Root subgroups on spherical varieties

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Sources

[AA] I. Arzhantsev, R. Avdeev, *Root subgroups on affine spherical varieties*, Selecta Math. (N.S.) **28** (2022), no. 3, Article 60; arXiv:2012.02088

[AZ1] R. Avdeev, V. Zhgoon, *On the existence of B-root subgroups on affine spherical varieties*, Dokl. Math. **105** (2022), no. 2, 51–55; arXiv:2112.14268

[AZ2] R. Avdeev, V. Zhgoon, *Root subgroups on horospherical varieties*, arXiv:2312.03377v2 (2024)

$\mathbb{G}_a$-subgroups

$\mathbb{K}$ the base field, $\overline{\mathbb{K}} = \mathbb{K}$, $\operatorname{char} \mathbb{K} = 0$

$\mathbb{G}_a := (\mathbb{K}, +)$ is the additive group of the field $\mathbb{K}$

X an irreducible algebraic variety

Definition

$\mathbb{G}_a \times X \to X$ a nontrivial action $\Rightarrow$
the image of $\mathbb{G}_a$ in $\operatorname{Aut} X$ is called a $\mathbb{G}_a$-subgroup on X.

$\mathbb{G}_a$-subgroups and LNDs

X an irreducible affine variety

$\mathbb{K}[X]$ is the algebra of regular functions on X

Definition

A derivation $\partial\colon \mathbb{K}[X] \to \mathbb{K}[X]$ is called locally nilpotent (for short: LND) if $\forall\, f \in \mathbb{K}[X]\ \exists\, k > 0 : \partial^k(f) = 0$.

∂ an LND on $\mathbb{K}[X]$ $\rightsquigarrow$
the $\mathbb{G}_a$-action on $\mathbb{K}[X]$ given by $s \mapsto \exp(s\partial)$ $\rightsquigarrow$
$\mathbb{G}_a$-subgroup φ_∂ on X

Fact

The map $\partial \mapsto \varphi_\partial$ induces a bijection
$\{$nonzero LNDs on $\mathbb{K}[X]\}/$proportionality $\leftrightarrow$
$\{\mathbb{G}_a$-subgroups in $\operatorname{Aut} X\}$.

Example. $X = \mathbb{A}^2$, $H : (s, (x, y)) \mapsto (x + s, y) \ \leftrightarrow\ \partial = \partial/\partial x$

Root subgroups

Let X be not necessarily affine; $H \subseteq \operatorname{Aut} X$ a $\mathbb{G}_a$-subgroup
$F \subseteq \operatorname{Aut} X$ an algebraic subgroup; $\mathfrak{X}(F)$ the character group of F

Definition

H is called an F-root subgroup if F normalizes H.

F naturally acts on $\operatorname{Lie} H \simeq \mathbb{K}^1$ via a character $\chi \in \mathfrak{X}(F)$

Definition

χ is called the weight of the F-root subgroup H.

Proposition [AA]

Suppose F is connected and acts on X with an open orbit O_F;
$HO_F \neq O_F \;\Rightarrow\; \exists!$ F-stable prime divisor $D \subseteq X$ such that
$HD \neq D$. («H moves D»)

X is affine $\;\Rightarrow\;$ H corresponds to an LND ∂ on $\mathbb{K}[X]$
H is normalized by F $\;\Leftrightarrow\;$ ∂ is normalized by F (with the same weight!)

Affine toric varieties

T an algebraic torus; $\mathfrak{X}(T)$ its character lattice

Definition

A T-variety X is called toric if X is normal, irreducible, and possesses an open T-orbit.

Given a sublattice $M \subset \mathfrak{X}(T)$, consider the dual lattice $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ and put $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$.

Fact

There is a bijection
$\{\text{affine toric } T\text{-varieties}\} \,/\, T\text{-equiv. isom.} \leftrightarrow \{(M, \mathcal{E})\}$,
where $M \subseteq \mathfrak{X}(T)$ is a sublattice and $\mathcal{E}$ is a strictly convex finitely generated cone in $N_{\mathbb{Q}}$.

Construction of affine toric varieties:
$(M, \mathcal{E}) \rightsquigarrow X = X(M, \mathcal{E}) := \mathrm{Spec}\, \mathbb{K}[\Gamma]$ where $\Gamma = \mathcal{E}^{\vee} \cap M$ and
$\mathbb{K}[\Gamma] = \bigoplus\limits_{\lambda \in \Gamma} \mathbb{K} f_{\lambda}$; $f_{\lambda} f_{\mu} := f_{\lambda+\mu}\ \forall\, \lambda, \mu \in \Gamma$

Demazure roots of a cone

Given a cone $\mathcal{E} \subset N_{\mathbb{Q}}$ as above, consider the set

$\mathcal{E}^1 := \{\rho \in N : \rho \text{ is primitive in } N,\ \mathbb{Q}_{\geqslant 0}\rho \text{ is a ray of } \mathcal{E}\}$

$\rho \in \mathcal{E}^1 \ \Rightarrow$

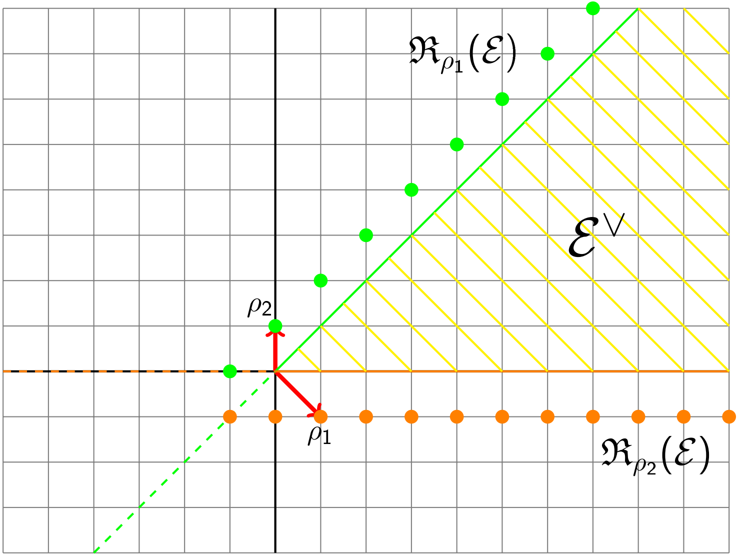
$\mathfrak{R}_\rho(\mathcal{E}) := \{e \in M \mid \langle \rho, e \rangle = -1,\ \langle \rho', e \rangle \geqslant 0 \quad \forall\, \rho' \in \mathcal{E}^1 \setminus \{\rho\}\}$

$\mathfrak{R}(\mathcal{E}) := \bigsqcup_{\rho \in \mathcal{E}^1} \mathfrak{R}_\rho(\mathcal{E})$

Definition

Elements of the set $\mathfrak{R}(\mathcal{E})$ are called Demazure roots of the cone $\mathcal{E}$.

Demazure roots: an example



Let X be an affine toric T-variety corresponding to a pair $(M, \mathcal{E})$.

$e \in \mathfrak{R}_\rho(\mathcal{E}) \;\rightsquigarrow\; \partial_e \colon \mathbb{K}[X] \to \mathbb{K}[X]$

$\partial_e(f_\lambda) := \langle \rho, \lambda \rangle f_{\lambda+e} \;\leftarrow\;$ a T-normalized LND

$\rightsquigarrow H_e \;\leftarrow\;$ a T-root subgroup on X

Theorem

1) Every T-root subgroup on X is uniquely determined by its weight.
2) The set of weights of all T-root subgroups on X is $\mathfrak{R}(\mathcal{E})$.
3) For every T-stable prime divisor D in X, there exists a T-root subgroup on X that moves D.

Arbitrary toric varieties

Definition

A fan in $N_{\mathbb{Q}}$ is a finite collection $\mathfrak{F}$ of strictly convex finitely generated cones in $N_{\mathbb{Q}}$ satisfying the following properties:
(1) if $\mathcal{E} \in \mathfrak{F}$, then every face of $\mathcal{E}$ is in $\mathfrak{F}$;
(2) if $\mathcal{E}_1, \mathcal{E}_2 \in \mathfrak{F}$, then $\mathcal{E}_1 \cap \mathcal{E}_2$ is a face of both $\mathcal{E}_1, \mathcal{E}_2$.

Fact

There is a bijection
$\{$toric T-varieties$\}$ / T-equiv. isom. $\leftrightarrow \{(M, \mathfrak{F})\}$,
where $M \subseteq \mathfrak{X}(T)$ is a sublattice and $\mathfrak{F}$ is a fan in $N_{\mathbb{Q}}$.

Remark. If X is an affine toric T-variety corresponding to a cone $\mathcal{E} \subset N_{\mathbb{Q}}$, then the corresponding fan $\mathfrak{F}$ consists of all faces of $\mathcal{E}$.

Demazure roots of a fan

Given a sublattice $M \subset \mathfrak{X}(T)$ and a fan $\mathfrak{F} \subset N_{\mathbb{Q}}$, consider the set

$\mathfrak{F}^1 := \{\rho \in N : \rho \text{ is primitive in } N,\ \mathbb{Q}_{\geqslant 0}\rho \in \mathfrak{F}\}$

Definition

An element $e \in M$ is called a Demazure root of the fan $\mathfrak{F}$ if the following conditions are satisfied:
(DR1) there is $\rho \in \mathfrak{F}^1$ such that $\langle \rho, e \rangle = -1$;
(DR2) $\langle \rho', e \rangle \geqslant 0$ for all $\rho' \in \mathfrak{F} \setminus \{\rho\}$;
(DR3) if a cone $\mathcal{K} \in \mathfrak{F}$ satisfies $\langle \mathcal{K}, e \rangle = 0$, then the cone generated by $\mathcal{K}$ and ρ belongs to $\mathfrak{F}$.

Remark. Condition (DR3) holds automatically if the fan $\mathfrak{F}$ is convex.

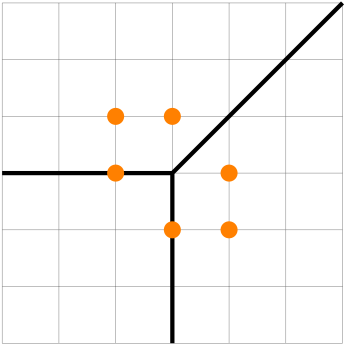
Let X be a toric T-variety corresponding to a pair $(M, \mathfrak{F})$.

Theorem (Demazure'1970)

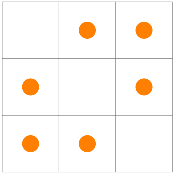
1) Every T-root subgroup on X is uniquely determined by its weight.
2) The set of weights of T-root subgroups on X is $\mathfrak{R}(\mathfrak{F})$.
3) If X is complete, then the group $(\operatorname{Aut} X)^0$ is generated by T and all T-root subgroups on X.

Example

$$X = \mathbb{P}^2,\; T = (\mathbb{K}^\times)^2,\; (t_1, t_2) \cdot [x_0 : x_1 : x_2] = [x_0 : t_1 x_1 : t_2 x_2]$$



The corresponding root subgroups in PGL_3:



A generalization

G a connected reductive algebraic group (e.g. SL_n)
$B \subseteq G$ a Borel subgroup (e.g. all upper-triangular matrices in SL_n)
$T \subseteq B$ a maximal torus (e.g. all diagonal matrices in SL_n)
$U := \mathrm{Rad}_u(B)$, a maximal unipotent subgroup in B (e.g. all upper-triangular matrices in SL_n with ones on the diagonal)
U^- the maximal unipotent subgroup in G opposite to U (e.g. all lower-triangular matrices in SL_n with ones on the diagonal)

$\Lambda^+ \subseteq \mathfrak{X}(B)$ the set of dominant weights

$\Lambda^+ \leftrightarrow$ {finite-dimensional simple G-modules}

Remark: in the toric case we have $G = B = T$, $U = \{e\}$, $\Lambda^+ = \mathfrak{X}(T)$

Spherical varieties

Definition

A G-variety X is called spherical if X is normal, irreducible, and possesses an open B-orbit.

Examples:
(1) toric varieties ($G = B = T$);
(2) (generalized) flag varieties G/P (P is a parabolic subgroup);
(3) horospherical varieties (the stabilizer in G of a point in general position contains a maximal unipotent subgroup of G);
(4) symmetric spaces.

[AA]: a proper generalization of root subgroups from the toric case to spherical varieties is given by B-root subgroups

Global problem

Describe all B-root subgroups on a given spherical G-variety.

Let X be a complete spherical G-variety, $\operatorname{Aut} X$ its automorphism group, and $A = (\operatorname{Aut} X)^0$ its connected component of the identity.

Fact

The group A is linear algebraic.

Now suppose that G acts on X effectively. Then $G \subset A$ and there is a G-module decomposition

$$\operatorname{Lie} A = \operatorname{Lie} G \oplus W \oplus \bigoplus_{i \in I} V_i, \text{ where}$$

• W is a G-module with trivial action, which corresponds to a torus centralizing G;

• V_i is a simple G-module whose highest weight vector v_i corresponds to a B-root subgroup in A.

Certain invariants of spherical varieties

Let X be a spherical G-variety.
$\mathbb{K}(X)$ is the field of rational functions on X

Weight lattice
$M := \{\text{weights of } B\text{-semiinvariant functions in } \mathbb{K}(X)\} \subseteq \mathfrak{X}(T)$
$\lambda \in M \;\rightsquigarrow\; f_\lambda \in \mathbb{K}(X)^{(B)}_\lambda$; may assume $f_\lambda f_\mu = f_{\lambda+\mu} \;\; \forall\, \lambda, \mu \in M$

$\mathcal{D}^B$ is the set of B-stable prime divisors in X
$\mathcal{D}^G$ is the set of G-stable prime divisors in X
$\mathcal{D} = \mathcal{D}^B \setminus \mathcal{D}^G$ is the set of colors of X

There is a natural map
$\varkappa\colon \mathcal{D}^B \to N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}), \quad \langle \varkappa(D), \lambda \rangle := \mathrm{ord}_D(f_\lambda)$.

$X \leftrightarrow$ colored fan

Basic properties of B-root subgroups

Let H be a B-root subgroup in X and let $\chi(H) \in \mathfrak{X}(T)$ be its weight.

Fact [AA], [AZ2]

$\chi(H) \in \Lambda^+$.

$H \rightsquigarrow$ a vector field ξ on X; all vector fields obtained in this way are called $\mathbb{G}_a$-integrable

Definition

H is called vertical if H preserves the open B-orbit in X.
H is called horizontal otherwise.

Remark

In the toric case all T-root subgroups are horizontal.

Reduction to $X_{\mathcal{D}}$

In what follows we assume that X is horospherical.

Put $X_{\mathcal{D}} = X \setminus \bigcup_{D \in \mathcal{D}} D$.

Fact [AA], [AZ1]

Every B-root subgroup on X preserves the set $X_{\mathcal{D}}$.

The problem of describing all B-root subgroups on X is divided into the following three:

(1) describe all B-root subgroups on $X_{\mathcal{D}}$;
(2) determine those B-root subgroups on $X_{\mathcal{D}}$ for which the corresponding vector fields extend to the whole X;
(3) determine which vector fields in (2) are $\mathbb{G}_a$-integrable (and hence correspond to B-root subgroups on the whole X).

Remark: all vector fields in (3) are automatically $\mathbb{G}_a$-integrable if X is affine or complete.

Let P be the stabilizer in G of the set $X_{\mathcal{D}}$.
$P = L \leftthreetimes P_u$ is a Levi decomposition of P.

Local structure theorem (Knop'1994)

1) There exists a closed L-stable subvariety $Z \subseteq X_{\mathcal{D}}$ such that the map $P_u \times Z \to X_{\mathcal{D}}$, $(p, z) \mapsto pz$, is a P-equivariant isomorphism, where the action of P on $P_u \times Z$ is given by the formula $lu(p, z) = (lupl^{-1}, lz)$ for all $l \in L$, $u, p \in P_u$, $z \in Z$.
2) The derived subgroup of L acts trivially on Z. (Then Z is automatically a toric T-variety.)

Important: in the horospherical case there is a canonical way to choose Z:
$Z =$ the set of fixed points w.r.t. the action of U^-

Using the local structure theorem, we obtained in [AZ1, AZ2] a complete description of B-root subgroups on $X_{\mathcal{D}}$. For affine X, we also obtained in [AZ1] several sufficient conditions for extending B-root subgroups from $X_{\mathcal{D}}$ to the whole X. As a result, we proved in [AZ1] the following result.

Theorem

Suppose that X is affine and not necessarily horospherical. Then every G-stable prime divisor in X can be connected with the open G-orbit in X via a proper B-root subgroup.

Standard B-root subgroups: the affine case [AA]

Suppose that X is affine (and horospherical). Then there is a strictly convex finitely generated cone $\mathcal{E} \subset N_{\mathbb{Q}}$ such that

$$\mathbb{K}[X] = \bigoplus_{\lambda \in \mathcal{E}^{\vee} \cap M} \mathbb{K}[X]_{\lambda}$$

where $\mathbb{K}[X]_{\lambda}$ is a simple G-module with highest weight λ. Moreover, this decomposition is a grading.

Fact: the weight of every horizontal B-root subgroup on X belongs to $\mathfrak{R}(\mathcal{E}) \cap \Lambda^{+}$.

Given $\mu \in \mathfrak{R}(\mathcal{E}) \cap \Lambda^{+}$, let $\rho \in \mathcal{E}^{1}$ be such that $\langle \rho, \mu \rangle = -1$. For every $\lambda \in \mathcal{E}^{\vee} \cap M$ and $g \in \mathbb{K}[X]_{\lambda}$ we define $\partial_{\mu}(g) = \langle \rho, \lambda \rangle g f_{\lambda}$. This is a generalization of the LND's in the affine toric case!

Proposition

The map ∂_{μ} is a B-normalized LND on $\mathbb{K}[X]$.

The LND ∂_{μ} and the corresponding B-root subgroup H_{μ} are called standard. H_{μ} is automatically horizontal.

Standard B-root subgroups: the general case [AZ2]

Proposition

For affine X, a B-root subgroup on X is standard if and only if it preserves the canonical section $Z \subset X_{\mathcal{D}}$.

For arbitrary X, by definition, a B-root subgroup is called standard if it preserves Z.
Let $\mathfrak{F}$ be the colored fan of X.

Theorem

For a weight $\mu \in \mathfrak{X}(T)$, the following conditions are equivalent:
(1) there exists a standard B-root subgroup on X of weight μ;
(2) $\mu \in \mathfrak{R}(\mathfrak{F}) \cap \Lambda^+$.
If the fan $\mathfrak{F}$ is convex (for example, for affine or complete X), then these conditions are equivalent to one more:
(3) there exists a horizontal B-root subgroup on X of weight μ.

The case $G = \mathrm{SL}_2 \times$ torus [AZ2]

Suppose $G = \mathrm{SL}_2 \times S$ where S is a torus.

For every spherical G-variety X we obtained a complete description of all vertical B-root subgroups on X.

Now assume that X is complete. Then we know all standard B-root subgroups and all vertical B-root subgroups $\Rightarrow$ for the group $A = (\operatorname{Aut} X)^0$ in the G-module decomposition

$$\operatorname{Lie} A = \operatorname{Lie} G \oplus W \oplus \bigoplus_{i \in I} V_i$$

we know all the V_i. The part W can be also easily found.

We computed the commutation relations between all the summands $\Rightarrow$ we recover the Lie algebra structure on $\operatorname{Lie} A \Rightarrow$ we recover the group A itself.

Final result: for $G = \mathrm{SL}_2 \times S$ we obtained a description of the group A for complete horospherical G-varieties similarly to Demazure's result for complete toric varieties.