Monday, 23 December, 2024 1:09 PM

Additive actions on Projective space with small modality

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I/ Notation.

We work on field $\mathbb{C}$.

- $\mathbb{P}^n$: Projective space
- $X \subseteq \mathbb{P}^n$: algebraic variety.
- $\mathbb{G}_a$: $(\mathbb{C}, +)$
- $\mathbb{G}_a^n$: $\mathbb{G}_a \times \dots \times \mathbb{G}_a$.

II/ Hassett - Tschinkel correspondence.

<u>Prob</u>: Classify equivariant compactifications of $\mathbb{G}_a^n$.

$\mathbb{C}^n \subseteq X$ as open affine.

$$\begin{array}{ccccl} \mathbb{G}_a^n \times & \mathbb{C}^n & \to & \mathbb{C}^n & \text{translation} \\ & \cap & & \cap\downarrow & \text{extended.} \\ \mathbb{G}_a^n \times & X & \to & X & \end{array}$$

<u>Def</u>: We call an effective action of $\mathbb{G}_a^n$ on X with open orbit an additive action on X.

<u>Thm</u> [Hassett and Tschinkel, 1999].

There is a one-to-one correspondence among the following.

i) The additive action on $\mathbb{P}^n$.

ii) The local Artinian algebra A of dimension $n+1$.

II/ Result.

<u>Def</u>: Modality of action of a connected algebraic group G on a variety X is the maximum of number of parameter in a continuous family of G-orbit on X.

$(A, m) \Leftrightarrow \mathbb{G}_a^n \times \mathbb{P}^n \to \mathbb{P}$ $\quad \operatorname{mod}(A) := \operatorname{mod}(\mathbb{G}_a^n, \mathbb{P}^n)$.

<u>Ex</u>: $A = k[x]/(x^3) \leftrightarrow \mathbb{G}_a^2 \times \mathbb{P}^2 \to \mathbb{P}^2$

$\operatorname{mod}(A) = 0$

<u>Def</u>: Hilbert function

$HF_A : \mathbb{N} \to \mathbb{N}$

$i \mapsto r_i(A) := \ell(m^i / m^{i+1})$

$\mathrm{mod}(A) = 0.$	$HF_A : \mathbb{N} \to \mathbb{N}$ $i \mapsto r_i(A) := \ell(m^i/m^{i+1})$ Socdegree of $A := s$, $m^s \neq 0$, $m^{s+1} = 0$.
Additive action $\rho: \mathbb{G}_a^n \times \mathbb{P}^n \to \mathbb{P}^n$	local algebra A.
1) $\mathbb{G}_a^n$-orbit on $\mathbb{P}^n$	$I \subseteq A$, $I = \langle f \rangle$, $f \neq 0$.
2) Set of fixed point $:= W \subseteq \mathbb{P}^n$	$\mathrm{Soc}(A)$
3) There is only one fixed point, $\lvert W \rvert = 1$.	A is Gorenstein
4 $\dim W \leq \dim_k A - s - 1$	$\Leftarrow$ A is teter $\Leftrightarrow$ $\exists\, G$: Gorenstein st $A \cong G/\mathrm{soc}\, G$
5) $\mathrm{mod}(A) = 0$	$A = k[x]/[x^{n+1}]$.
6) $\mathrm{mod}(A) = 1$	8 isomorphism types.
7. $\mathrm{mod}(A) = 2$ $\mathrm{mod}(A) \geq r_i(A) - 1 = \ell(m^i/m^{i+1}) - 1$ $r_i(A) \leq 3$.	
7.1. $HF_A: (1\ 3)$	$A = k[x,y,z]/(x^2, y^2, z^2, xy, xz, yz)$.
7.2. HF_A $(1\ 3\ 1 \ldots)$	$A = k[x,y,z]/(x^{s+1}, xy, xz, y^2 - \alpha x^s, z^2 - \beta x^s, yz - \gamma x^s)$ $\alpha, \beta, \gamma \in k \setminus \{0\}$.
7.3 HF_A $(1\ 3\ 2 \ldots)$	25 isomorphism type.
7.4 HF_A $(1\ 3\ 3, \ldots)$	infinite isomorphism types
7.5 HF_A $(1\ 2\ 2 \ldots)$	4 isomorphism type.
7.6 HF_A $(1\ 2\ 3 \ldots)$	? open

Sketch of proof.

Step 1: classify Algebra A with fixed Hilbert polinomial. ρ

Step 2: study $\mathrm{mod}(A)$,

$\mathrm{mod}(A) = n - 1.$ $\longleftarrow$ A is stretched ring $\Leftrightarrow$ $m^2 = \langle f \rangle$.

$HF_A (1, n, 1, 1, \ldots, 1)$.

$HF_A(1, n, 2, 2, \dots, 1)$.

$\uparrow$ $m^2 = \langle f_1, f_2 \rangle$.

$\text{mod}(A) = n-1$. $\leftarrow$ A is <u>almost stretched</u> Gorenstein =

$HF_A(1, n, 2, \dots 2, 1 \dots 1)$.