X such that $\bigwedge^n \mathcal{T}_X$ is ample ($n = \dim X$).

In other words: $\exists$ closed embedding $X \hookrightarrow \mathbb{P}^N$ and a global tensor field $\sigma \in H^0(X, (\bigwedge^n \mathcal{T}_X)^{\otimes m})$ such that

$$\{x \in X \mid \sigma(x) = 0\} = X \cap H, \qquad H \subset \mathbb{P}^N \text{ is a general hyperplane.}$$

Fano manifolds are of importance in algebraic geometry (Minimal Model Program, etc).

Picard group $\operatorname{Pic} X \simeq H^2(X, \mathbb{Z}) \simeq \mathbb{Z}^\rho$, $\rho = \rho_X$, *Picard number*.

Restriction: $\rho = 1$ (*unipolar* Fano manifolds)

Flag manifolds

Example

Generalized flag manifold (= homogeneous rational projective variety) $X = G/P$ is Fano.

Here G is a semisimple Lie group, P is a parabolic subgroup.

$\rho_X = \dim P/[P,P] =$ maximal length of $(P = P_1 \subset P_2 \subset \cdots \subset P_\rho \subset G)$

X unipolar $\iff$ P maximal parabolic

Grassmannians

Unipolar flag manifolds: examples

1. Grassmannians:

$$G = SL_n, \quad P = \begin{array}{c|c|c|} & k & \\ \cline{2-3} k & * & * \\ \cline{2-3} & 0 & * \\ \cline{2-3} \end{array}, \quad X = \mathrm{Gr}_k(\mathbb{C}^n) = \{U \subset \mathbb{C}^n \mid \dim U = k\}$$

2. Symplectic (isotropic) Grassmannians:

$$G = Sp_n = Sp(\mathbb{C}^n, \omega) \quad (n \text{ even}, \ \omega \text{ a symplectic form})$$

$$P = \begin{array}{c|c|c|c|c} & k & & & \\ \cline{2-4} k & * & * & * & \\ \cline{2-4} & 0 & * & * & \\ \cline{2-4} & 0 & 0 & * & k \\ \cline{2-4} & & & k & \end{array}, \quad X = \mathrm{IGr}_k(\mathbb{C}^n) = \{U \subset \mathbb{C}^n \mid \dim U = k, \ \omega|_U = 0\}$$

Rational curves

Assume: X Fano, $\rho_X = 1$.

Degree of an algebraic curve $C \subset X$
(w.r.t. the ample generator of $\operatorname{Pic} X \simeq \mathbb{Z}$):

$$\deg C = \frac{1}{m_X} |C \cap H|, \qquad X \subset \mathbb{P}^N, \quad H \subset \mathbb{P}^N \text{ is a general hyperplane,}$$

$$m_X \in \mathbb{N}.$$

Rational curves: $C \simeq \mathbb{P}^1$.

X is covered by rational curves. Countably many families of such curves:

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\ \mu\ } & X, \\ {\scriptstyle \pi}\downarrow & {\scriptstyle \mathbb{P}^1\text{-bundle}} & \\ \mathcal{K} & \text{quasiprojective, irreducible} & \end{array} \qquad C = \mu(\pi^{-1}(c)), \quad c \in \mathcal{K}.$$

Minimal rational curves:

Rational curves play an important role in the geometry of Fano manifolds (goes back to S. Mori).

Families of *minimal* rational curves:

1. $\mu(\mathcal{P}) \supset X^0$, a dense open subset of X (= *dominating family*);
2. $\deg C = \min$ over all families $\mathcal{K}$ with (1).

Analogy (Yum-Tong Siu et al):

Geodesic curves in Riemannian geometry	Minimal rational curves in the geometry of Fano manifolds
$\exists$ geodesic curve through any point in any direction	Not true in general

VMRT

$x \in X^0 \rightsquigarrow \mathcal{K}_x = \pi(\mu^{-1}(x))$
parameterizes minimal rational curves through x.

Tangent map (proper, birational)

$$\begin{aligned} \tau_x : \mathcal{K}_x &\longrightarrow \mathbb{P}(\mathcal{T}_{X,x}) \\ c &\longmapsto \mathbb{P}(\mathcal{T}_{C,x}), \qquad C = \mu(\pi^{-1}(c)) \end{aligned}$$

Variety of minimal rational tangents (VMRT) $\mathcal{C}_x = \operatorname{Im} \tau_x \subset \mathbb{P}(\mathcal{T}_{X,x})$, embedded projective variety.

Particular case: $X \subset \mathbb{P}^N$, $\forall x \in X^0\ \exists$ line $C \subset X$, $C \ni x$.

Then: minimal rational curves = lines in $X \implies \tau_x : \mathcal{K}_x \xrightarrow{\sim} \mathcal{C}_x$.

Holds for $X = G/P$ (unipolar flag manifold); lines in X form a single family $\mathcal{K}$.

Example: VMRT of Grassmannians

Example 1

$X = \mathrm{Gr}_k(\mathbb{C}^n) \hookrightarrow \mathbb{P}(\bigwedge^k \mathbb{C}^n)$ (Plücker embedding)
$x = U = \langle v_1, \dots, v_k \rangle \mapsto \langle v_1 \wedge \cdots \wedge v_k \rangle$

Lines:
$C = \{U \mid U' \subset U \subset U''\}$; U', U'' fixed, $\dim U' = k-1$, $\dim U'' = k+1$.

VMRT: $\mathcal{C}_x = \mathbb{P}(U^*) \times \mathbb{P}(\mathbb{C}^n/U) \xhookrightarrow{\text{Segre}} \mathbb{P}(U^* \otimes \mathbb{C}^n/U) = \mathbb{P}(\mathcal{T}_{X,x})$.

Example 2

$X = \mathrm{IGr}_k(\mathbb{C}^n) \hookrightarrow \mathbb{P}(\bigwedge^k \mathbb{C}^n)$

Lines: as in Example 1 with $\omega(U', U'') = 0$. This equation cuts out $\mathcal{C}_x \subset \mathbb{P}(U^*) \times \mathbb{P}(\mathbb{C}^n/U)$ and $\mathbb{P}(\mathcal{T}_{X,x}) \subset \mathbb{P}(U^* \otimes \mathbb{C}^n/U)$.

Fano manifolds via VMRT

Principle (J.-M. Hwang, N. Mok, 90's)

Geometry of a unipolar Fano manifold X with many symmetries is controlled by projective geometry of its VMRT $\mathcal{C}_x \subset \mathbb{P}^{n-1}$ ($n = \dim X$) at a general point $x \in X$.

Program

Recognize X by $\mathcal{C}_x$.

Theorem (S. Mori, 1979; K. Cho and Y. Miyaoka, 1998)

$\mathcal{C}_x = \mathbb{P}^{n-1} \implies X = \mathbb{P}^n$

Further developments (J.-M. Hwang, N. Mok, and collaborators): implementing the program for unipolar flag manifolds.

Main theorem

Main Theorem (J.-M. Hwang, N. Mok, J. Hong, Q. Li, D.T.)

Suppose $Y = G/P$ is a unipolar flag manifold, $y \in Y$.

If X is a unipolar Fano manifold such that, for (some) $x \in X^0$,

$$\begin{array}{ccc} \mathcal{C}_x & \xrightarrow{\sim} & \mathcal{C}_y \\ \cap & & \cap \\ \mathbb{P}(\mathcal{T}_{X,x}) & \xrightarrow{\sim} & \mathbb{P}(\mathcal{T}_{Y,y}), \end{array}$$

then $X \simeq Y$.

Remark

$\mathcal{C}_y$ is locally rigid.

Scheme of the proof

Proof combines ideas and techniques from: algebraic geometry, differential geometry, Lie algebras and algebraic groups, spherical varieties.

Scheme of the proof:

$$\begin{aligned}&\text{VMRT} \rightsquigarrow \text{differential-geometric structure on } X \\ &\qquad \rightsquigarrow G' \curvearrowright X \supset X^0 = G'/P',\ \operatorname{codim}(X \setminus X^0) \geq 2 \\ &\qquad\qquad \implies G' = G,\ P' = P,\ X = Y\end{aligned}$$

May assume $G = (\operatorname{Aut} Y)^\circ$, $P = P(\alpha_k)$, α_k is a simple root of G.

Two cases:

1. α_k long; example: $Y = \mathrm{Gr}_k(\mathbb{C}^n)$;
2. α_k short; example: $Y = \mathrm{IGr}_k(\mathbb{C}^n)$, $1 < k < n/2$.

Step 1: VMRT of Y

Lie algebra grading:

$$\mathfrak{g} = \operatorname{Lie} G = \overbrace{\underbrace{\mathfrak{g}_{-d} \oplus \cdots \oplus \mathfrak{g}_{-1}}_{\mathfrak{p}_{\mathrm{nil}}} \oplus \mathfrak{g}_0}^{\mathfrak{p} = \operatorname{Lie} P} \oplus \underbrace{\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_d}_{\mathfrak{p}_{\mathrm{nil}}^{+} \simeq \mathcal{T}_{Y,y}}$$

1. Long root case: $\mathcal{C}_y =$ the unique closed orbit of $G_0 \curvearrowright \mathbb{P}(\mathfrak{g}_1)$.
2. Short root case: $\mathcal{C}_y = \mathcal{C}_0 \sqcup \mathcal{C}_1 \subset \mathbb{P}(\mathfrak{g}_1 \oplus \mathfrak{g}_2)$, $\mathcal{C}_0 =$ the open P-orbit, $\mathcal{C}_1 = \mathcal{C}_y \cap \mathbb{P}(\mathfrak{g}_1) =$ the closed P-orbit.

VMRT of Y: examples

Examples (▸ VMRT of ▸ Grassmannians)

1. $Y = \mathrm{Gr}_k(\mathbb{C}^n)$, $\mathfrak{g} = \mathfrak{sl}_n = \begin{array}{c|c|c} & k & \\ \hline k & 0 & -1 \\ \hline & 1 & 0 \end{array}$,

   $$\mathcal{C}_y = \begin{array}{|c|c|} \hline 0 & 0 \\ \hline \mathrm{rk} = 1 & 0 \\ \hline \end{array} = \mathbb{P}(\mathbb{C}^k)^* \times \mathbb{P}(\mathbb{C}^n/\mathbb{C}^k).$$

2. $Y = \mathrm{IGr}_k(\mathbb{C}^n, \omega)$ $(1 < k < n/2)$, $\mathfrak{g} = \mathfrak{sp}_n = \begin{array}{c|c|c|c|c} & k & & & \\ \hline k & 0 & -1 & -2 & \\ \hline & 1 & 0 & -1 & \\ \hline & 2 & 1 & 0 & k \\ \hline & & & k & \end{array}$,

   $$\mathcal{C}_y = \{(U', U'') \mid \omega(U', U'') = 0\},$$
   $\mathcal{C}_0 = \{\text{U'' non-isotropic}\}$, $\mathcal{C}_1 = \{\text{U'' isotropic}\}$.

Step 2: symbol algebra

$$\widehat{\mathcal{C}}_x \subset \mathcal{T}_{X,x} \rightsquigarrow \mathcal{W}_x = \langle \widehat{\mathcal{C}}_x \rangle \subset \mathcal{T}_{X,x} \rightsquigarrow \mathcal{W} \subset \mathcal{T}_{X^0}$$

$$\mathcal{W} = \mathcal{W}^1 \subset \mathcal{W}^2 \subset \cdots \subset \mathcal{W}^k \subset \cdots \subset \mathcal{W}^d = \mathcal{T}_{X^0}$$

$$\mathcal{W}^k = \mathcal{W}^{k-1} + [\mathcal{W}^{k-1}, \mathcal{W}] \cdot \mathcal{O}_X$$

$$\rho_X = 1 \Rightarrow \mathcal{W}^d = \mathcal{T}_{X^0}$$

$$[\mathcal{W}^k, \mathcal{W}^l] \subset \mathcal{W}^{k+l}$$

Definition

Symbol algebra $\mathfrak{g}_x = \mathcal{W}_x^1 \oplus \mathcal{W}_x^2/\mathcal{W}_x^1 \oplus \cdots \oplus \mathcal{W}_x^d/\mathcal{W}_x^{d-1}$

Properties:

1. Generators: $\xi \in \mathcal{W}_x$.
2. Defining relations: $[\xi, \eta] = 0$ whenever $\langle \xi, \eta \rangle$ is tangent to $\widehat{\mathcal{C}}_x$.
3. $\mathfrak{g}_x$ depends on $\mathcal{C}_x \simeq \mathcal{C}_y$ only.
4. $\mathfrak{g}_x \simeq \mathfrak{p}^+_{\mathrm{nil}}$ in the long root case.

Digression: Lie algebras from projective varieties

General construction:

Let $Z \subset \mathbb{P}^{n-1} = \mathbb{P}(V)$ be a non-degenerate irreducible projective variety.

Then: $L \subset \mathbb{P}(V)$, a line tangent to Z at some $z \in Z^{\mathrm{reg}}$

$\implies L = \mathbb{P}(T)$, $T = \langle \xi, \eta \rangle$, a 2-plane tangent to $\widehat{Z}$, $\langle \xi \rangle = z$.

Definition

Associated Lie algebra:

$$\mathfrak{L} = \mathfrak{L}(Z) =$$

$$\mathrm{Lie}\Big(V \;\Big|\; [\xi, \eta] = 0,\ \forall T = \langle \xi, \eta \rangle \text{ s.t. } L = \mathbb{P}(T) \text{ tangent to } Z \text{ at } z \in Z^{\mathrm{reg}} \Big)$$

$$= \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \mathfrak{L}_3 \oplus \cdots$$

$$\mathfrak{L}_1 = V, \qquad \mathfrak{L}_2 = \textstyle\bigwedge^2 V / \langle \xi \wedge \eta,\ \forall T \text{ as above} \rangle$$

Associated Lie algebras

Questions:

1. What is the structure of $\mathfrak{L}$?
2. Is it always true that $\dim \mathfrak{L} < \infty$?

Observation: $Z \subset \mathbb{P}(V)$ big enough
$\implies \bigwedge^2 V = \sum \bigwedge^2 T$ over all T tangent to $\widehat{Z}$
$\implies \mathfrak{L} = \mathfrak{L}_1$ is Abelian.

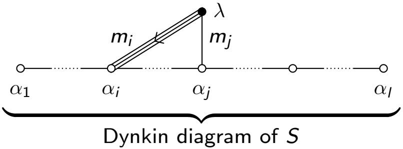
In particular, $\mathfrak{L}$ is Abelian for smooth Z in the cases:

- $\dim Z > \frac{n-1}{2}$ (Hwang–Mok);
- Z is a complete intersection (Hwang).

Associated Lie algebras for flag varieties

Case of flag varieties: $Z = S/Q$, S semisimple, $Q \subset S$ parabolic.
$V = V(\lambda)$, simple S-module of lowest weight λ.
$V \ni v_\lambda$, lowest weight vector, $Z = S\langle v_\lambda \rangle$.

Extended Dynkin diagram:



Multiplicities of edges at λ: $m_i = -\langle \lambda, \alpha_i^\vee \rangle$, so that $\lambda = -\sum_i m_i \omega_i$.

Kac–Moody algebra:

$$\mathfrak{g} = \operatorname{Lie}\Big(e_i, f_i, h_i \mid i = 0, 1, \dots, l,\ \{e_i, f_i, h_i\} \text{ are } \mathfrak{sl}_2\text{-triples} + \text{other relations}\Big)$$

$$= \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k, \qquad \deg e_0 = 1,\ \deg f_0 = -1,$$

$$\deg h_0 = \deg h_i = \deg e_i = \deg f_i = 0, \quad i = 1, \dots, l.$$

Associated Lie algebras for flag varieties

$\mathfrak{g}_0$ is reductive, $\quad [\mathfrak{g}_0, \mathfrak{g}_0] = \mathfrak{s} = \operatorname{Lie} S, \quad \mathfrak{g}_1 \simeq V(\lambda)$.

Put: $\quad \mathfrak{n} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots, \quad \mathfrak{u}_0 = \operatorname{Lie}(e_1, \dots, e_l) \subset \mathfrak{s}$

$$\implies \mathfrak{u} = \mathfrak{u}_0 \oplus \mathfrak{n} = \operatorname{Lie}\Big(e_0, e_1, \dots, e_l \mid (\operatorname{ad} e_i)^{1+m_{ij}} e_j = 0, \ \forall i \neq j\Big),$$

$$m_{ij} = \begin{cases} 0, & \alpha_i \text{ not linked to } \alpha_j, \\ \text{multiplicity of edge } \alpha_i - \alpha_j, & \alpha_i \text{ short}, \\ 1, & \text{otherwise}. \end{cases} \qquad (\alpha_0 = \lambda).$$

Theorem

$\mathfrak{L} \twoheadleftarrow \mathfrak{n}, \quad \operatorname{Ker} = \operatorname{ideal}\big((\operatorname{ad} e_0)^2 e_\alpha = 0, \ \alpha > 0\big)$
$= 0 \quad$ *if* $\dim \mathfrak{g} < \infty$ (▸ symbol algebra) and in some other cases.

Theorem (A. Zavadskii, 2023)

$\sum_{i>0} m_i > 3 \implies \dim \mathfrak{L} = \infty.$

Fine symbol algebra

Recall:

- $\mathfrak{g}_x$ depends on $\mathcal{C}_x \simeq \mathcal{C}_y$ only;
- $\mathfrak{g}_x \simeq \mathfrak{p}_{\mathrm{nil}}^{+}$ in the long root case.

Short root case: $\mathfrak{g}_x$ sometimes too coarse $\implies$ modify the definition.

Definition

Fine symbol algebra $\mathfrak{g}_x = \mathcal{U}_x^1 \oplus \mathcal{U}_x^2/\mathcal{U}_x^1 \oplus \cdots$, where:

$$\mathcal{U}^1 \subset \mathcal{U}^2 = \mathcal{W}^1 \subset \mathcal{U}^3 \subset \mathcal{U}^4 = \mathcal{W}^2 \subset \cdots$$

$\mathcal{U}_x^1 = \langle \widehat{\mathcal{C}}_1 \rangle$ (recall the structure of VMRT), $\quad \mathcal{U}^{2k+1} = \mathcal{W}^k + [\mathcal{W}^k, \mathcal{U}] \cdot \mathcal{O}_X$.

Properties of the fine symbol algebra:

- $\mathfrak{g}_x$ may depend on X, not only on $\mathcal{C}_x$;
- $\mathfrak{g}_x \simeq \mathfrak{p}_{\mathrm{nil}}^{+}$ or its degeneration: finitely many possibilities.

Step 3: universal prolongation

Problem

Given a positively graded finite-dimensional nilpotent Lie algebra $\mathfrak{h}^+ = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_d$, *embed it into a* $\mathbb{Z}$*-graded Lie algebra* $\mathfrak{h} = \mathfrak{h}^+ \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_{-1} \oplus \mathfrak{h}_{-2} \oplus \cdots$ *such that* $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{h}^+) \subset \mathfrak{h}^+$ *and* $\mathfrak{h}_0$ *is prescribed.*

Definition

We call such an $\mathfrak{h}$ a *prolongation* of $(\mathfrak{h}^+, \mathfrak{h}_0)$.

Definition-Proposition

Among all prolongations $\mathfrak{h}$ there exists the biggest one, called the *universal prolongation* of $(\mathfrak{h}^+, \mathfrak{h}_0)$. (May happen $\dim \mathfrak{h} = \infty$!)

Criterion

A prolongation $\mathfrak{h}$ *is universal iff* $H^1(\mathfrak{h}^+, \mathfrak{h})_{\deg<0} = 0$.

Universal prolongation of the symbol algebra

Our situation:

$\mathfrak{g}_x$ (coarse or fine) $\hookrightarrow$ universal prolongation

$$\mathfrak{g}' = \mathfrak{g}_x \oplus \overbrace{\mathfrak{g}'_0 \oplus \mathfrak{g}'_{-1} \oplus \mathfrak{g}'_{-2} \oplus \cdots}^{\mathfrak{p}'},$$

$$\mathfrak{g}'_0 = \operatorname{Lie}\operatorname{Aut}(\mathfrak{g}_x, \widehat{\mathcal{C}}_x).$$

Proposition

- $\dim \mathfrak{g}' < \infty$
- *Long root case:* $\mathfrak{g}' = \mathfrak{g}$, $\mathfrak{p}' = \mathfrak{p}$.
- *Short root case:* $(\mathfrak{g}', \mathfrak{p}') = (\mathfrak{g}, \mathfrak{p})$ *or its degeneration.*
- *Any case:* $\dim G'/P' = \dim G/P = \dim X$.

Step 4: Cartan connection

Definition

A *Cartan connection* of type $(\mathfrak{g}', \mathfrak{p}')$ on X^0 is a principal P'-bundle $\mathcal{P} \to X^0$ together with a $\mathfrak{g}'$-valued 1-form $\gamma : \mathcal{T}_{\mathcal{P}} \to \mathfrak{g}'$ such that:

- γ is P'-equivariant;
- $\gamma : \mathcal{T}_{\mathcal{P},p} \xrightarrow{\sim} \mathfrak{g}'$, $\forall p \in \mathcal{P}$;
- $\gamma : \mathcal{T}_{\mathcal{P}_x} \xrightarrow{\sim} \mathfrak{p}'$ is the Maurer–Cartan form, $\forall x \in X^0$.

Proposition

- $\exists$ *Cartan connection of type* $(\mathfrak{g}', \mathfrak{p}')$ *on a dense open subset* $X^0 \subset X$*;*
- *curvature* $K = d\gamma + \frac{1}{2}\gamma \wedge \gamma = 0$.

Proof: inductive construction starting from a principal G_0'-bundle $\mathcal{P}^0 \to X^0$ associated with $\widehat{\mathcal{C}} \to X^0$, based on N. Tanaka's method. Obstructions are sections of a vector bundle with fiber $H^2(\mathfrak{g}_x, \mathfrak{g}')_{\deg<0}$.

Step 5: Transitive group action

Proposition

$\exists$ action $G' \curvearrowright X$ with a dense open orbit $X^0 \simeq G'/P'$, $\operatorname{codim}(X \setminus X^0) \geq 2$.

Proof is based on flatness of the Cartan connection and Cartan–Fubini type extension theorem (J.-M. Hwang, N. Mok, 2001).

Case 1: $G' = G,\ P' = P \implies X^0 = Y \implies X = X^0$.

Case 2: G' non-reductive (only short root case) $\implies G' = G'_{\mathrm{uni}} \rtimes G'_{\mathrm{red}}$, Borel subgroup $B \subset G'_{\mathrm{red}}$ acts on X^0 with a dense open orbit, i.e., X^0 is a spherical G'_{red}-variety.

Theory of spherical varieties $\implies X^0$ admits no G'_{red}-equivariant compactifications with small boundary.

This excludes Case 2 and completes the proof.

Some references

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