ON GORENSTEIN GRAPHS

Tran Nam Trung
(institute of Mathematics, VAST, Vietnam)

Online seminar Flexibility and Computational Methods
(HSE university, Moskow, Russia and
institute of Mathematics, VAST, Hanoi, Vietnam)

From 21/12/2024 To 24/12/2024

Outline:

1. Simplicial complexes
2. Cohen-Macaulay / Gorenstein property
3. Edge ideals
4. Gorenstein graphs vs girth
5. Othe classes of Gorenstein graphs
6. Construct new Gorenstein graphs from the old ones

1. Simplicial complexes:

$V = \{1, 2, \dots, n\}$; $n \geqslant 1$

Δ is a simplicial complex on V :

$$\sigma \in \Delta,\ \tau \subseteq \sigma \Rightarrow \tau \in \Delta$$

$F \in \Delta$: $\quad \dim(F) = |F| - 1$

$\dim(\Delta) = \max\{\dim(F) \mid F \in \Delta\}$

K – a field

$R = K[x_1, \dots, x_n]$;

$\tau \subseteq [n]$: $\quad x_\tau = \prod_{i \in \tau} x_i .$

$I_\Delta = (x_\tau \mid \tau \in \Delta)$ – the Stanley-Reisner ideal of Δ

$K[\Delta] = R/I_\Delta$: the Stanley-Reisner ring of Δ

Problem. Understanding algebraic properties of $K[\Delta]$ interms of Δ.

Δ is Cohen-Macaulay complex if $K[\Delta]$ is CM
(CM for short)

Δ is Gorenstein complex if $K[\Delta]$ is Gorenstein

2. Cohen-Macaulay / Gorenstein property

2. Cohen-Macaulay / Gorenstein property

$I \subseteq R$ - a monomial ideal with $\dim R/I = d$.

Minimal free resolution:

$$0 \leftarrow F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots \leftarrow F_p \leftarrow 0$$

- I is CM $\Leftrightarrow$ $n - p = d$.
- I is Gorenstein $\Leftrightarrow$ $n-p = d$ and $\operatorname{rank}(F_p) = 1$

Reisner's criterion.

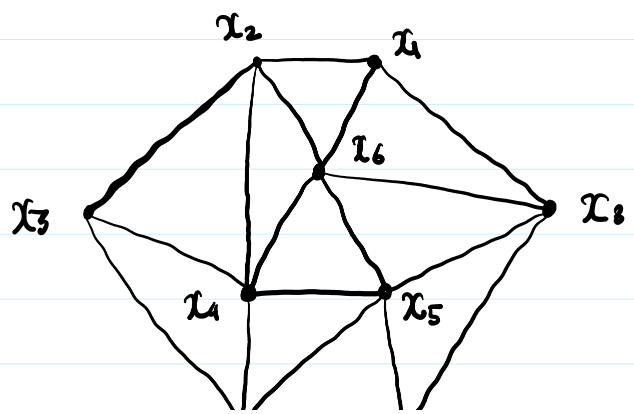
The link: $F \in \Delta$, $lk_\Delta(F) = \{G \mid G \cup F \in \Delta \ \& \ F \cap G = \phi\}$

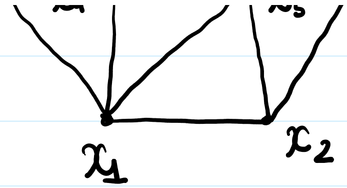
Theorem. (Reisner) Δ is CM $\Leftrightarrow$ $\widetilde{H}_i(lk_\Delta(F); K) = 0$ $\forall F \in \Delta$, $i < \dim lk_\Delta(F)$.

Remark. The CM property of Δ depends only on the topology of $|\Delta|$ (geometric realization)

(Munkres - Stanley).

Example. (1) Δ be a triangulation of $\mathbb{P}^2_{\mathbb{R}}$:





$$\tilde{H}_1(\mathbb{P}^2_{\mathbb{R}}; K) = \begin{cases} K & \text{if } \operatorname{char}(K) = 2, \\ 0 & \text{otherwise} \end{cases}$$

Δ is CM $\Leftrightarrow$ $\operatorname{char}(K) \neq 2$.

Stanley's criterion.

The star: $\quad st_\Delta(v) = \{F \in \Delta \mid x \in \Delta\}$

if $st_\Delta(v) = \Delta$, then Δ is a cone over v

$\Rightarrow \tilde{H}_i(\Delta; K) = 0 \ \forall i.$

$\operatorname{core}(V) = \{v \in V \mid st_\Delta(v) \neq \Delta\}$

$\operatorname{core}(\Delta) = \Delta|_{\operatorname{core}(\Delta)} = \{F \in \Delta \mid F \subseteq \operatorname{core}(V)\}$

$f = (f_{-1}, f_0, f_1, \dots, f_{d-1})$ be the f vector of Δ

The Euler characteristic of Δ:

$$\tilde{\chi}(\Delta) = \sum_{i=-1}^{d-1} (-1)^i f_i(\Delta)$$

Δ is Euler:

$$\tilde{\chi}(lk_\Delta(F)) = (-1)^{\dim(F)} \quad \forall F \in \Delta.$$

Thm. Δ is Gorenstein if $\operatorname{core}(\Delta)$ is CM & Eulerian.

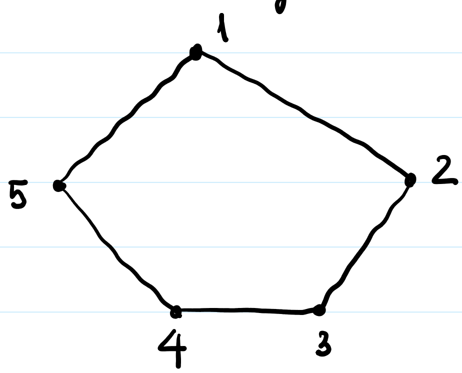
Example 2. Δ is a triangulation of $\mathbb{P}^3_{\mathbb{R}}$. Then Δ is Gorenstein $\Leftrightarrow$ $\operatorname{char}(K) \neq 2$.

3. Edge ideals.

$G = (V, E)$ a simple graph: $V = \{1, 2, \dots, n\}$

$I(G) = (x_i x_j \mid \{i,j\} \in E) \subseteq R$

the edge ideal of G



$\Rightarrow I(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1)$

$S = \{1, 3\}$.

$S \subseteq V$: an independent set of G if $\forall x, y \in S \Rightarrow \{x, y\} \notin E$.

$\Delta(G)$ = the set of all independent sets of G

— the independence complex of G

$I(G) = I_{\Delta(G)}$.

Def. G is CM (or Gorenstein) $\Leftrightarrow$ so is $I(G)$.

Def. G is CM (or Gorenstein) $\Leftrightarrow$ so is $I(G)$.

Problem. 1) Characterize CM graphs G in terms of G.

2) Characterize Gorenstein graphs G in terms of G

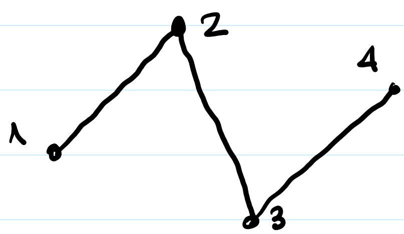
Remarks: 1) $\exists G$: G is CM $\Leftrightarrow$ $\mathrm{char}(K) \neq 2$.

2) $\exists G$: G is Gorenstein $\Leftrightarrow$ $\mathrm{Char}(K) \neq 2$

<u>cover</u>: $C \subset V$ is a cover of G if $e \cap C \neq \emptyset$ $\forall e \in C$

a minimal cover means w.r.t. inclusion

Note: a set $S \subseteq V$ is an independent set $\Leftrightarrow$ $V \setminus S$ is a cover.



$\{1,3\}, \{1,4\}, \{2,4\}$: max. ind. sets

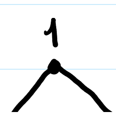
$\{2,4\}, \{2,3\}, \{1,3\}$: min. covers

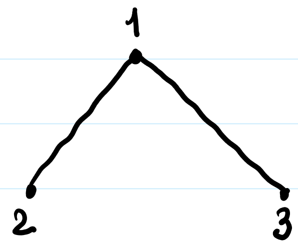
a CM graph.

$$I(G) = \bigcap_{C - \text{min cover}} (x_i \mid i \in C) \text{ — min. primary decomp.}$$

Thus, G is CM $\Rightarrow |C| = \text{const.}$ $\forall$ min. cover C of G

we say: G is well-covered.





$\{1\}, \{2,3\}$: min. covers
$\Rightarrow$ not well-covered. $\Rightarrow$ $\boxed{\text{not CM.}}$

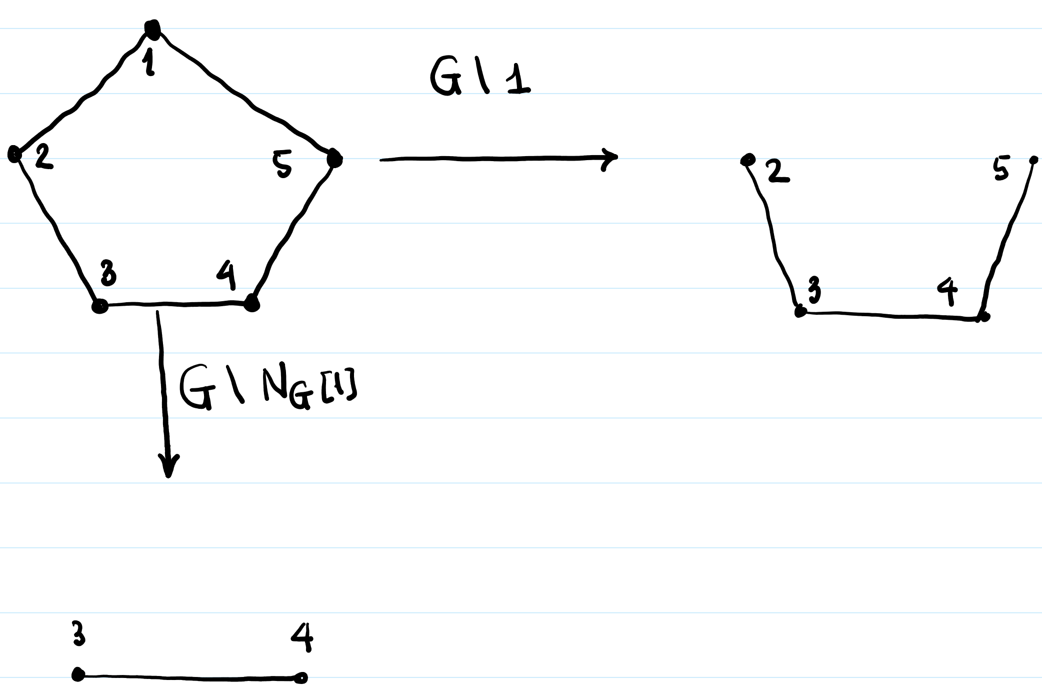
$$\alpha(G) = \max\{|S| \mid S \text{ is an ind. set of } G\}$$

Let $S \subseteq V$. Denote $G \setminus S$ to be the graph (V', E')

- $V' = V \setminus S$
- $E' = \{e \in E \mid e \subseteq V'\}$

$v \in V$: $N_G(v) = \{u \mid \{u,v\} \in E\}$ – the neighbors of v

$N_G[v] = \{v\} \cup N_G(v)$ – the closed neighbors of v



Def. G is in the class $W_2 \Leftrightarrow$ G is well-covered and

$\forall v \in V$: $G \setminus v$ is still well-covered & $\alpha(G\setminus v) = \alpha(G)$.

Remark. G is Gorenstein $\Rightarrow$ G is in W_2.

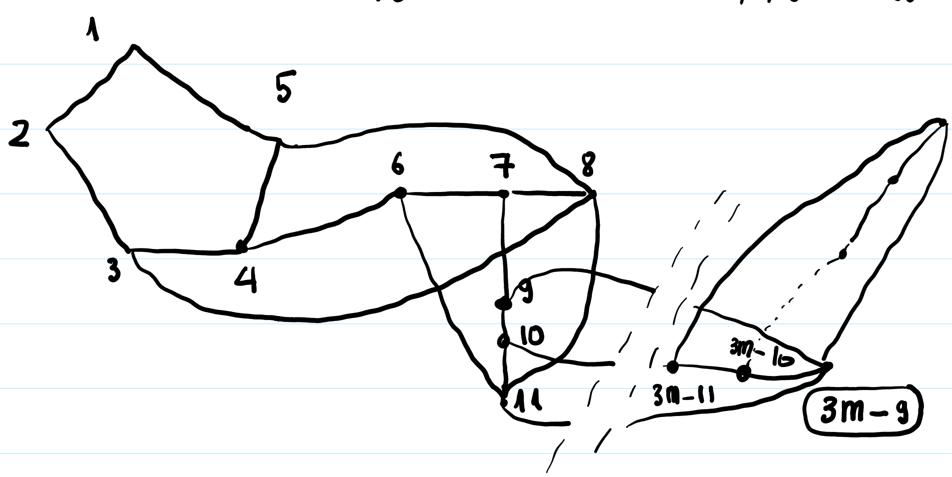
4. Gorenstein graphs vs girth.

- Herzog – Hibi: if G is bipartite, then G is Gorenstein $\Leftrightarrow$ G is just an edge.

- Planar graphs:

Pinter construct a family of planar graphs of girth ≥ 4 in W_2: G_m

- $G_1 = K_2$, $G_2 = C_5$
- for $m \geq 2$: . Let $x, y, u \in V(G)$ with $\deg(x) = \deg(y) = 2$ and $x, y \in N(u)$.
  . Take new 3 vertices: a, b, c: join: $a \to b \to c \to x$; $a \to x$, $b \to y$ $\Big|$ G_m.



Minh – Hoang – T: if $\text{girth}(G) \geq 4$ & G is planar, then G is Gorenstein $\Leftrightarrow$ $G \cong G_m$ for $m = \alpha(G)$.

Hoang – T: if $\text{girth}(G) \geq 4$, then G is Gorenstein $\Leftrightarrow$ G is in W_2

Hoang_T: girth(G) ≥ 4, then G is Gorenstein $\Leftrightarrow$ G is in W_2

. Proof:

$*$ $G \in W_2 \Rightarrow \Delta(G)$ is Eulian.

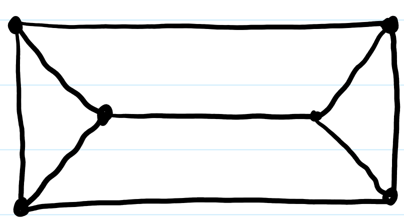
$*$ $G \in W_2 \Rightarrow \Delta(G)$ satifies Reisner's criterion

5. Other classes of Gorenstein graphs

. Planar graphs: (T)

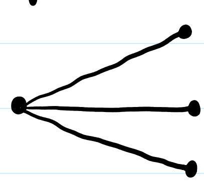
G is Gorenstein $\Leftrightarrow$ $G \cong G_m$ with $m = \alpha(G)$

or G is the following:



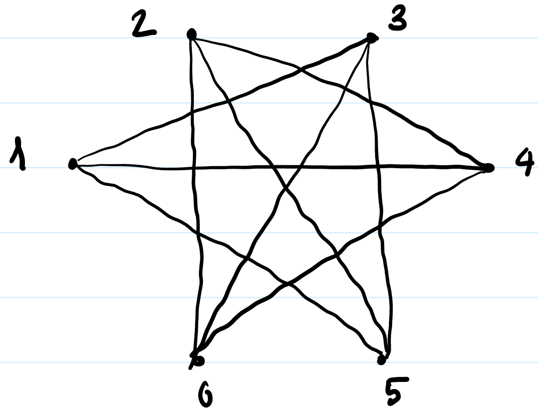
. Claw-free graphs: T

G is claw-free graph if G has no induced subgraph $K_{1,3}$.



Thm (T): Let G be a claw-free graph. Then G is Gorenstein $\Leftrightarrow$ G is one of K_1, K_2, and C_n^c with $n \geq 5$.

C_6^c :



- Circulant graphs:

Assume $n \geq 3$ and $S \subseteq \{1, \dots, \lfloor n/2 \rfloor\}$

The circulant graph $C_n(S)$:

$$\{i,j\} \in C_n(S) \iff \min\{|i-j|,\ n-|i-j|\} \in S.$$

A. Nikseresht – M.R. Oboudi: if either $\deg_G(v) \leq 4 \ \forall v$, or

$G = C_n(\{1, \dots, d\})$ for some $d \leq n/2$, then G is Gorenstein

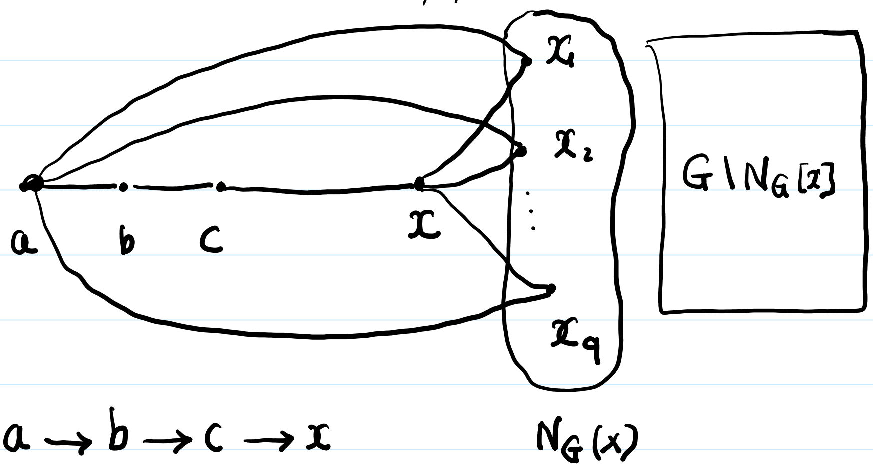
$\iff$ G is one of tK_2, $t\overline{C_6}$ or $tC_{13}(1,5)$ for some t and $n \geq 4$.

6. Construct new Gorenstein graphs from the old ones

Construction: • G – a Gorenstein graph

$x \in G$: $\deg_G(x) \geq 2$

- Take 3 new vertices: a, b, c:

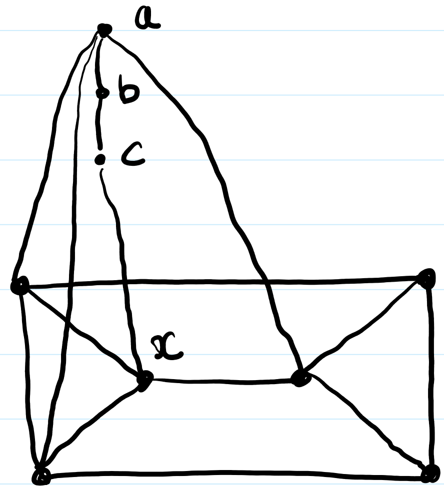


- Join: * $a \to b \to c \to x$

  * a to every vertex in $N_G(x)$

$\Rightarrow$ The obtained graph is Gorenstein.

For example:



Gorenstein graph

The end