Additive actions on projective spaces and other varieties

Yulia Zaitseva

HSE University, Moscow

23.12.2024

Additive actions

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $\mathbb{G}_a = (\mathbb{K}, +)$.

The **vector group** $\mathbb{G}_a^n := \underbrace{\mathbb{G}_a \times \ldots \times \mathbb{G}_a}_{n \text{ times}} = (\mathbb{K}^n, +)$.

Definition

An **additive action** on a variety X is an effective action $\mathbb{G}_a^n \times X \to X$ with an open orbit.

Hirzebruch's question (1954)

Problem 26. Describe all analytic compactifications of $\mathbb{C}^2$.

Problem 27. Describe all analytic compactifications of $\mathbb{C}^n$ with the second Betti number 1.

Equivariant version

B. Hassett and Yu. Tschinkel'99: the systematic study of algebraic compactifications $\mathbb{C}^n \hookrightarrow X$ such that the action of the group $\mathbb{G}_a^n = (\mathbb{C}^n, +)$ of $\mathbb{C}^n$ by parallel translations extends to $\mathbb{G}_a^n \times X \to X$.

Motivation: distribution of rational points, Manin's conjecture (Chambert-Loir-Tschinkel'02,'12)

“Additive analogue” of toric geometry

$\mathbb{G}_m = (\mathbb{K}^\times, \times)$ and $\mathbb{G}_a = (\mathbb{K}, +)$

Definition

A **toric** variety is an irreducible variety X with an effective action $T \times X \to X$ of an algebraic torus $T = \mathbb{G}_m^n$ with an open orbit.

Additive actions: replace an algebraic torus $\mathbb{G}_m^n$ by a vector group $\mathbb{G}_a^n$

Main differences:

- such an action is no longer unique on X;
- the number of orbits is no longer finite;
- orbits of a unipotent group on affine varieties are closed $\Rightarrow$

$\Rightarrow$ invariant open affine covering on X no longer exists.

Two additive actions $\mathbb{G}_a^2 \times \mathbb{P}^2 \to \mathbb{P}^2$

Example 1

$$(a_1, a_2) \cdot [z_0 : z_1 : z_2] = [z_0 : z_1 + a_1 z_0 : z_2 + a_2 z_0]$$

∞ orbits: $\{z_0 \neq 0\}$ and a line of fixed points $z_0 = 0$

Example 2

$$(a_1, a_2) \cdot [z_0 : z_1 : z_2] = \left[z_0 : z_1 + a_1 z_0 : z_2 + a_1 z_1 + \left(a_2 + \frac{a_1^2}{2}\right) z_0\right]$$

3 orbits: $\{z_0 \neq 0\}$, $\{z_0 = 0,\ z_1 \neq 0\}$, $\{[0 : 0 : 1]\}$

Local algebras

Let A be a finite-dimensional commutative associative algebra with unity over $\mathbb{K}$.

Definition

An algebra A is **local** if A contains a unique maximal ideal $\mathfrak{m}$.

In this case $A = \mathbb{K} \oplus \mathfrak{m}$, all elements in $\mathfrak{m}$ are nilpotent, and all elements in $A^\times := A \setminus \mathfrak{m}$ are invertible.

We have $A \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \ldots \supset \mathfrak{m}^d \supset \mathfrak{m}^{d+1} = 0$.

Moreover, any A is a direct sum $A_1 \oplus \ldots \oplus A_s$ of local algebras.

Example

$$A = \mathbb{K}[x, y]/(x^4, x^2y, x^3 - y^2)$$

Knop-Lange Theorem

Theorem (Knop-Lange'84)

There is a bijection between

(a) commutative algebras A with unity and $\dim A = n+1$;

(b) effective actions $G \times \mathbb{P}^n \to \mathbb{P}^n$ with an open orbit, where G connected commutative linear algebraic group.

Knop-Lange Theorem

Theorem (Knop-Lange'84)

There is a bijection between

(a) commutative algebras A with unity and $\dim A = n+1$;

(b) effective actions $G \times \mathbb{P}^n \to \mathbb{P}^n$ with an open orbit, where G connected commutative linear algebraic group.

(a) $\to$ (b)

- denote $A^\times$ the group of invertible elements in A;
- $A^\times$ is a connected commutative linear algebraic group that is open in A;
- $G = \mathbb{P}(A^\times) := A^\times / \mathbb{K}^\times$ connected commutative linear algebraic group;
- G acts on $\mathbb{P}(A) = \mathbb{P}^n$ with an open orbit isomorphic to $\mathbb{P}(A^\times)$.

Knop-Lange Theorem

Theorem (Knop-Lange'84)

There is a bijection between

(a) commutative algebras A with unity and $\dim A = n+1$;

(b) effective actions $G \times \mathbb{P}^n \to \mathbb{P}^n$ with an open orbit, where G connected commutative linear algebraic group.

(b) $\to$ (a) page 1/2

- $\dim G = n$;
- $G \subseteq \mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(\mathbb{K})$;
- denote $\pi\colon \mathrm{GL}_{n+1}(\mathbb{K}) \to \mathrm{PGL}_{n+1}(\mathbb{K})$, let $H := \pi^{-1}(G)$;
- $\dim H = \dim G + \dim \operatorname{Ker} \pi\big|_H = n+1$;
- H is connected since $\pi(H) = G$ and $\operatorname{Ker} \pi\big|_H$ are connected;
- H is commutative since the commutator subgroup

$[H,H] \subseteq \operatorname{Ker} \pi\big|_H = \mathbb{K}^\times$ consists of matrices with $\det = 1$;

Knop-Lange Theorem

Theorem (Knop-Lange'84)

There is a bijection between

(a) commutative algebras A with unity and $\dim A = n+1$;
(b) effective actions $G \times \mathbb{P}^n \to \mathbb{P}^n$ with an open orbit, where G connected commutative linear algebraic group.

(b) $\to$ (a) page 2/2

- so $H \subseteq \mathrm{GL}_{n+1}(\mathbb{K})$ is connected commutative, $\dim H = n+1$;
- denote $A \subseteq \mathrm{Mat}_{n+1}(\mathbb{K})$ subalgebra generated by $H \subseteq \mathrm{GL}_{n+1}(\mathbb{K})$;
- A is commutative and with unity;
- the action $H \times \mathbb{K}^{n+1} \to \mathbb{K}^{n+1}$ has an open orbit since $G \times \mathbb{P}^n \to \mathbb{P}^n$ has;
- $\dim A = n+1$ since the action $A^\times \times \mathbb{K}^{n+1} \to \mathbb{K}^{n+1}$ is effective, has an open orbit and $A^\times$ is commutative.

Corollary 1

Bijection between G-orbits on $\mathbb{P}^n$ and nonzero principal ideals in the algebra A.

Proof: G-orbits on $\mathbb{P}^n$ $\leftrightarrow$ association classes of nonzero elements in the algebra A $\leftrightarrow$ generators of principal ideals.

Corollary 2

There is a unique action $\mathbb{G}_a^n \times \mathbb{P}^n \to \mathbb{P}^n$ with finitely many orbits. It corresponds to $A = \mathbb{K}[x] \,/\, (x^{n+1})$.

Example 1

$A = \mathbb{K}^{n+1}$ with the coordinatewise multiplication
$\Rightarrow$ the group $A^\times = (\mathbb{K}^\times)^{n+1}$, $A^\times / \mathbb{K}^\times \cong \mathbb{G}_m^n$
$\Rightarrow$ compute $(1, t_1, \ldots, t_n)(z_0, \ldots, z_n) \in A$
$\Rightarrow$ action $(t_1, \ldots, t_n) \cdot [z_0 : z_1 : \ldots : z_n] = [z_0 : t_1 z_1 : \ldots t_n z_n]$.
$2^{n+1} - 1$ orbits

Example 2

The local algebra $A = \mathbb{K}[x_1, \ldots, x_n] / (x_i x_j,\ i \leqslant j)$
$\Rightarrow$ the group $A^\times / \mathbb{K}^\times = (1 + \mathfrak{m}, \times) \cong (\mathfrak{m}, +) \cong \mathbb{G}_a^n$ via exp
$\Rightarrow$ compute $\exp(a_1 x_1 + \ldots + a_n x_n)(z_0 + z_1 x_1 + \ldots z_n x_n) =$

$$z_0 + (z_1 + a_1 z_0) x_1 + \ldots + (z_n + a_n z_0) x_n \in A$$

$\Rightarrow$ action
$(a_1, \ldots, a_n) \cdot [z_0 : z_1 : \ldots : z_n] = [z_0 : z_1 + a_1 z_0 : \ldots : z_n + a_n z_0]$.
Open orbit $\{z_0 \neq 0\}$ and the hyperplane of fixed points $\{z_0 = 0\}$

Rank and local summands

Any connected commutative linear algebraic group $G \cong \mathbb{G}_m^r \times \mathbb{G}_a^s$ for some $r, s \in \mathbb{Z}_{\geqslant 0}$. The **rank** of G: $r = \operatorname{rk} G$.

Remark

In Knop-Lange theorem, A contains exactly $\operatorname{rk} G + 1$ maximal ideals.

Proof: If $A = \mathbb{K} \oplus \mathfrak{m}$ is local, its group of invertible elements equals $A^\times = \mathbb{K}^\times \oplus \mathfrak{m} = \mathbb{K}^\times \times (1 + \mathfrak{m})$, where $(1 + \mathfrak{m}, \times) \cong (\mathfrak{m}, +) \cong \mathbb{G}_a^n$ via exponential map and $\mathbb{K}^\times \cong \mathbb{G}_m$.
Any commutative algebra A is a sum of local algebras $\Rightarrow$
$\operatorname{rk} A^\times$ = the number of local summands = the number of maximal ideals.
By construction, $A^\times = H$ and $\operatorname{rk} H = \operatorname{rk} G + 1$.

Hassett-Tschinkel correspondences

Theorem (Knop-Lange'84, Hassett-Tschinkel'1999)

There is a bijection between
(a) local commutative algebras A with unity and $\dim A = n+1$;
(b) additive actions $\mathbb{G}_a^n \times \mathbb{P}^n \to \mathbb{P}^n$.

Theorem (Hassett-Tschinkel'99)

There is a bijection between
(a) pairs (A, U), where
A is a local commutative algebra with unity, $\dim A = m$,
$U \subseteq \mathfrak{m}$ is a subspace generating A, $\dim U = n$;
(b) faithful cyclic representations $\mathbb{G}_a^n \to \mathrm{GL}_m(\mathbb{K})$.

$m = n + 1$

Hassett-Tschinkel correspondence

(a) → (b)

The representation of $(\exp U, \times) \cong (U, +) = \mathbb{G}_a^n$ on A by multiplication with cyclic vector $1 \in A$.

(b) → (a)

$$\rho\colon \mathbb{G}_a^n \to \mathrm{GL}_m(\mathbb{K}),$$
$$d\rho\colon \mathfrak{g} = \langle x_1, \ldots, x_n \rangle \to \mathfrak{gl}_m(\mathbb{K}),$$
$$\psi\colon \mathbb{K}[x_1, \ldots, x_n] \to \mathrm{Mat}_m(\mathbb{K})$$

Let $A := \psi(\mathbb{K}[x_1, \ldots, x_n]) \cong \mathbb{K}[x_1, \ldots, x_n]/I$, $U := \psi(\langle x_1, \ldots, x_n \rangle)$.
Here $I = \mathrm{Ker}(\psi) = \{g \in \mathbb{K}[x_1, \ldots, x_n] \mid \psi(g)e = 0\}$,
where $e \in \mathbb{K}^n$ is a cyclic vector. So, $A \cong \mathbb{K}^m$.

Two algebras and two actions

There are two 3-dimensional local algebras, so we have two additive actions $\mathbb{G}_a^2 \times \mathbb{P}^2 \to \mathbb{P}^2$:

Example 1

Algebra $A = \mathbb{K}[x, y]/(x^2, xy, y^2) = \langle 1, x, y \rangle$
$\Rightarrow$ compute $\exp(a_1 x + a_2 y)(z_0 + z_1 x + z_2 y) \in A$
$\Rightarrow$ action $(a_1, a_2) \cdot [z_0 : z_1 : z_2] = [z_0 : z_1 + a_1 z_0 : z_2 + a_2 z_0]$

Example 2

Algebra $A = \mathbb{K}[x]/(x^3) = \langle 1, x, x^2 \rangle$
$\Rightarrow$ compute $\exp(a_1 x + a_2 x^2)(z_0 + z_1 x + z_2 x^2) \in A$
$\Rightarrow$ action

$$(a_1, a_2) \cdot [z_0 : z_1 : z_2] = \left[z_0 : z_1 + a_1 z_0 : z_2 + a_1 z_1 + \left(a_2 + \frac{a_1^2}{2}\right) z_0\right]$$

Classifications of local algebras

$\dim A$	1	2	3	4	5	6	$\geqslant 7$
#isom. classes of loc. alg.	1	1	2	4	9	25	∞

For $\dim A = 7$, take

$$A_\alpha = \mathbb{K}[x_1, x_2, x_3, x_4] \,/\, (x_1^2 + x_3^2 - 2x_2^2, x_4^2 - x_2^2 - \alpha(x_3^2 - x_2^2), x_i x_j, i \neq j)$$

For any $\alpha \in \mathbb{K}$ $\exists$ a finite number of algebras of this form $\cong A_\alpha$.

Case of Gorenstein algebra

Definition

A local algebra A is **Gorenstein** if its socle

$$\mathrm{Soc}(A) := \{a \in A \,|\, a\mathfrak{m} = 0\}$$

has dimension one, i.e. $\mathrm{Soc}(A) = \mathfrak{m}^d$ and $\dim \mathfrak{m}^d = 1$.

Theorem

An additive action $\mathbb{G}_a^n \times \mathbb{P}^n \to \mathbb{P}^n$ has a unique fixed point $\Leftrightarrow$

$$A \text{ is Gorenstein}$$

.

Additive actions on projective hypersurfaces

Hassett-Tschinkel correspondence for $m = n + 2$.
Induced additive actions on hypersurfaces in $X \subseteq \mathbb{P}^{n+1} \leftrightarrow$
$\leftrightarrow$ pairs (A, U), where A is a local commutative algebra, $\dim A = n + 2$, the hyperplane $U \subseteq \mathfrak{m}$ generates A.

$X = \overline{p(\exp U)}$, where $p \colon A \to \mathbb{P}(A)$

Theorem (Sharoiko'09)

Any smooth quadric $Q \subseteq \mathbb{P}^n$ admits a unique additive action.

Arzhantsev-Sharoiko'11, Arzhantsev-Popovskiy'14: results on additive actions on degenerate quadrics.
Bazhov'13: results on additive actions on cubics.

Non-degenerate projective hypersurface

Definition

A hypersurface $X = \{f(z_0, \ldots, z_n) = 0\} \subseteq \mathbb{P}^n$ is **non-degenerate** if f involves all variables after any linear transformation of variables in $\mathbb{P}^n$.

Theorem (Arzhantsev-Z.'22)

A pair (A, U) corresponds to a non-degenerate projective hypersurface $\Leftrightarrow$ the algebra A is Gorenstein and $\mathfrak{m} = U \oplus \mathrm{Soc}(A)$.

Theorem (Arzhantsev-Z.'22, Beldiev'23)

Let $X \subseteq \mathbb{P}^n$ be a hypersurface admitting an additive action. Then such an action is unique $\Leftrightarrow$ X is non-degenerate.

Multilinear forms

Theorem (Arzhantsev-Sharoiko'11)

Let $X \subseteq \mathbb{P}^{n+1}$ be a projective hypersurface admitting an induced additive action, and (A, U) be the corresponding pair. Then the degree of the hypersurface X equals the maximal d with $\mathfrak{m}^d \not\subseteq U$.

Arzhantsev-Popovsky'14: d-linear form $F\colon \underbrace{A \times \ldots \times A}_{d} \to \mathbb{K}$ is **invariant** if
1) $F(1, \ldots, 1) = 0$;
2) for any $u \in U$, $z^{(1)}, \ldots, z^{(d)} \in A$, we have
$F(uz^{(1)}, z^{(2)}, \ldots, z^{(d)}) + \ldots + F(z^{(1)}, z^{(2)}, \ldots, uz^{(d)}) = 0$.

Lemma (Arzhantsev-Z.'22)

$\operatorname{Ker} F = \{x \in A \mid F(x, z^{(2)}, \ldots, z^{(d)}) = 0 \;\; \forall z^{(2)}, \ldots, z^{(d)} \in A\}$ is the maximal ideal of A contained in U, where $f \leftrightarrow F$.

If f is non-degenerate, then $\operatorname{Ker} F = 0$.

Additive actions on flag varieties

Theorem (Arzhantsev-Popovskiy'14)

Let X be a complete variety admitting an additive action. Assume that the group $\mathrm{Aut}(X)^0$ is a reductive linear algebraic group. Then X is a flag variety G/P.

Theorem (Arzhantsev'11)

Let G be a simple group and P be a parabolic subgroup of G. Then the flag variety G/P admits an additive action if and only if the unipotent radical P_u is commutative + few explicit exceptions.

Theorem (Fu-Hwang'14, Devyatov'15)

Let G/P be a flag variety admitting an additive action. If G/P is not isomorphic to a projective space, then such an action is unique.

Additive actions on toric varieties

Definition

An additive action $\mathbb{G}_a^n \times X \to X$ is **normalized** if the image of $\mathbb{G}_a^n$ in $\mathrm{Aut}(X)$ is normalized by the acting torus.

Definition

A fan Σ is **bilateral** if there is a basis $p_1, \ldots, p_n$ of the lattice N such that n rays of Σ are generated by the vectors $p_1, \ldots, p_n$, and the remaining rays lie in the negative orthant with respect to this basis.

Theorem (Arzhantsev-Romaskevich'17)

Let X be a complete toric variety. TFAE:

(a) the variety X admits an additive action;

(b) the variety X admits a (unique) normalized additive action;

(c) the fan Σ of X is bilateral.

Euler-symmetric varieties

Theorem (Fu-Hwang'20, Shafarevich'23)

A projective toric variety X admits an additive action $\Leftrightarrow$ X is Euler-symmetric, i.e. for a general smooth point $x \in X$ there is a $\mathbb{G}_m$-action on X such that x is an isolated fixed point and the induced $\mathbb{G}_m$-action on the tangent space $T_x X$ is by scalar multiplication.

Further results

Theorem (Dzhunusov'21)

Any complete toric surface admits at most two additive actions

Theorem (Dzhunusov'22)

A criterion for a complete toric variety to admit a unique (normalized) additive action.

Theorem (Shakhmatov'21)

There is a smooth toric 3-dimensional complete non-projective variety admitting an additive action.

Theorem (Shafarevich'21)

A classification of additive actions on toric projective hypersurfaces.

General classifications

Classifications of varieties with an additive action:

Derenthal-Loughran'10: singular del Pezzo surfaces

Hassett-Tschinkel'99: smooth projective 3-folds of Picard number 1 (only $\mathbb{P}^3$ and Q_3)

Huang-Montero'20: smooth projective 3-folds of Picard number $\geqslant 2$ (13 toric and 4 non-toric)

Fu-Montero'19: n-dimensional smooth projective variety of Picard number 1 with index $\geqslant n-2$

References

[1] Arzhantsev. Flag varieties as equivariant compactifications of $\mathbb{G}_a^n$. Proc. Amer. Math. Soc. 139 (2011), 783-786

[2] Arzhantsev-Kotenkova. Equivariant embeddings of commutative linear algebraic groups of corank one. Doc. Math. 20 (2015), 1039-1053

[3] Arzhantsev-Popovskiy. Additive actions on projective hypersurfaces. Proc. Math. Stat. 79, Springer, 2014, 17-33

[4] Arzhantsev-Romaskevich. Additive actions on toric varieties. Proc. Amer. Math. Soc. 145 (2017), 1865-1879

[5] Arzhantsev-Sharoyko. Hassett-Tschinkel correspondence: Modality and projective hypersurfaces. J. Algebra 348 (2011), 217-232

[6] Arzhantsev-Zaitseva. Equivariant completions of affine spaces. Russian Math. Surveys 77 (2022), 571-650

[7] Beldiev. Gorenstein algebras and uniqueness of additive actions. Results Math. 78 (2023), article 192

References

[8] Borovik-Gaifullin-Trushin. Commutative actions on smooth projective quadrics. Comm. Algebra 50 (2022), 5468-5476

[9] Chambert-Loir-Tschinkel. On the distribution of points of bounded height on equivariant compactifications of vector groups. Invent. Math. 148 (2002), 421-452

[10] Chambert-Loir-Tschinkel. Integral points of bounded height on partial equivariant compactifications of vector groups. Duke Math. J. 161 (2012), 2799-2836

[11] Cheong. Equivariant compactifications of a nilpotent group by G/P. Transform. Groups 22 (2017), 163-186

[12] Derenthal-Loughran. Singular del Pezzo surfaces that are equivariant compactifications. J. Math. Sci. 171 (2010), 714-724

[13] Derenthal-Loughran. Equivariant compactifications of two-dimensional algebraic groups. Proc. Edinb. Math. Soc. (2) 58 (2015), 149-168

[14] Devyatov. Unipotent commutative group actions on flag varieties and nilpotent multiplications. Transform. Groups 20 (2015), 21-64

References

[15] Dzhunusov. Additive actions on complete toric surfaces. Internat. J. Algebra Comput. 31 (2021), 19-35

[16] Dzhunusov. On uniqueness of additive actions on complete toric varieties. J. Algebra 609 (2022), 642-656

[17] Feigin. $\mathbb{G}_a^M$ degeneration of flag varieties. Selecta Math. (N.S.) 18 (2012), 513-537

[18] Fu-Hwang. Uniqueness of equivariant compactifications of $\mathbb{C}^n$ by a Fano manifold of Picard number 1. Math. Res. Lett. 21 (2014), 121-125

[19] Fu-Hwang. Euler-symmetric projective varieties. Algebr. Geom. 7 (2020), 377-389

[20] Fu-Montero. Equivariant compactifications of vector groups with high index. C.R. Math. 357 (2019), 455-461

[21] Hassett-Tschinkel. Geometry of equivariant compactifications of $\mathbb{G}_a^n$. Int. Math. Res. Not. IMRN (1999), 1211-1230

[22] Huang-Montero. Fano threefolds as equivariant compactifications of the vector group. Michigan Math. J. 69 (2020), 341-368

References

[23] Knop-Lange. Commutative algebraic groups and intersections of quadrics. Math. Ann. 267 (1984), 555-571

[24] Liu. Additive actions on hyperquadrics of corank two. Electron. Res. Arch. 30 (2022), 1-34

[25] Shafarevich. Additive actions on toric projective hypersurfaces. Results Math. 76 (2021), article 145

[26] Shafarevich. Euler-symmetric projective toric varieties and additive actions. Indag. Math. (N.S.) 34 (2023), 42-53

[27] Shakhmatov. Smooth non-projective equivariant completions of affine spaces. Math. Notes 109 (2021), no. 6, 884-895

[29] Shalika-Tschinkel. Height zeta functions of equivariant compactifications of unipotent groups. Comm. Pure Appl. Math. 69 (2016), 693-733

[30] Sharoiko. Hassett-Tschinkel correspondence and automorphisms of a quadric. Sb. Math. 200 (2009), no. 11, 1715-1729