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# On the cocharacter closedness and cocharacter closure of rational orbits for algebraic group actions over valued fields

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# Content

## Set up

Let $k$ be a field, and let $G$ be a linear algebraic group acting on an (affine) algebraic variety $V$ defined over $k$, $v \in V(k)$, and let $\lambda : \mathbb{G}_m \to G$ be a $k$-cocharacter of $G$.
When $k$ is a valued field (e.g., $k = \mathbb{Q}_p, \mathbb{F}_q((T)), \ldots$), we may endow $G(k)$ and $V(k)$ with the $v$-adic topology induced from that of the base field $k$.
We consider the following types of closedness of the orbit $G.v$ as well as $G(k).v$:

- The Zariski closedness of $G.v$.
- The Hausdorff closedness of $G(k).v$ in $V(k)$, i.e. $G(k).v$ is closed with respect to the topology induced from $k$.
- We say that the orbit $G(k).v$ is cocharacter closed over $k$ if whenever $\lim_{\alpha \to 0} \mu(\alpha).v = v'$ for a $k$-cocharacter $\mu$, then $v' \in G(k).v$.

## Remarks

- If $G$ is a reductive group, then the orbit $G.v$ is Zariski closed if and only if $G(\overline{k}).v$ is cocharacter-closed over $\overline{k}$. (The Hilbert-Mumford Theorem saying that *if* $S \subseteq \overline{G.v} \setminus G.v$ *is closed,* $G$*-stable, then there exists a cocharacter* $\lambda : \mathbb{G}_m \to G$ *such that* $\lambda(\alpha).v \to v' \in S$.)
- Generally, $G(k).v \subsetneq (G.v)(k)$. Furthermore, if the stabilizer $G_v$ is a smooth subgroup scheme, then there is a bijection between the set of $G(k)$-orbits in $(G.v)(k)$ and the kernel $\mathrm{Ker}\left(\mathrm{H}^1(k, G_v) \to \mathrm{H}^1(k, G)\right)$ of the natural map between Galois cohomologies. In fact, this bijection was considered to obtain some landmark results in the arithmetic of hyperelliptic curves due to M. Bhargava, B. Gross (2013, 2014).
- (Bate-Martin-Roehrle 2005) Cocharacter closedness provides a key geometric realization for the notion of (relative) complete reducibility of subgroups due to J.-P. Serre (1997).

## $G$-complete reducible subgroups

**Definitions.** A subgroup $H$ of $G$ is called $G$-complete reducible over a field $k$ if whenever $H \leq P_\lambda$ for some $k$-cocharacter $\lambda : \mathbb{G}_m \to G$, there exists $u \in R_u(P_\lambda)(k)$ such that $H \leq uL_\lambda u^{-1}$.

**Remark.** For each cocharacter $\lambda : \mathbb{G}_m \to G$, we associate this cocharacter with an R-parabolic subgroup of the form

$$P_\lambda := \{g \in G \mid \text{ There exists limit } \lim_{\alpha\to 0} \lambda(\alpha)g\lambda(\alpha)^{-1} \in G\}.$$

In the case that $G$ is connected, parabolic subgroups and R-parabolic subgroups are the same.
$P_\lambda$ contains its unipotent radical and a Levi subgroup as follows.

- $R_u(P_\lambda) := \{g \in G \mid lim_{\alpha\to 0}\lambda(\alpha)g\lambda(\alpha)^{-1} = 1\}$,
- $L_\lambda := \{g \in G \mid lim_{\alpha\to 0}\lambda(\alpha)g\lambda(\alpha)^{-1} = g\}$.

We have Levi decompostion $P_\lambda = L_\lambda R_u(P_\lambda) = R_u(P_\lambda) \rtimes L_\lambda$.

## R-parabolic subgroups-An example

Let $\lambda : \mathbb{G}_m \to G$ be a $k$-cocharacter of $G = \mathrm{GL}_3$ given by

$$\lambda(\alpha) = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix}.$$

Then

$$\begin{aligned} P_\lambda &:= \{g \in G \mid lim_{\alpha \to 0} \lambda(\alpha) g \lambda(\alpha)^{-1} \text{ exists}\}, \\ &= \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}, \\ R_u(P_\lambda) &:= \{g \in G \mid lim_{\alpha \to 0} \lambda(\alpha) g \lambda(\alpha)^{-1} = 1\}, \\ &= \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\}. \end{aligned}$$

## $G$-complete reducible subgroups (continued)

**Example.** Let $k$ be algebraically closed, and $H \leq G = \mathrm{GL}_n(k)$. Then $H \leq G$ is $G$-cr if and only if $H$ acts completely reducibly on $k^n$.

**Proposition.** (J.-P. Serre (1997)) Assume $G$ is reductive, and $H$ is $G$-cr. Then $H$ is reductive. The converse is also true if $\mathrm{char}.k = 0$. Particularly, if $G$ is reductive, then a subgroup $H$ of $G$ is reductive if and only if it is $G$-cr.

## Relationship between cocharacter closedness and complete reducible subgroups

**Definition.** Let $H$ be a subgroup of $G$, then $\overline{h} = (h_1, \ldots, h_n) \in H^n$ is called a *generic tuple* of $H$ if $H = \langle h_1, \ldots, h_n \rangle$.

**Theorem (Bate-Herpel-Martin-Roehrle 2005, 2017)** We consider silmutaneous conjugation $G \curvearrowright G^n$:

$$g \cdot (g_1, g_2, \ldots, g_n) = (gg_1g^{-1}, \ldots, gg_ng^{-1}).$$

Let $\overline{h}$ be a generic tuple for a subgroup $H$ of $G$. Then $H \leq G$ is $G$-cr over $k \Leftrightarrow G(k).\overline{h}$ is cocharacter-closed over $k$.

## A rational Hilbert-Mumford Theorem

**Remark.** The cocharacter closure $\overline{G(k).v}^c$ can be defined by

$$\overline{G(k).v}^c = \bigcup G(k).w,$$

where $G(k).w$ is accessible from $G(k).v$.

**Definitions.**

- $G(k).v'$ is called 1-accessive from $G(k).v$ ($G(k).v \stackrel{1-acc}{\longrightarrow} G(k).v'$) if there exists a $k$-cocharacter $\lambda : \mathbb{G}_m \to G$ such that $\lim \lambda(\alpha).v = v''$ exists and belongs to $G(k).v'$.
- $G(k).v'$ is called accessible from $G(k).v$ if there exists $n \in \mathbb{Z}^+$ such that $G(k).v'$ is $n$-accessive from $G(k).v$, i.e.

$$G(k).v \stackrel{1-acc}{\longrightarrow} G(k).v_2 \stackrel{1-acc}{\longrightarrow} \cdots \stackrel{1-acc}{\longrightarrow} G(k).v_{n+1} = G(k).v'.$$

## A rational Hilbert-Mumford Theorem (continued)

**Theorem.** (Bate-Herpel-Martin-Roehrle 2017)
There is a unique cocharacter closed orbit $\mathcal{O} \subseteq \overline{G(k).v}^c$, and furthermore $\mathcal{O}$ is 1-accessible from $G(k).v$, i.e. there exists a $k$-cocharacter $\lambda : \mathbb{G}_m \to G$ such that $\lambda(\alpha).v \to v' \in \mathcal{O}$.

**Corollary.** Assume that $X$ is a $G(k)$-stable, cocharacter closed subset of $V(k)$, which meets $\overline{G(k).v}^c$. Then there is a $k$-cocharacter $\lambda : \mathbb{G}_m \to G$ such that $\lim \lambda(\alpha).v$ exists and lies in $X$.

## Relationship between closures

**Remark.**
We denote by $\overline{G(k).v}^H$ (resp., $\overline{G(k).v}^c$, $\overline{G(k).v}^Z$) the closure with respect to Hausdorff topology (resp., cocharacter closed topology, Zariski topology).

(Bate-Herpel-Martin-Roehrle 2017, Example 11.1): Although

$$G(\overline{k}).v \text{ is Zariski closed} \Leftrightarrow G(\overline{k}).v \text{ is cocharacter closed},$$

$\overline{G(\overline{k}).v}^c$ might be strictly smaller than $\overline{G(\overline{k}).v}^Z$.

## Problems

**Problem 1.**
Let $G$ be a linear algebraic group acting on an affine variety $V$ defined over a local field $k$, and let $v \in V(k)$ be a rational point. What are the relationships between the following types of closedness?

**(1)** $G.v$ is Zariski closed in $V$.

**(2)** $G(k).v$ is Hausdorff closed in $V(k)$, i.e. it is closed with respect to the topology induced from $k$.

**(3)** $G(k).v$ is cocharacter closed.

## Problems (continued)

**Problem 2.**
What is the relationship between the closures $\overline{G(k).v}^H$ and $\overline{G(k).v}^c$? More concretely, we consider the following cases:

**(1)** $\overline{G(k).v}^H = \overline{G(k).v}^c$ if $G$ is a $k$-torus?

**(2)** $\overline{G(k).v}^H = \overline{G(k).v}^c$ if $G$ is a commutative $k$-group?

**(3)** $\overline{G(k).v}^H = \overline{G(k).v}^c$ if $G$ is a nilpotent $k$-group?

## Problems (continued)

**Problem 3.**
We consider the conjugate action of $G = \mathrm{GL}(W)$ on $V = \mathrm{End}(W)$. Is this true that $\overline{\mathrm{GL}(W)(k).f}^{H} = \overline{\mathrm{GL}(W)(k).f}^{c}$?

## Results (B.-Hien-Hoang-Duc (in progress))

**(a)** Assume that $G$ is reductive and $v \in V(k)$. Let $k_i = k^{p^{-\infty}} = \cup_{n=1}^{\infty} k^{p^{-n}}$. Then the following conditions are equivalent:

- **(1)** $G.v$ is Zariski closed.
- **(2)** $G(k_i).v$ is cocharacter closed.
- **(3)** $G(k_i).v$ is Hausdorff closed.

## A key ingredient-Result of Kraft-Kuttler's type

- Let $k$ be a field, and let $G$ be a reductive acting on an algebraic variety $V$ defined over $k$, $v \in V(k)$, and let $\lambda : \mathbb{G}_m \to G$ be a $k$-cocharater of $G$. Assume that $\lim_{\alpha \to 0} \lambda(\alpha).v = v' \in G.v$. Then
- $v' \in R_u(P_\lambda)(k_i).v$. *(Since $v \in V(k)$, the stabilizer $G_v$ is $k$-closed, hence defined over $k_i$.)*
- **Question:** $v' \in R_u(P_\lambda)(k).v$?

## Example

Let $G = \mathrm{SL}_2$ acts on itself $V = G$ by conjugation. Choose $b \in k$, and $a \in k^\times \setminus \{\pm 1\}$. Let

$$v = \begin{pmatrix} a & b(a^{-1} - a) \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \in V(k).$$

We define the cocharater $\lambda : \mathbb{G}_m \to G$ by

$$\lambda(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}.$$

## Example (continued)

Then

$$P_\lambda = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}, R_u\left(P_\lambda\right) = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}.$$

It is easy to see that

$$\begin{aligned} v' = \lim_{\alpha \to 0} \lambda(\alpha).v &= \lim_{\alpha \to 0} \begin{pmatrix} a & \alpha^2 b(a^{-1} - a) \\ 0 & a^{-1} \end{pmatrix} \\ &= \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \\ &= u.v \in R_u(P_\lambda)(k).v, \end{aligned}$$

where

$$u = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}.$$

## Example (continued)

Furthermore, if we choose $b \in k_s \setminus k$, and $a \in k^\times \setminus \{\pm 1\}$. Let

$$v = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} \notin V(k), v' = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in V(k),$$

and define $\lambda \in X_*(G)_k$ by

$$\lambda(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}.$$

It is easy to see that

$$\begin{cases} \lim_{\alpha \to 0} \lambda(\alpha).v = v' \in R_u(P_\lambda)(k_s).v \setminus R_u(P_\lambda)(k).v, \\ G_v(k_s) \text{ is not } \Gamma\text{-stable.} \end{cases}$$

## Remark (cf. B.-Hien 2022)

**(b)** Assume that $G$ is nilpotent. Then the following conditions are equivalent:

- **(1)** $G.v$ is Zariski closed.
- **(2)** $G(k).v$ is Hausdorff closed.
- **(3)** $G(k).v$ is cocharacter closed.

**Key ingredient.** The stabilizer group scheme $G_v$ satisfy the Gabber condition. *(We say that a $k$-group $H$ satisfies the $(*)$-condition (or the Gabber condition) if all $\overline{k}$-tori of $H_{\overline{k}}$ are contained in $(H^+)_{\overline{k}}$. Here, $H^+$ is the largest smooth subgroup of $H$.)*

## Remark

**Example.** (Gabber/Gille/Morret-Bailly 2014)
Let $k = \mathbb{F}_q((T))$ be the imperfect local function field with $T$-adic topology. Assume that $G = \mathbb{G}_a \rtimes \mathbb{G}_m$ the semidirect product with the operation $(x, y) \cdot (x', y') = (x + yx', yy')$, and let $G$ act on the affine line $\mathbb{A}^1$ by $(x, y) \cdot z = x^p + y^p z$. Let $v = T \in k \setminus k^p$ be a rational point of $\mathbb{A}^1$. Then we have the following

**(a)** $G(\overline{k}).v = \overline{k}^p + (\overline{k}^{\times p})v = \overline{k}$ is Zariski closed in $\mathbb{A}^1$.

**(b)** $G(k).v = k^p + (k^\times)^p v \subsetneq k$, and $G(k).v$ is not Hausdorff closed in the $T$-adic topology, as well as is not cocharacter closed. In fact, $0 \in \overline{G(k).v}^c \setminus G(k).v$.

**(c)** $G_v = \{(x, y) \in \mathbb{G}_a \rtimes \mathbb{G}_m \mid x^p + (y-1)^p T = 0\}$ does not satisfy the Gabber condition.

**Question:** If $G$ is solvable, then is it true that
$G(k).v$ is cocharacter closed $\not\Rightarrow$ $G(k).v$ is Hausdorff closed?

## Results (relationship between closures)

**(c)** Assume that $G = T$ is a split $k$-torus. Then

$$\overline{T(k).v}^H = \overline{T(k).v}^c.$$

More precisely, if $w \in \overline{T(k).v}^H$, then $T(k).v \stackrel{1-acc}{\longrightarrow} T(k).w$, i.e. there is a $k$-cocharacter $\lambda : \mathbb{G}_m \to T$ such that $\lim_{\alpha \to 0} \lambda(\alpha).v = v' \in T(k).w$.

## One example for (c)

Assume that $k$ is a nonarchimedean local field, $T = \mathbb{G}_m \times \mathbb{G}_m$ acts on $\mathbb{A}^2$ by

$$\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t_1^{i_1} x \\ t_2^{i_2} y \end{pmatrix},$$

with $i_1, i_2 > 0$. Let $v = (1,1)^T$, $w = (a,0)^T \in \overline{T(k).v}^H$. Then $a = s^{i_1}$, we choose $\lambda(\alpha) = \mathrm{diag}(1,\alpha)$, and obtain $\lim \lambda(\alpha).v = v' = (1,0) \in T(k).w$. Hence, $w = \left(s^{i_1}, 0\right)^T \in \overline{T(k).v}^c$. Therefore, $\overline{T(k).v}^H = \overline{T(k).v}^c$.

## Results (relationship between closures, continued)

**(d)** Assume that $G = T$ is a $k$-torus. Then

$$\overline{T(k).v}^H = \overline{T(k).v}^c.$$

*Idea of the proof.* Consider the cases that $T$ is $k$-split, $T$ is $k$-anisotropic. Then consider the general case where $T = T_a \cdot T_s$ (almost direct product). Here, $T(k)$ is approximate to $T_a(k)T_s(k)$.

## Results (relationship between closures, continued)

**(e)** Assume that $G = U$ is a unipotent group. Then $\overline{G(k).v}^{H} = \overline{G(k).v}^{c}$. *(In fact, in B.-Hien (2022), we obtained a rational version of the Rosenlicht Theorem saying that* $U(k).v$ *is always Hausdorff closed.)*

## Results (relationship between closures, continued)

**(f)** If $G$ is solvable, generally, $\overline{G(k).v}^c \subsetneq \overline{G(k).v}^H$. If $G = \mathbb{G}_a \rtimes \mathbb{G}_m$, we may identify $G$ with

$$\left\{ \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} \mid x \in \mathbb{G}_a, y \in \mathbb{G}_m \right\}$$

via

$$(x, y) \mapsto \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}.$$

We consider the natural action of $G = \mathbb{G}_a \rtimes \mathbb{G}_m$ on $\mathbb{A}^2$. Then

$$G(k).v = k \times k^\times,$$

and

$$\overline{G(k).v}^c = k \times k^\times \cup \{(1, 0)\} \subsetneq \overline{G(k).v}^H = k \times k.$$

**(g)** $\overline{\mathrm{GL}(W)(k).v}^H = \overline{\mathrm{GL}(W)(k).v}^c$ when $n \in \{2, 3\}$, or $v$ is semisimple. What happen if $v$ is nilpotent?

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THANK YOU FOR YOUR ATTENTION !