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Flexibility of horospherical varieties

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Basic settings

$\mathbb{K}$ – algebraically closed field of characteristic zero.

$\mathbb{K}[X]$ is the algebra of regular functions on an affine algebraic variety X.

All varieties are irreducible affine algebraic varieties.

All actions of algebraic groups are assumed to be algebraic.

All automorphisms of varieties are regular. $\mathrm{Aut}(X)$ is the group of regular automorphisms of a variety X.

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2. G-actions with an open orbit
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LND

Let B be a $\mathbb{K}$-algebra.

Definition

A linear operator

$$\delta\colon B \to B$$

is called a derivation if it satisfies the Leibniz's rule.

$$\delta(ab) = a\delta(b) + b\delta(a).$$

Definition

A derivation $\delta\colon B \to B$ is called a locally nilpotent (LND) if for each $b \in B$ there exists $n \in \mathbb{Z}_{>0}$ such that

$$\delta^n(b) = 0.$$

If $B = \mathbb{K}[X]$, we say that δ is an LND on X.

Exponent of an LND

Suppose $\delta\colon B \to B$ is an LND. Then one can define the exponent of δ:

$$\exp(\delta)(b) = b + \frac{\delta(b)}{1!} + \frac{\delta^2(b)}{2!} + \dots$$

It is easy to show that $\exp(\delta) \in \mathrm{Aut}(B)$. Moreover,

$$H_\delta = \{\exp(t\delta) \mid t \in \mathbb{K}\}$$

is a $\mathbb{G}_a$-subgroup in $\mathrm{Aut}(B)$ i.e. an algebraic subgroup isomorphic to $(\mathbb{K}, +)$.

Well-known theorem

Each $\mathbb{G}_a$-subgroup corresponds to an LND.

The group of special automorphisms $\mathrm{SAut}(X)$ is the subgroup in $\mathrm{Aut}(X)$ generated by all $\mathbb{G}_a$-subgroups.

Flexibility

Definition

A regular point $x \in X$ is called flexible, if the tangent space $T_x X$ is spannad by tangent vectors to orbits $H_\delta \cdot x$ for various $\mathbb{G}_a$-actions.

A variety is flexible if all regular points are flexible.

Theorem (Arzhantsev-Flenner-Kaliman-Kutzschebauch-Zaidenberg, 2013)

For an irreducible affine variety X of dimension ≥ 2, the following conditions are equivalent.

1. The group $\mathrm{SAut}(X)$ acts transitively on X^{reg};
2. The group $\mathrm{SAut}(X)$ acts infinitely transitively on X^{reg};
3. X is a flexible variety.

Recall that infinitely transitivity means transitivity on m-tuples of distinct points.

Generically flexibility

Definition

If a variety X possesses a flexible points then it is called generically flexible.

Let us define the field Makar-Limanov invariant

$$\mathrm{FML}(X) = \bigcap_{D \in \mathrm{LND}(X)} \mathrm{Quot}(\mathrm{Ker}\, D) = \mathbb{K}(X)^{\mathrm{SAut}(X)}.$$

Proposition

The following conditions are equivalent

1. X is generically flexible;
2. $\mathrm{FML}(X) = \mathbb{K}$;
3. X admits an open $\mathrm{SAut}(X)$-orbit.

If X is not generically flexible, there is an $\mathrm{SAut}(X)$-invariant prime divisor on X.

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G-actions with an open orbit

Let G be a linear algebraic group. Suppose G acts on X with an open orbit $\mathcal{O}$. Then each G-orbit consists of either only regular (a regular orbit) or only singular points (a singular orbit) of X. $\mathcal{O}$ is regular.

To prove flexibility of X we can proceed in two stages:

1. To prove that X is generically flexible;
2. To prove that for any regular G-orbit $O \neq \mathcal{O}$ we can move a point $x \in O$ to a point in a higher-dimensional G-orbit by an automorphism of X.

If the number of G-orbits on X is finite, then to prove 2 it is sufficient to prove that for any regular G-orbit $O \neq \mathcal{O}$ its closure $\overline{O}$ is not $\mathrm{SAut}(X)$-invariant.

A technique for proving generically flexibility

Theorem (G.-Shafarevich, 2019)

Let T be an algebraic subtorus in $\mathrm{Aut}(X)$. Denote by H the subgroup in $\mathrm{Aut}(X)$ generated by T and $\mathrm{SAut}(X)$. If a normal G-variety X with an open G-orbit is not generically flexible, then there exists an H-invariant prime divisor D.

Idea of the proof. If X is not generically flexible, then there exists an $\mathrm{SAut}(X)$-invariant prime divisor D_0. If D_0 is principle, i.e. $D_0 = \mathrm{div}(f)$, then $f \in \mathbb{K}[X]^{\mathrm{SAut}(X)}$. Since $\mathrm{SAut}(X) = \mathrm{ML}(X)$ is a normal subgroup of $\mathrm{Aut}(X)$, the subalgebra $\mathrm{ML}(X)$ is T-invariant. This implies that there exists nonconstant T-semi-invariant (i.e. H-semi-invariant) in $\mathrm{ML}(X)$.
If D_0 is not principle, we move to Cox realization of X. The preimage of D_0 in the total coordinate space is principle.

A technique for moving a point outside of the orbit

Let us consider an action of the multiplicative group of $\mathbb{K}$. Such actions we call $\mathbb{G}_m$-actions. This action corresponds to a $\mathbb{Z}$-grading on $\mathbb{K}[X]$. This action is hyperbolic if there exist $a > 0$ and $b < 0$ such that $\mathbb{K}[X]_a \neq \{0\}$ and $\mathbb{K}[X]_b \neq \{0\}$.

Theorem (G.-Shafarevich, 2019)

Let Z be the set of $\mathbb{G}_m$-fixed points for a non-hyperbolic $\mathbb{G}_m$-action on a normal affine irreducible variety X. Assume $Z \cap X^{\mathrm{reg}} \neq \varnothing$. Then $Z^{\mathrm{reg}} \cap X^{\mathrm{reg}} \neq \varnothing$ and for every $z \in Z^{\mathrm{reg}} \cap X^{\mathrm{reg}}$ the tangent space $T_z X$ is spanned by $T_z Z$ and tangent vectors to orbits of regular $\mathbb{G}_a$-actions on X.

If $Z = \overline{O}$ is a closure of an orbit, then $\overline{O}$ is not $\mathrm{SAut}(X)$-invariant.

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Horospherical varieties

Let G be a connected linear algebraic group.

Definition

An irreducible G-variety X is called horospherical, if for a generic point $x \in X$ the stabilizer of x contains a maximal unipotent subgroup $U \subseteq G$.
If X contains an open G-orbit, then X is called complexity-zero horospherical.

Suppose that X is an affine complexity-zero horospherical variety. It is easy to see that the unipotent radical of G acts trivially on X. Hence, we may assume that G is reductive. Taking a finite covering, we may assume that $G = T \times G'$, where T is an algebraic torus and G' is a semisimple group.
We have the following sequence of inclusions

$$\mathbb{K}[X] \hookrightarrow \mathbb{K}[\mathcal{O}] \hookrightarrow \mathbb{K}[G].$$

Horospherical varieties

Let B be a Borel subgroup of G and let $M = \mathfrak{X}(B)$ be the group of characters of B. For a $\Lambda \in M$ we put

$$S_\Lambda = \{f \in \mathbb{K}[G] \mid f(gb) = \Lambda(b)f(g) \text{ for all } g \in G, b \in B\}.$$

Then

$$S_\Lambda S_{\Lambda'} = S_{\Lambda+\Lambda'}.$$

The set $\mathfrak{X}^+(B)$ of dominant weights consists of all Λ such that $S_\Lambda \neq \{0\}$. Popov and Vinberg proved in 1972 that for an affine complexity-zero horospherical G-variety X there is a decomposition

$$\mathbb{K}[X] = \bigoplus_{\Lambda \in P} S_\Lambda$$

for some submonoid $P \in \mathfrak{X}^+(B)$.
Denote by $\sigma^\vee$ the cone in $M_\mathbb{Q}$ spanned by P. The variety X is normal if and only if P is saturated, i.e. $\sigma^\vee \cap \mathbb{Z}(P) = P$.

There is a one-to-one correspondence between faces of σ and G-orbits of X. More precisely, if $O_\tau \subseteq X$ is the G-orbit in X corresponding to a face τ of the cone $\sigma^\vee$, then the ideal of functions vanishing on O_τ has the form

$$I(O_\tau) = \bigoplus_{\Lambda \in P \setminus \tau} S_\Lambda.$$

This ideal vanishes on the closure $\overline{O_\tau}$. Then

$$O_\tau = \overline{O_\tau} \setminus \left(\bigcup_{\gamma \prec \tau} \overline{O_\gamma} \right).$$

An important particular case of horospherical varieties give toric varieties.

Definition

A toric variety is a variety X admitting an action of an algebraic torus $T \simeq (\mathbb{K}^\times)^n$ with open orbit.

Remark

Often by toric variety one mean a normal toric variety. We do not a-priori assume a toric variety to be normal.

So, toric variety is a horospherical (complexity-zero) variety corresponding to $G \cong (\mathbb{K}^\times)^n$. For a toric variety each nonzero homogeneous component has dimension one. We have

$$\mathbb{K}[X] = \bigoplus_{m \in P} \mathbb{K}\chi^m,$$

where $\chi^m = t_1^{m_1} \cdot \ldots \cdot t_n^{m_n}$ is the character of the torus T corresponding to a point $m = (m_1, \ldots, m_n)$.

$$\{\text{Toric varieties}\}$$
$$\cap$$
$$\{\text{Horospherical varieties}\}$$
$$\cap$$
$$\{\text{Spherical varieties}\}$$

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List of results

- Non-degenerate (i.e. $\mathbb{K}[X]^\times = \mathbb{K}^\times$) normal toric varieties are flexible. (Arzhantsev-Kuyumzhiyan-Zaidenberg, 2012)
- Horospherical varieties (not necessary normal) with a semisimple group G are flexible. (Shafarevich, 2017)
- Normal horospherical varieties with $\mathbb{K}[X]^\times = \mathbb{K}^\times$ are flexible. (G.-Shafarevich, 2019)
- Criterion for not necessary normal toric varieties to be flexible. (G.-Boldyrev, 2022)

Ideas of proof (for results on horospherical varieties)

Step 1. X is generically flexible.
– If G is semisimple, then the image of G in $\mathrm{Aut}(X)$ is contained in $\mathrm{SAut}(X)$. Therefore, there is an open $\mathrm{SAut}(X)$-orbit.
– If $G = G' \times T$ and X is normal, then we denote by H the group generated by $\mathrm{SAut}(X)$ and T and use the fact that there is no H-invariant prime divisor D on X. The last statement is true since if D is H-invariant, then it is G-invariant. We have only finite number of orbits, hence, D is a closure of G-orbit O. But $\overline{O}$ is not $\mathrm{SAut}(X)$-invariant (see Step 2).
Step 2. For each G-orbit $O \neq \mathcal{O}$, $\overline{O}$ is not $\mathrm{SAut}(X)$-invariant.
$O = O_\tau$ and we have a linear function $\alpha\colon M_{\mathbb{Q}} \to \mathbb{Q}$ such that $\alpha|_\tau = 0$, $\alpha|_{\sigma^\vee} \geq 0$ and $\alpha|_M \in \mathbb{Z}$. Then α gives a $\mathbb{Z}$-grading on $\mathbb{K}[X]$ corresponding to a non-hyperbolic $\mathbb{G}_m$-action with the set of stable points $Z = \overline{O_\tau}$.

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Non-normal toric varieties

For a toric variety $\mathbb{K}[X] = \bigoplus_{m \in P} \mathbb{K}\chi^m$.
Denote by $\sigma^\vee$ the cone in $M_\mathbb{Q}$ spanned by P. The saturation of P is $P_{\mathrm{sat}} = \sigma^\vee \cap \mathbb{Z}(P)$. The variety X is normal if and only if P is saturated, i.e. $P_{\mathrm{sat}} = P$. Otherwise $q \in P_{\mathrm{sat}} \setminus P$ are called gaps.

Definition

A point $p \in P$ is called a saturation point if $p + \sigma^\vee \cap P_{\mathrm{sat}} \subseteq P$.
A face τ is called almost saturated if it contains a saturation point.

Lemma (G.-Boldyrev, 2022)

For a face τ of codimension one the following conditions are equivalent:

- τ is almost saturated;
- the corresponding T-orbit is regular (i.e. consists of regular points)

Non-normal toric varieties

We can consider the cone σ in $N_{\mathbb{Q}} = M^*_{\mathbb{Q}}$, which is dual to $\sigma^\vee$.

$$\sigma = \{v \in N_{\mathbb{Q}} \mid \langle u, v\rangle \geq 0 \text{ for all } u \in \sigma^\vee\}.$$

We construct a cone γ removing in σ all extremal rays that correspond to faces of $\sigma^\vee$, which are not almost saturated. (I.e. correspond to singular orbits.)

Theorem (G.-Boldyrev, 2022)

Let X be a toric variety and σ be the corresponding cone. Let us remove all extremal rays of the cone σ, that correspond to orbits, consisting of singular points. Let γ be the cone generated by all other extremal rays. The following conditions are equivalent

- The variety X is flexible;
- The cone γ is not contained in any hyperspace.
- There is no $f \in \mathbb{K}[X] \setminus \mathbb{K}$ such that $\{f = 0\} \subset X^{\mathrm{sing}}$.

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Problems and conjections

Problem

To obtain a criterion of flexibility for a non-normal horospherical variety of non-semisimple group. (Here we need only Step 1, i.e. to check generic flexibility.)

Conjecture

Let a reductive group G acts on a normal affine variety X with open orbit. Suppose $\mathbb{K}[X]^\times = \mathbb{K}^\times$. Then X is flexible.

Problem

To check the previous conjecture in case of spherical varieties. (Here we need only Step 2, since generic flexibility follows from result due to Avdeev and Zhgoon.)

Thank you!