INTRODUCTION TO ALGEBRAIC GROUPS AND INVARIANT THEORY

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Preface

These notes originate in a series of eleven lectures and eight seminars given at the Eberhard Karls Universität Tübingen in the Spring of 2007. The material here is based on special courses and seminars given by Professor Ernest Vinberg, Dmitri Timashev and myself at the Faculty of Mechanics and Mathematics of Moscow State University.

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1. Affine Algebraic Groups

1.1. **Examples and first properties.** Let $\mathbb{K}$ be an algebraically closed field, and put $V = \mathbb{K}^n$. The elements of the polynomial algebra $\mathbb{K}[x_1, \dots, x_n]$ can be viewed as $\mathbb{K}$-valued functions on V. They form the algebra $\mathcal{O}(V)$ of *regular functions* on V. *An algebraic subvariety* $X \subseteq V$ is the set of solutions of a system of polynomial equations on V with the algebra of regular functions $\mathcal{O}(X)$. By definition, functions in $\mathcal{O}(X) = \mathcal{O}(V)/\mathbb{I}(X)$ are restrictions of elements of $\mathcal{O}(V)$ to X, and $\mathbb{I}(X) = \{f \in \mathcal{O}(V) : f \mid_X \equiv 0\}$. There are natural notions of morphisms and isomorphisms between algebraic subvarieties. *An affine variety* is an isomorphism class of algebraic subvarieties.

Definition 1.1.1. *An affine algebraic group* is a group G equipped with a structure of an affine variety such that the multiplication map $\mu : G \times G \to G$, $\mu(g_1, g_2) = g_1 g_2$ and the inverse map $i : G \to G$, $i(g) = g^{-1}$ are morphisms of affine varieties.

Example 1.1.2 (The general linear group)**.** Consider the group $\mathrm{GL}(n, \mathbb{K}) = \mathrm{GL}(n)$ of all invertible $(n \times n)$-matrices over $\mathbb{K}$. It has a structure of an affine variety. Namely, take the vector space $\mathbb{K}^{n^2+1}$ with coordinates $(t, a_{ij}), i, j = 1, \dots, n$. Then $\mathrm{GL}(n)$ may be realized as a subset of $\mathbb{K}^{n^2+1}$ defined by the equation $\det(a_{ij})t = 1$. This implies $\mathcal{O}(\mathrm{GL}(n)) = \mathbb{K}[a_{ij}][\frac{1}{\det}]$.

The multiplication map μ is given as $\mu(A, B) = AB = C$, and if $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, then $c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$. As we know from the course of linear algebra, $i(A) = A^{-1} = (d_{ij})$ with $d_{ij} = \frac{(-1)^{i+j}}{\det(A)} m_{ji}$, where m_{ji} are minors of A. This shows that the maps $\mu : \mathrm{GL}(n) \times \mathrm{GL}(n) \to \mathrm{GL}(n)$ and $i : \mathrm{GL}(n) \to \mathrm{GL}(n)$ are polynomial, thus $\mathrm{GL}(n)$ is an affine algebraic groups.

Remark 1.1.3. In this course we consider only affine algebraic groups, so the adjective "affine" will be sometimes omitted.

Proposition 1.1.4. *Let $G \subseteq \mathrm{GL}(n)$ be a subgroup. Assume that G is a (closed) subvariety in $\mathrm{GL}(n)$. Then G is an algebraic group.*

Proof. The multiplication and inverse maps for G are the restrictions to $G \times G$ and G of the corresponding maps for $\mathrm{GL}(n)$. Clearly, such restrictions are morphisms. □

A closed (in Zariski topology) subgroup of $\mathrm{GL}(n)$ is called a *linear algebraic group*. As we shall prove later (see Theorem 1.3.20 and Remark 1.3.21), any affine algebraic group admits a realization as a closed subgroup of some $\mathrm{GL}(n)$. So, the notion "affine algebraic group" and "linear algebraic group" turn out to be synonyms. On the other hand, some authors use the term "linear algebraic group" in order to fix the matrix realization of a group G.

Example 1.1.5 (Classical Groups)**.** Besides $\mathrm{GL}(n)$, there are four other series of algebraic groups that are called *classical*.

(i) The *special linear group* $\mathrm{SL}(n)$ consists of the matrices of determinant 1 in $\mathrm{GL}(n)$. It is clearly a subgroup defined by the equation $\det(a_{ij}) - 1 = 0$ in $\mathrm{GL}(n)$. Thus $\mathrm{SL}(n)$ is an algebraic group. Since it is a hypersurface in the space of matrices $\mathrm{Mat}(n \times n, \mathbb{K})$, the dimension of $\mathrm{SL}(n)$ equals $n^2 - 1$.

(ii) Let $q : V \times V \to \mathbb{K}$ be a bilinear symmetric non-degenerate form on the space $V = \mathbb{K}^n$. Define a group

$$\mathrm{O}(q) := \{A \in \mathrm{GL}(n) : q(Av_1, Av_2) = q(v_1, v_2) \text{ for any } v_1, v_2 \in V\}.$$

Let Q be a symmetric $(n \times n)$-matrix associated with q, i.e. $q(v_1, v_2) = v_1^T Q v_2$ for any $v_1, v_2 \in V$. Then

$$q(Av_1, Av_2) = (Av_1)^T Q (Av_2) = v_1^T A^T Q A v_2,$$

and $A \in \mathrm{O}(q)$ if and only if $A^T Q A = Q$. The last equality may be considered as a system of quadratic equations on a_{ij}, where, as usual, $A = (a_{ij})$. This shows that $\mathrm{O}(q)$ is an algebraic group. It is called the *orthogonal group* associated with the form q. For example, if $v_1 = (x_1, \ldots, x_n)$, $v_2 = (y_1, \ldots, y_n)$, and the form q is defined as $q(v_1, v_2) = x_1 y_1 + \cdots + x_n y_n$, then $Q = E$ and $\mathrm{O}(q) = \{A \in \mathrm{GL}(n) : A^T A = E\}$. In this case we denote $\mathrm{O}(q)$ as $\mathrm{O}(n)$.

(iii) The equation $A^T Q A = Q$ implies $\det(A) = \pm 1$ (Q is non-degenerate !). Define the *special orthogonal group* as

$$\mathrm{SO}(q) = \{A \in \mathrm{O}(q) : \det(A) = 1\}.$$

Clearly, $\mathrm{SO}(q)$ is a subgroup in $\mathrm{O}(q)$, and this subgroup is proper (see Exercise 1.1.11).

(iv) Assume that $\mathrm{char}\,\mathbb{K} \neq 2$. Let $V = \mathbb{K}^{2n}$, $\omega : V \times V \to \mathbb{K}$ be a bilinear skew-symmetric non-degenerate form on V, and Ω be the associated skew-symmetric $(2n \times 2n)$-matrix. Define the *symplectic group* as

$$\mathrm{Sp}(\omega) = \{A \in \mathrm{GL}(2n) : \omega(Av_1, Av_2) = \omega(v_1, v_2) \text{ for any } v_1, v_2 \in V\} =$$
$$= \{A \in \mathrm{GL}(2n) : A^T \Omega A = \Omega\}.$$

Again for the standard skew-symmetric form

$$\omega(v_1, v_2) = x_1 y_2 - x_2 y_1 + \cdots + x_{2n-1} y_{2n} - x_{2n} y_{2n-1}$$

we reserve the notation $\mathrm{Sp}(\omega) = \mathrm{Sp}(2n)$.

Example 1.1.6 (Finite groups)**.** Recall that any finite set X with n elements admits a canonical structure of an affine algebraic variety (over $\mathbb{K}$). This variety has n irreducible one-point components and the algebra of regular functions $\mathcal{O}(X)$ is the direct sum of n copies of the field $\mathbb{K}$: $\mathcal{O}(X) = \mathbb{K} \oplus \cdots \oplus \mathbb{K}$. In particular, any $\mathbb{K}$-valued function on X is regular, and any map $X \to Y$ to another affine variety Y is a morphism. This shows that any finite group G has a canonical structure of an affine algebraic group.

Example 1.1.7 (Additive and multiplicative groups)**.** The *additive group* G_a is the affine line $\mathbb{K}^1$ with group low $\mu(x, y) = x + y$ and $i(x) = -x$. The *multiplicative group* G_m is the affine open subset $\mathbb{K}^\times \subset \mathbb{K}$ with $\mu(x, y) = xy$, $i(x) = x^{-1}$. Clearly, they are commutative one-dimensional algebraic groups. The group G_m may be realized as $\mathrm{GL}(1)$, but for a matrix realization of G_a one needs 2×2-matrices:

$$\left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}, \ c \in \mathbb{K} \right\}.$$

Proposition 1.1.8. *Let G_1 and G_2 be affine algebraic groups. Then the direct product $G_1 \times G_2$ has a canonical structure of an affine algebraic group.*

Proof. If X_1, X_2, Y_1, Y_2 are affine varieties and $\phi_1 : X_1 \to Y_1$, $\phi_2 : X_2 \to Y_2$ are morphisms, then $\phi : X_1 \times X_2 \to Y_1 \times Y_2$, $\phi(x_1, x_2) = (\phi_1(x_1), \phi_2(x_2))$ is again a morphism. This shows that the multiplication map and the inverse map:

$$\mu : (G_1 \times G_2) \times (G_1 \times G_2) \to G_1 \times G_2, \quad \mu((g_1, g_2), (g_1', g_2')) = (g_1 g_1', g_2 g_2'),$$
$$i : G_1 \times G_2 \to G_1 \times G_2, \quad i(g_1, g_2) = (g_1^{-1}, g_2^{-1})$$

are morphisms. ∎

Proposition 1.1.8 allows us to construct many examples of algebraic groups. In particular, the direct product $T^k = G_m^k = \mathbb{K}^\times \times \cdots \times \mathbb{K}^\times$ (k times) is a commutative algebraic group called an *algebraic torus*. Note that, if t_i is the coordinate in the ith copy of $\mathbb{K}^\times$, then $\mathcal{O}(T^k) = \mathbb{K}[t_1, t_1^{-1}, \ldots, t_k, t_k^{-1}]$ is the algebra of Laurent polynomials.

We finish this subsection with some further examples of linear algebraic groups that will be used frequently in this course.

Example 1.1.9 (Some other matrix groups). Let $D(n)$, $B(n)$ and $U(n)$ be the subgroups of diagonal, upper triangular, and upper triangular unipotent matrices in $\mathrm{GL}(n)$ respectively:

$$D(n) = \left\{ \begin{pmatrix} * & 0 & \ldots & 0 \\ 0 & * & \ldots & 0 \\ \ldots & \ldots & \ldots & 0 \\ 0 & 0 & \ldots & * \end{pmatrix} \right\}, \ B(n) = \left\{ \begin{pmatrix} * & * & \ldots & * \\ 0 & * & \ldots & * \\ \ldots & \ldots & \ldots & * \\ 0 & 0 & \ldots & * \end{pmatrix} \right\}, \ U(n) = \left\{ \begin{pmatrix} 1 & * & \ldots & * \\ 0 & 1 & \ldots & * \\ \ldots & \ldots & \ldots & * \\ 0 & 0 & \ldots & 1 \end{pmatrix} \right\}.$$

Note that these subgroups are defined in $\mathrm{GL}(n)$ by equations of the form:

$$a_{ij} = \ 0 \text{ or } 1 \text{ for some } i, j,$$

so they are algebraic subgroups.

Exercises to subsection 1.1.

Exercise 1.1.10. Check that if $A, B \in \mathrm{Mat}(n \times n)$ and $v_1^T A v_2 = v_1^T B v_2$ for any $v_1, v_2 \in \mathbb{K}^n$, then $A = B$.

Exercise 1.1.11. Show that the group $\mathrm{O}(q)$ contains matrices A_1 and A_2 with $\det(A_1) = 1$ and $\det(A_2) = -1$.

Exercise 1.1.12. Check that $\mathrm{SO}(2)$ is commutative, but $\mathrm{O}(2)$ is not.

Exercise 1.1.13. Prove that for any bilinear symmetric non-degenerate form q on $V = \mathbb{K}^n$ the subgroup $\mathrm{O}(q)$ (resp. $\mathrm{SO}(q)$) is conjugate to $\mathrm{O}(n)$ (resp. to $\mathrm{SO}(n)$) in $\mathrm{GL}(n)$. Moreover, for any bilinear skew-symmetric non-degenerate form ω on $V = \mathbb{K}^{2n}$ the subgroup $\mathrm{Sp}(\omega)$ is conjugate to $\mathrm{Sp}(2n)$ in $\mathrm{GL}(2n)$.

Exercise 1.1.14 (*)**.** Prove that $\det(A) = 1$ for any $A \in \mathrm{Sp}(\omega)$.

Exercise 1.1.15 (*)**.** Show that $\mathrm{SL}(2) = \mathrm{Sp}(2)$, but $\mathrm{Sp}(2n)$ is a proper subgroup of $\mathrm{SL}(2n)$ for any $n > 1$.

Exercise 1.1.16. Show that $\mathrm{GL}(n, \mathbb{R})$ is not a closed subgroup in $\mathrm{GL}(n, \mathbb{C})$.

Exercise 1.1.17. Find the linear span of $\mathrm{SL}(n)$ in the vector space $\mathrm{Mat}(n \times n)$.

Exercise 1.1.18. Find the center of a) $\mathrm{GL}(n)$; b) $\mathrm{SL}(n)$; c) $B(n)$; d) $U(n)$.

Exercise 1.1.19. Calculate the dimension of a) $D(n)$; b) $B(n)$; c) $U(n)$.

Exercise 1.1.20. Describe elements of finite order in G_a and G_m.

Exercise 1.1.21. Do all elements of finite order form a subgroup in $\mathrm{GL}(n)$?

Exercise 1.1.22. Find a matrix realization for $G_m^k \times G_a^s$.

Exercise 1.1.23. Let A be a finite-dimensional $\mathbb{K}$-algebra. Prove that the automorphism group of A is an affine algebraic group.

Exercise 1.1.24 (*)**.** Let G be a group with a structure of an affine algebraic variety such that the multiplication map $\mu : G \times G \to G$ is a morphism. Prove that the inverse map $i : G \to G$ is a morphism.

Exercise 1.1.25 (**)**.** Is the group $\mathrm{SL}(2)$ isomorphic to $\mathbb{A}^3$ as an affine variety ?

1.2. **Connected components and homomorphisms.** Let G be an algebraic group. Our aim is to show that the group structure on G imposes strong restrictions on its geometry. The reason for this is "homogeneity" or "equal status of points". More precisely, with any element $g \in G$ one associates two mappings:

$$L_g : G \to G,\ L_g(x) = gx, \quad R_g : G \to G,\ R_g(x) = xg.$$

By definition of an algebraic group, these mappings are morphisms. Moreover, $L_{g^{-1}}$ (resp. $R_{g^{-1}}$) is inverse to L_g (resp. R_g), so L_g and R_g are automorphisms of the variety G. For any $x, y \in G$ the map $L_{yx^{-1}}$ sends x to y. In particular, the automorphism group of the variety G acts on G transitively.

Lemma 1.2.1. *Let G be an algebraic group. Then the variety G is smooth.*

Proof. Let $x \in G$ and y be a smooth point of G. Then the automorphism $L_{xy^{-1}}$ sends y to x, so x is also a smooth point. □

Lemma 1.2.2. *If H is an (abstract) subgroup of an algebraic group G, then its closure $\overline{H}$ is an algebraic subgroup of G.*

Proof. Since for any $h \in H$ the automorphism $L_h : G \to G$ preserves H, it also preserves $\overline{H}$. This implies $H\overline{H} \subseteq \overline{H}$. So for any $g \in \overline{H}$ one has $Hy \subset \overline{H}$. But $\overline{H}y$ coincides with the closure of the subset Hy and is contained in $\overline{H}$. This proves that $\overline{H}\overline{H} \subset \overline{H}$. The map i is an automorphism of G and, if it preserves a subset H, then it preserves its closure $\overline{H}$. So $\overline{H}$ is a subgroup in G, and by Proposition 1.1.4 this subgroup is algebraic. □

Recall that a subset Z of a variety X is called *locally closed* if Z is open in its closure $\overline{Z}$.

Lemma 1.2.3. *Let G be an algebraic group and $H \subset G$ a locally closed subset that is an (abstract) subgroup of G. Then H is closed.*

Proof. By Lemma 1.2.2, for any $g \in \overline{H}$ the coset gH is contained in $\overline{H}$. Moreover, gH is open in $\overline{H}$ as a translate of an open subset $H \subset \overline{H}$. But H is dense in $\overline{H}$, so $H \cap gH \neq \emptyset$. If $x \in H \cap gH$, then $g = xh^{-1}$ for some $h \in H$, thus $g \in H$. □

Theorem 1.2.4. *Let G be an algebraic group and $e \in G$ the unit. Then*

(i) *irreducible components of G coincide with connected components;*
(ii) *the connected component G^0 containing e is a normal subgroup of a finite index in G, and all other connected components are cosets of G by G^0.*

Proof. Let $G^0, \dots, G^k$ be the irreducible components of G. By definition, there exists $g \in G^0 \setminus (G^1 \cup \dots \cup G^k)$. The image of g under any automorphism of the variety G also belongs to a unique irreducible component of G. This shows that any element of G lies in a unique irreducible component, so $G^i \cap G^j = \emptyset$ for any $i \neq j$, and (i) is proved.

Since the variety $G^0 \times G^0$ is irreducible, so is its image $\mu(G^0 \times G^0)$. Then $\mu(G^0 \times G^0)$ is contained in an irreducible component of G. But $e \in \mu(G^0 \times G^0)$, and thus $\mu(G^0 \times G^0) = G^0$. The same arguments show that $i(G^0) = G^0$, so G^0 is a subgroup. Now consider the morphism $\phi : G^i \times G^0 \to G$, $\phi(g, h) = ghg^{-1}$. The variety $G^i \times G^0$ is irreducible, $\phi(g, e) = e$, thus $\phi(G^i \times G^0) \subseteq G^0$, or $ghg^{-1} \in G^0$ for any $g \in G, h \in G^0$, and G^0 is normal in G.

For any $g \in G^i$ the automorphism L_g maps G^0 isomorphically to some irreducible component of G. But $L_g(e) = g$, thus L_g maps G^0 to G^i. Moreover, the number of irreducible components of any variety is finite, so G^0 has a finite index in G, and the proof of (ii) is completed. $\square$

Lemma 1.2.5. *Let G be a connected algebraic group and $U, V \subset G$ be two non-empty open subsets. Put $UV := \{uv : u \in U, v \in V\}$. Then $G = UV$.*

Proof. Since the map i is an automorphism of the variety G, the subset $V^{-1} := \{g^{-1} : g \in V\}$ is open. By Theorem 1.2.4, the group G is irreducible. For any $g \in G$ the intersection of two open subsets gV^{-1} and U is non-empty, so there are $v \in V$, $u \in U$ such that $gv^{-1} = u$, or $g = uv$. $\square$

Proposition 1.2.6. *Let G be an algebraic group and $\{M_i \subset G : i \in I\}$ be a family of subsets such that*

1) *$e \in M_i$ for any $i \in I$;*
2) *M_i are irreducible (in the induced topology);*
3) *M_i contains an open subset of $\overline{M_i}$.*

Then the subgroup $H \subseteq G$ generated by the subsets M_i is closed and connected.

Proof. Consider $M_{i_1\dots i_k}^{\epsilon_1\dots\epsilon_k} := \mu^{\epsilon_1\dots\epsilon_k}(M_{i_1} \times \dots \times M_{i_k})$, where $\mu^{\epsilon_1\dots\epsilon_k}(g_1, \dots, g_k) := g_1^{\epsilon_1} \dots g_k^{\epsilon_k}$ with $\epsilon_i = \pm 1$. Clearly, $M_{i_1\dots i_k}^{\epsilon_1\dots\epsilon_k}$ satisfies 1). Since the image and the product of irreducible subsets are irreducible, we also have 2). Finally, take open subsets $U_i \subset M_i \subset \overline{M_i}$. By Theorem 3.0.24, the image of

$$\overline{M_{i_1}} \times \dots \times \overline{M_{i_k}} \to \overline{M_{i_1\dots i_k}^{\epsilon_1\dots\epsilon_k}}$$

contains an open subset of $\overline{M_{i_1\dots i_k}^{\epsilon_1\dots\epsilon_k}}$, and we get 3). Note that

$$H = \bigcup M_{i_1\dots i_k}^{\epsilon_1\dots\epsilon_k}$$

and

$$M_{i_1\dots i_k}^{\epsilon_1\dots\epsilon_k}, M_{j_1\dots j_s}^{\delta_1\dots\delta_s} \subseteq M_{i_1\dots i_k j_1\dots j_s}^{\epsilon_1\dots\epsilon_k\delta_1\dots\delta_s} = M_{i_1\dots i_k}^{\epsilon_1\dots\epsilon_k} M_{j_1\dots j_s}^{\delta_1\dots\delta_s}.$$

Set $M := M_{i_1\dots i_k}^{\epsilon_1\dots\epsilon_k}$ with the maximal $\dim \overline{M_{i_1\dots i_k}^{\epsilon_1\dots\epsilon_k}}$. Then any $M_{j_1\dots j_s}^{\delta_1\dots\delta_s}$ is contained in $\overline{M_{i_1\dots i_k j_1\dots j_s}^{\epsilon_1\dots\epsilon_k\delta_1\dots\delta_s}} = \overline{M}$ (Theorem 3.0.25), thus $\overline{H} = \overline{M}$. This shows that $\overline{H}$ is connected. By Lemma 1.2.2, $\overline{H}$ is a subgroup. Applying Lemma 1.2.5 to $\overline{H}$ and $U = V$ an open subset of M, we get $\overline{H} = UU \subseteq H$, so H is closed. $\square$

Corollary 1.2.7. *The groups $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ are connected.*

Proof. As we know from linear algebra, any non-degenerate matrix is a product of elementary matrices. Thus one may apply Proposition 1.2.6 to irreducible curves $M_{ij} = \{E + cE_{ij} : c \in \mathbb{K}\}$ $(i \neq j)$ and $M_{ii} = \{E + (\lambda - 1)E_{ii} : \lambda \in \mathbb{K}^\times\}$ which generate $\mathrm{GL}(n)$. In the case of $\mathrm{SL}(n)$ the curves M_{ij} $(i \neq j)$ are sufficient. $\square$

Remark 1.2.8. For $\mathrm{GL}(n)$, there is a simpler proof: it is irreducible as an open subset of $\mathrm{Mat}(n \times n)$.

Corollary 1.2.9. *Let G be a connected algebraic group. Then the commutant $[G, G]$ is a closed connected subgroup of G.*

Proof. Consider a morphism $\gamma : G \times G \to G$, $\gamma(x, y) = xyx^{-1}y^{-1}$ and apply Proposition 1.2.6 to $\{M_i\} = \{\gamma(G)\}$. $\square$

Now we come to the definition of a morphism in the category of algebraic groups.

Definition 1.2.10. Let G and F be algebraic groups. A morphism $\phi : G \to H$ is said to be a *homomorphism* if $\phi(g_1 g_2) = \phi(g_1)\phi(g_2)$, i.e. ϕ is a homomorphism of abstract groups.

Example 1.2.11. For any integral k the map $A \to \det(A)^k$ is a homomorphism from $\mathrm{GL}(n)$ to $\mathbb{K}^\times$.

On the other hand, the map $\mathbb{C} \to \mathbb{C}^\times$, $a \to e^a$ is a homomorphism of abstract groups, but not a morphism.

Theorem 1.2.12. *Let $\phi : G \to F$ be a homomorphism of algebraic groups. Then $\mathrm{Ker}(\phi) \subseteq G$ and $\mathrm{Im}(\phi) \subseteq F$ are closed subgroups. Moreover,*

$$\dim G = \dim \mathrm{Ker}(\phi) + \dim \mathrm{Im}(\phi).$$

Proof. Since $\mathrm{Ker}(\phi) = \phi^{-1}(e)$, it is a closed normal subgroup of G. Assume that G is connected. The image $\mathrm{Im}(\phi)$ is a subgroup of F, and, by Theorem 3.0.24, it contains an open subset U of $\overline{\mathrm{Im}(\phi)}$. By Lemmas 1.2.2 and 1.2.5, $\overline{\mathrm{Im}(\phi)} = UU = \mathrm{Im}(\phi)$.

If $G^0, G^1, \ldots, G^k$ are connected components of G and $g_1 \in G^1, \ldots, g_k \in G^k$, then

$$\mathrm{Im}(\phi) = \phi(G^0) \cup \phi(g_1)\phi(G^0) \cup \cdots \cup \phi(g_k)\phi(G^0)$$

is closed.

In order to prove the dimension formula, one may assume that $\phi(G) = F$. Any fiber of $\phi|_{G^0} : G^0 \to F^0$ is a coset of G^0 by $\mathrm{Ker}(\phi)_0 := \mathrm{Ker}(\phi) \cap G^0$. By Theorem 3.0.26, one has $\dim G^0 = \dim \mathrm{Ker}(\phi)_0 + \dim F^0$. Clearly, $\dim G = \dim G^0$ and $\dim F = \dim F^0$. We claim that $\dim \mathrm{Ker}(\phi) = \dim \mathrm{Ker}(\phi)_0$. Indeed, the connected component $\mathrm{Ker}(\phi)^0$ is contained in G^0, thus

$$\dim \mathrm{Ker}(\phi)^0 \leq \dim \mathrm{Ker}(\phi)_0 \leq \dim \mathrm{Ker}(\phi) = \dim \mathrm{Ker}(\phi)^0.$$

□

Definition 1.2.13. A homomorphism $\phi : G \to F$ is called an *isomorphism* if there is a homomorphism $\psi : F \to G$ such that $\phi \circ \psi = \mathrm{id}_F$ and $\psi \circ \phi = \mathrm{id}_G$.

Remark 1.2.14. If $\mathrm{char}\,\mathbb{K} = 0$, then by Theorem 3.0.27 any bijective homomorphism is an isomorphism. For $\mathrm{char}\,\mathbb{K} = p > 0$ this is not the case: one may consider the Frobenius homomorphism $\phi : G_a \to G_a$, $\phi(x) = x^p$.

Exercises to subsection 1.2.

Exercise 1.2.15. Construct two non-isomorphic structures of an algebraic group on the variety $\mathbb{A}^3$.

Exercise 1.2.16 (*). Show that the variety $\mathbb{A}^1 \setminus \{0,1\}$ does not admit a structure of an algebraic group.

Exercise 1.2.17. Show that SO(2) is isomorphic to G_m, and the second connected component of O(2) consists of elements of order two.

Exercise 1.2.18. Let H be a subgroup of a finite index in G. Prove that H contains G^0.

Exercise 1.2.19. Give an example of an algebraic group G such that the center $Z(G)$ is finite, but $Z(G^0)$ is infinite.

Exercise 1.2.20. By Theorem 1.2.4, for any algebraic group G the quotient group $F = G/G^0$ is finite. Give an example of G which is not isomorphic to a semidirect product of F and G^0.

Exercise 1.2.21. Let $\mathbb{K}$ be a field. Prove that the polynomial $\det(a_{ij})-1 \in \mathbb{K}[a_{11}, \ldots, a_{nn}]$ is irreducible.

Exercise 1.2.22. Calculate the commutant of a) $\mathrm{GL}(n)$; b) $\mathrm{SL}(n)$; c) $B(n)$; d) $U(n)$.

Exercise 1.2.23. Let $G \subset \mathrm{GL}(n)$ be a closed connected subgroup with a finite center $Z(G)$. Prove that $Z(G/Z(G)) = \{e\}$. It is true for a non-connected G ?

Exercise 1.2.24 (*). Let G be an algebraic group. Prove that $[G,G]$ is a closed subgroup of G.

Exercise 1.2.25 (*). Give an example of a homomorphism of Lie groups $\phi : G_1 \to G_2$ such that $\mathrm{Im}(\phi)$ is not a (closed) Lie subgroup of G_2.

1.3. **Actions and representations of algebraic groups.** Let X be an algebraic variety and G an algebraic group.

Definition 1.3.1. A morphism $\alpha : G \times X \to X$ is said to be an (algebraic) *action*, if it satisfies the following properties:

(i) $\alpha(e, x) = x$ for any $x \in X$;
(ii) $\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$ for any $g_1, g_2 \in G$, $x \in X$.

Further we abbreviate $\alpha(g, x)$ as $g \cdot x$. The fact that we have an action of G on X will be denoted as $G : X$, and a variety with a G-action will be called shortly a G-variety.

Example 1.3.2. There are three remarkable actions of a group G on itself. Namely, $\alpha_L(g, g_1) = gg_1$, $\alpha_R(g, g_1) = g_1 g^{-1}$, $\alpha_C(g, g_1) = gg_1g^{-1}$.

Definition 1.3.3. Assume that a group G acts on a variety X. For any $x \in X$ define its *orbit* $Gx := \{y \in X : y = g \cdot x \text{ for some } g \in G\}$ and its *stabilizer* $G_x := \{g \in G : g \cdot x = x\}$.

Theorem 1.3.4. *Let an algebraic group G act on a variety X. Then for any $x \in X$ the stabilizer G_x is a closed subgroup of G and the orbit Gx is a smooth locally closed subvariety of X. Moreover,*

$$\dim G = \dim \overline{Gx} + \dim G_x.$$

Proof. Consider a morphism

$$\tau : G \times X \to X \times X, \ \tau(g, x) = (x, g \cdot x)$$

and the embedding $r_x : G \to G \times X$, $r_x(g) = (g, x)$. Clearly, $G_x = r_x^{-1}(\tau^{-1}(x, x))$, thus G_x is closed.

For the orbit morphism $G^0 \to X$, $g \to g \cdot x$, the image G^0x contains an open subset U of $\overline{G^0x}$ (Theorem 3.0.24). Then g_iU is open in $g_i\overline{G^0x} = \overline{g_iG^0x} = \overline{G^ix}$. This implies that Gx contains an open subset $W = \cup_{i=0}^k g_iU$ of $\overline{Gx} = \cup_{i=0}^k \overline{G^ix}$ (here $g_0 = e$). But then $Gx = \cup_{g \in G} gW$ is open in $\overline{Gx}$. Smoothness of Gx follows from transitivity of the G-action (by automorphisms) on Gx.

The dimension of any fiber of the orbit morphism $G^0 \to G^0x$ equals $\dim(G^0)_x$, and Theorem 3.0.26 implies $\dim G^0 = \dim(G^0)_x + \dim \overline{G^0x}$. Finally, one can easily check that $\dim G = \dim G^0$, $\dim G_x = \dim(G^0)_x$, and $\dim \overline{Gx} = \dim \overline{G^0x}$. □

Example 1.3.5. The group $\mathrm{GL}(n)$ acts on the space of symmetric $n \times n$-matrices as $(A, S) \to ASA^T$. Non-degenerate symmetric matrices form an open orbit and the stabilizer of the unit matrix equals $\mathrm{O}(n)$. By Theorem 1.3.4, $\dim \mathrm{GL}(n) = \dim \mathrm{O}(n) + \frac{n(n+1)}{2}$, thus $\dim \mathrm{O}(n) = \frac{n(n-1)}{2}$.

Corollary 1.3.6. *For any action $G : X$ there is a closed G-orbit.*

Proof. The orbit Gx is open and dense in $\overline{Gx}$. Hence $\dim(\overline{Gx} \setminus Gx) < \dim \overline{Gx}$ and any orbit of the smallest dimension is closed in X. □

Corollary 1.3.7. *For any subset $A \subseteq G$ define the centralizer $Z_G(A) := \{g \in G : gx = xg \text{ for any } x \in A\}$. Then $Z_G(A)$ is a closed subgroup of G.*

Proof. For any $a \in A$ the centralizer $Z_G(a)$ is the stabilizer of $a \in G$ with respect to the action α_C (see Example 1.3.2). Thus $Z_G(a)$ is closed, and $Z_G(A) = \cap_{a \in A} Z_G(a)$. □

Proposition 1.3.8. *For any action $G : X$ and any integer k the subsets*

$$X_k := \{x \in X : \dim G_x \geq k\} \text{ and } X^k := \{x \in X : \dim Gx \leq k\}$$

are closed.

Proof. Since the sets X_k and X^k coincide for G and G^0, and G^0 preserves irreducible components of X, one may reduce the proof to the case when G in connected and X is irreducible. Again consider $\tau : G \times X \to X \times X$, $\tau(g, x) = (x, g \cdot x)$. By Theorem 3.0.26, there exist an integer m and a non-empty open subset $U \subseteq \overline{\tau(G \times X)}$ such that

(i) for any point $(x, y) \in U$ any component of the fiber $\tau^{-1}(x, y)$ has dimension m;
(ii) any component of any non-empty fiber $\tau^{-1}(x, y)$ has dimension $\geq m$.

Note that $\tau^{-1}(x, y) = \{(g, x) : g \cdot x = y\} \cong gG_x \cong G_x$. The image of the projection of U to the first factor contains a non-empty open subset (Theorem 3.0.24). This shows that $X_k = X$ for $k \leq m$ and there is an open subset $W \subset X$ which is contained in $X \setminus X_{m+1}$. Since X_{m+1} is a G-invariant subset, the open G-invariant subset GW also is contained in $X \setminus X_{m+1}$. Consider the decomposition into irreducible components: $X \setminus GW = Y(1) \cup \cdots \cup Y(s)$. Any $Y(i)$ is an irreducible G-variety with $\dim Y(i) < \dim X$. Arguing by induction on the dimension, one may assume that $Y(i)_k$ is closed for any k. On the other hand, for $k > m$ one has $X_k = Y(1)_k \cup \cdots \cup Y(s)_k$.

Finally, $X^k = X_{\dim G - k}$ (Theorem 1.3.4). □

Proposition 1.3.9. *Let an algebraic group G act on a variety X. Then the subset X^G of G-fixed points in closed in X.*

Proof. For any $g \in G$ define a morphism $\psi_g : X \to X \times X$, $\psi_g(x) = (x, g \cdot x)$. The set X^g of g-fixed points is the preimage of the diagonal in $X \times X$, thus is closed. The set X^G is the intersection of closed subsets: $X^G = \cap_{g \in G} X^g$. □

Definition 1.3.10. Let V be a finite-dimensional $\mathbb{K}$-vector space. A (rational) *representation* of an algebraic group G in the space V is a homomorphism

$$\rho : G \to \mathrm{GL}(V)$$

of algebraic groups. Here V is said to be a (finite-dimensional) rational G-module. A representation is called *faithful*, if $\mathrm{Ker}(\rho) = \{e\}$.

Definition 1.3.11. Any representation $\rho : G \to \mathrm{GL}(V)$ defines an action $\alpha_\rho : G \times V \to V$, $g \cdot v = \rho(g)v$. Such actions are called *linear*.

Remark 1.3.12. A rational representation of $\mathrm{GL}(n)$ in a space V is a homomorphism $\rho : \mathrm{GL}(n) \to \mathrm{GL}(V)$ such that the matrix entries of $\rho(A)$ are polynomials in $a_{ij}, \frac{1}{\det(A)}$. The presence of $\frac{1}{\det(A)}$ motivates the term "rational".

Remark 1.3.13. Standard constructions of representation theory (restrictions to invariant subspaces, quotient and dual representations, direct sums, tensor products, symmetric and exterior powers,...) allow to produce numerous rational G-modules from given ones.

Lemma 1.3.14. *Any rational representation $\rho : G \to GL(V)$ defines a natural algebraic action on the projective space $\mathbb{P}(V)$: $g \cdot [v] := [\rho(g)v]$.*

Proof. Let $x_1, \ldots, x_n$ be a coordinate system on V. The preimage of the open chart $U_i : x_i \neq 0$ on $\mathbb{P}(V)$ under the action map $\alpha : G \times \mathbb{P}(V) \to \mathbb{P}(V)$ is $U_i' = \{(g, [v]) : x_i(\rho(g)v) \neq 0\}$. The inverse image of the coordinate function $\frac{x_j}{x_i}$ on U_i is

$$\alpha^*(\frac{x_j}{x_i})(g, [v]) = \frac{x_j(\rho(g)v)}{x_i(\rho(g)v)} = \frac{\sum_k \rho_{kj}(g)x_j(v)}{\sum_k \rho_{ki}(g)x_i(v)},$$

which is a regular function on U_i'. This proves that α is a morphism. □

Now we came to actions of an algebraic group G on an affine variety X. There is a natural action of G on regular functions:

$$(g \cdot f)(x) := f(g^{-1} \cdot x), \quad f \in \mathcal{O}(X), \ x \in X, \ g \in G. \quad (*)$$

Lemma 1.3.15. *Formula $(*)$ gives a well-defined linear G-action on $\mathcal{O}(X)$ by automorphisms of a $\mathbb{K}$-algebra.*

Warning: in general, the $\mathbb{K}$-space $\mathcal{O}(X)$ has infinite dimension, so at the moment we speak only about linear actions of an abstract group.

Proof. For any $g \in G$ the function $g \cdot f$ is a composition $X \to X \to \mathbb{K}$ of the automorphism $L_{g^{-1}}$ and the function f, so $g \cdot f \in \mathcal{O}(X)$. Clearly, $e \cdot f = f$ and $(g_1 \cdot (g_2 \cdot f))(x) = (g_1 \cdot f)(g_2^{-1} \cdot x) = f(g_2^{-1} \cdot (g_1^{-1} \cdot x)) = f((g_1g_2)^{-1} \cdot x) = (g_1g_2 \cdot f)(x)$, thus we have an action of an abstract group. Moreover,

$$(g \cdot (\lambda_1 f_1 + \lambda_2 f_2))(x) = (\lambda_1 f_1 + \lambda_2 f_2)(g^{-1} \cdot x) =$$
$$= (\lambda_1 f_1)(g^{-1} \cdot x) + (\lambda_2 f_2)(g^{-1} \cdot x) = (\lambda_1(g \cdot f_1) + \lambda_2(g \cdot f_2))(x),$$

and $g \cdot (f_1 f_2) = (g \cdot f_1)(g \cdot f_2)$ by the same arguments. □

Theorem 1.3.16. *The algebra $\mathcal{O}(X)$ is a union $\bigcup_{i=1}^{\infty} W_i$ of a finite dimensional rational G-submodules $W_i \subset \mathcal{O}(X)$.*

Proof. The morphism $\alpha : G \times X \to X$ corresponds to a homomorphism of coordinate algebras:

$$\alpha^* : \mathcal{O}(X) \to \mathcal{O}(G \times X) \cong \mathcal{O}(G) \otimes_{\mathbb{K}} \mathcal{O}(X), \quad \alpha^*(F) = \sum_{i=1}^{m} f_i \otimes h_i$$

with

$$F(g \cdot x) = \sum_{i=1}^{m} f_i(g)h_i(x).$$

This proves that the linear span $\langle GF \rangle$ of the orbit GF is contained in $\langle h_1, \ldots, h_m \rangle$, so is finite-dimensional. One may assume that $h_1, \ldots, h_m$ form a basis in $\langle GF \rangle$. Since this subspace is G-invariant, the following formula holds for any $g \in G$:

$$(g \cdot h_i)(x) = h_i(g^{-1} \cdot x) = \sum_{j=1}^{m} p_{ij}(g^{-1})h_j(x).$$

for some $p_{ij} \in \mathcal{O}(G)$. There are regular functions $p'_{ij} \in \mathcal{O}(G)$, such that $p'_{ij}(g) = p_{ij}(g^{-1})$. Thus an element $g \in G$ acts in the space $\langle GF \rangle$ by the matrix $(p'_{ji}(g))$, and the G-module $\langle GF \rangle$ is rational. Finally, take a countable $\mathbb{K}$-basis $F_1, F_2, F_3, \ldots$ in $\mathcal{O}(X)$ and set $W_i = \sum_{j=1}^{i} \langle GF_j \rangle$. □

Definition 1.3.17. A G-module V is called *rational* if any element $v \in V$ is contained in a finite-dimensional rational submodule.

Corollary 1.3.18. *The G-module $\mathcal{O}(X)$ is rational.*

Let G be a group and X, Y be two G-sets. Recall that a map $\phi : X \to Y$ is called (G-)*equivariant* if $\phi(g \cdot x) = g \cdot \phi(x)$ for all $g \in G$ and $x \in X$. Moreover, if $g \cdot y = y$ for all $y \in Y$, then an equivariant map $\phi : X \to Y$ is called (G-)*invariant*.

Theorem 1.3.19. *Let G be an algebraic group acting on an affine variety X. Then there are a rational finite-dimensional G-module V and a closed G-equivariant embedding $I : X \hookrightarrow V$.*

Proof. Since the algebra $\mathcal{O}(X)$ is finitely generated, there is a finite-dimensional rational G-submodule $U \subset \mathcal{O}(X)$ which generates $\mathcal{O}(X)$. Let $V = U^*$ be the module dual to U. Define the map $I : X \to V$, $I(x) = l_x \in U^*$, where $l_x(u) := u(x)$. By definition,

$$l_x(\lambda_1 u_1 + \lambda_2 u_2) = (\lambda_1 u_1 + \lambda_2 u_2)(x) = \lambda_1 u_1(x) + \lambda_2 u_2(x) = \lambda_1 l_x(u_1) + \lambda_2 l_x(u_2),$$

and the function l_x is linear. In order to prove that I is a closed embedding it is sufficient to check that the dual homomorphism $I^* : \mathcal{O}(V) \to \mathcal{O}(X)$ is surjective. The regular functions on V form the symmetric algebra $\mathrm{Sym}(U)$ of the space U. By definition, for any $u \in U$ one has $I^*(u)(x) = u(I(x)) = u(l_x) = l_x(u) = u(x)$. This shows that $I^*(U)$ coincides with the subspace $U \subset \mathcal{O}(X)$, thus generates $\mathcal{O}(X)$, and the homomorphism $I^* : \mathcal{O}(V) = \mathrm{Sym}(U) \to \mathcal{O}(X)$ is surjective.

We need to check that I is equivariant. This is straightforward:

$$I(g \cdot x)(u) = l_{g \cdot x}(u) = u(g \cdot x) = (g^{-1} \cdot u)(x) = l_x(g^{-1} \cdot u) = (g \cdot l_x)(u) = (g \cdot I(x))(u).$$

□

Theorem 1.3.20. *Any affine algebraic group admits a faithful representation.*

Proof. Consider the action α_L of G on itself be left translations: $g \cdot g_1 = gg_1$. By Theorem 1.3.19, there are a rational finite-dimensional G-module V and an equivariant closed embedding $I : G \hookrightarrow V$. Since the action α_L is effective, the corresponding representation $\rho : G \to \mathrm{GL}(V)$ is faithful. □

Remark 1.3.21. It follows from the above theorem that any affine algebraic group may be realized as a closed subgroup in some $\mathrm{GL}(n)$. Indeed, the homomorphism $\rho : G \to \rho(G)$ is bijective and $\rho(G)$ is closed in $\mathrm{GL}(V)$ (Theorem 1.2.12). If $\mathrm{char}\,\mathbb{K} = 0$, then ρ automatically defines an isomorphism between G and $\rho(G)$. In general, one may argue as follows: Take the unit $e \in G$ and consider the orbit morphism $\phi : \rho(G) \to I(G)$, $\phi(\rho(g)) = \rho(g) \cdot I(e)$. Since $I : G \to I(G)$ is an isomorphism, the composition $I^{-1} \circ \phi \circ \rho : G \to G$ is the identity map. This shows that $I^{-1} \circ \phi$ is inverse to ρ.

Let an algebraic group G act on an algebraic variety X We are going to explain that the algebra of regular functions $\mathcal{O}(X)$ is a rational G-module (with respect to the action defined by (*)).

Proposition 1.3.22. *Let X and Y be algebraic varieties. Then*

$$\mathcal{O}(X \times Y) \cong \mathcal{O}(X) \otimes_{\mathbb{K}} \mathcal{O}(Y).$$

Proof. Assume that X is affine and $Y = \cup_{i=1}^{s} Y_i$ be an affine covering of Y. Take any $f \in \mathcal{O}(X \times Y)$. Let f_i be the restriction of f to $X \times Y_i$. Fix a $\mathbb{K}$-basis $g_1, g_2, \ldots$ of $\mathcal{O}(X)$. Then $f_i = \sum_k g_k \otimes h_{ik}$ with $h_{ik} \in \mathcal{O}(Y_i)$. Since $f_i - f_j$ is identically zero on the affine open subset $Y_i \cap Y_j$, and $h_{ik}, h_{jk} \in \mathcal{O}(Y_i \cap Y_j)$, one has $0 = \sum_k g_k \otimes (h_{ik} - h_{jk})$, thus h_{ik} and h_{jk} coincide on $Y_i \cap Y_j$. This shows that h_{ik} glue together to a regular function $h_k \in \mathcal{O}(Y)$, and $f = \sum_k g_k \otimes h_k$. So the natural homomorphism $\mathcal{O}(X) \times_{\mathbb{K}} \mathcal{O}(Y) \to \mathcal{O}(X \times Y)$ is surjective. If some $\sum_k g_k \otimes h_k$ maps to zero, then we may restrict this equality to any affine chart $X \times Y_i$ and get $h_k = 0$ for all k.

For an arbitrary variety X, fix an affine covering $X = \cup_i X_i$. Taking any $f \in \mathcal{O}(X \times Y)$, restricting it to all $X_i \times Y$, and repeating the above arguments, we get the statement.

Now the proof of Theorem 1.3.16 works also for:

Theorem 1.3.23. *Let G be an algebraic group. For any G-variety X the algebra $\mathcal{O}(X)$ is a rational G-module.*

Exercises to subsection 1.3.

Exercise 1.3.24. Prove that for any $A \in \mathrm{GL}(n)$ the centralizer $Z_{\mathrm{GL}(n)}(A)$ is connected. Is it true for SL(2) ?

Exercise 1.3.25. For any closed subset $A \subseteq G$ define the normalizer $N_G(A) = \{g \in G : gxg^{-1} \in A \text{ for any } x \in A\}$. Check that $N_G(A)$ is a closed subgroup of G.

Exercise 1.3.26. Assume that there is an algebraic action of G on a vector space V. Show that this action is linear (in the sense of Definition 1.3.11) if and only if

$$g \cdot (\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 (g \cdot v_1) + \lambda_2 (g \cdot v_2) \text{ for any } g \in G,\ \lambda_1, \lambda_2 \in \mathbb{K},\ v_1, v_2 \in V.$$

Exercise 1.3.27. Describe orbits of the tautological linear actions: (a) $\mathrm{SL}(n) : \mathbb{K}^n$; (b) $D(n) : \mathbb{K}^n$; (c) $B(n) : \mathbb{K}^n$; (d) $U(n) : \mathbb{K}^n$; (e) $\mathrm{O}(n) : \mathbb{K}^n$; (f) $\mathrm{SO}(n) : \mathbb{K}^n$; (g) $\mathrm{Sp}(2n) : \mathbb{K}^{2n}$.

Exercise 1.3.28. Following Example 1.3.5, calculate $\dim \mathrm{Sp}(2n)$.

Exercise 1.3.29 (*). Prove that the groups $\mathrm{SO}(n)$ and $\mathrm{Sp}(2n)$ are connected, and $\mathrm{O}(n)$ consists of two connected components.

Exercise 1.3.30 (*). For any pair of groups listed below construct explicitly a surjective two-sheeted homomorphism:

(a) $\mathrm{SL}(2) \to \mathrm{SO}(3)$;
(b) $\mathrm{SL}(2) \times \mathrm{SL}(2) \to \mathrm{SO}(4)$;
(c) $\mathrm{Sp}(4) \to \mathrm{SO}(5)$;
(d) $\mathrm{SL}(4) \to \mathrm{SO}(6)$.

Exercise 1.3.31. Let $\mathbb{K}$ be a field. Prove that the polynomial $\det(a_{ij}) \in \mathbb{K}[a_{11}, \ldots, a_{nn}]$ is irreducible.

Exercise 1.3.32. Let V be a finite-dimensional rational G-module. Prove that the natural G-action on the variety $\mathcal{F}(V)$ of complete flags in V is algebraic.

Exercise 1.3.33 (*). Define an algebraic action of an algebraic group G on a prevariety X. Show that in this case the subset X^G need not be closed in X.

Exercise 1.3.34. Give an example of a non-linear algebraic action of the group G_a on $\mathbb{K}^n$.

Exercise 1.3.35. Let G be a finite group. Prove that any G-module is rational.

Exercise 1.3.36. Assume that an algebraic group G acts on an irreducible variety X. Then the formula $(g \cdot f)(x) := f(g^{-1} \cdot x)$ defines a structure of a G-module on the field of rational functions $\mathbb{K}(X)$. Give an example where this module is not rational.

Exercise 1.3.37 (*). Let G be an algebraic group acting of a quasiaffine variety Y. Prove that there are an affine G-variety X and an open equivariant embedding $J : Y \hookrightarrow X$.

1.4. **Homogeneous spaces.** Let G be an affine algebraic group and H a closed subgroup of G. The set of left cosets G/H admits a natural transitive G-action: $g \cdot g_1 H = gg_1 H$. The aim of this section is to introduce a structure of an algebraic variety on G/H such that the action above becomes algebraic. This structure is based on the following result.

Theorem 1.4.1 (Chevalley's Theorem (1951))**.** *Let G be an affine algebraic group and H a closed subgroup of G.*

(1) *There are a representation $\rho : G \to \mathrm{GL}(V)$ and a vector $v \in V$ such that*
$$H = \{g \in G : \rho(g)v \in \langle v \rangle\}.$$
(2) *If the subgroup H is normal, then there is a representation $\rho' : G \to \mathrm{GL}(V')$ such that $H = \mathrm{Ker}(\rho')$.*

Proof. Denote by $\mathbb{I}(H)$ the ideal of all functions $f \in \mathcal{O}(G)$ that are zero on the (closed) subvariety $H \subset G$. As any ideal of $\mathcal{O}(G)$, $\mathbb{I}(H)$ is finitely generated: $\mathbb{I}(H) = (f_1, \ldots, f_s)$. Consider the action $G : G$ by left translations. Clearly,
$$g \in H \;\Leftrightarrow\; gH = H \;\Leftrightarrow g \cdot \mathbb{I}(H) = \mathbb{I}(H).$$
Fix a finite-dimensional (rational) G-submodule $W \subset \mathcal{O}(X)$ containing $f_1, \ldots, f_s$ and set $U = W \cap \mathbb{I}(H)$, $\dim U = k$. We claim that $g \in H$ if and only if $g \cdot U = U$. Indeed, the subspace U contains $f_1, \ldots, f_s$ and is contained in $\mathbb{I}(H)$, thus the set of common zeroes of functions from U is H, and $g \cdot U = U \Rightarrow gH = H$. Conversely, $gH = H$ implies $g \in H$.

Let us recall a lemma from linear algebra (see Exercise 1.4.9).

Lemma 1.4.2. *Assume that A is a linear operator on a vector space W and U is a k-dimensional subspace of W. The operator A acts naturally on the kth exterior power $\bigwedge^k W$ of W, and A preserves the subspace $U \subset W$ if and only if it preserves the line $\bigwedge^k U \subset \bigwedge^k W$.*

Let us fix a basis $u_1, \ldots, u_k$ of the subspace U. In order to prove (1), one should set $V = \bigwedge^k W$ and $v = u_1 \wedge \cdots \wedge u_k$.

Now we came to (2). By the above construction, the vector v in an eigenvector for H, and there exists a homomorphism $\chi_0 : H \to \mathbb{K}^\times$ such that $h \cdot v = \chi_0(h)v$. Let us consider all homomorphisms $\chi_i : H \to \mathbb{K}^\times$ such that there is a non-zero $v_i \in V$ with $h \cdot v_i = \chi_i(h)v_i$ for any $h \in H$. The subspaces
$$V_i = \{w \in V : h \cdot w = \chi_i(h)w \text{ for any } h \in H\}$$
form a direct sum in V. Denote this sum as:
$$\tilde{V} = \bigoplus_{i=0}^{k} V_i.$$
If $w \in V_i$, then
$$h \cdot (g \cdot w) = g \cdot ((g^{-1}hg) \cdot w) = g \cdot \chi_i(g^{-1}hg)w = \chi_i(g^{-1}hg)(g \cdot w),$$
thus $g \cdot w$ belongs to some V_j. This implies that $\tilde{V}$ is G-invariant. Then G acts naturally on the space $L(\tilde{V})$ of linear operators on $\tilde{V}$: if $\tilde{\rho} : G \to \mathrm{GL}(\tilde{V})$ is our representation and $C \in L(\tilde{V})$, then $g \cdot C := \tilde{\rho}(g)C\tilde{\rho}(g)^{-1}$. Since the group G permutes the summands V_i in $\tilde{V}$, the subspace $L_0 := \oplus_{i=0}^{k} L(V_i)$ of $L(\tilde{V})$ is G-invariant. Let us consider the representation $\rho' : G \to \mathrm{GL}(L_0)$. We claim that $\mathrm{Ker}(\rho') = H$. Indeed, elements of H are sent to operators that are scalar on any V_i. These are precisely the operators that commutes with each element of $\oplus_{i=0}^{k} L(V_i)$.

On the other hand, if an element $g \in G$ is sent to an operator which is scalar on any V_i, then, in particular, g preserves the line $\langle v \rangle$, and, by assumption, belongs to H. □

Let us suppose for the rest of this section that $\operatorname{char} \mathbb{K} = 0$.

Corollary 1.4.3. *The set G/H of left cosets admits a unique structure of a quasiprojective algebraic variety such that the natural action $G : G/H$ is algebraic.*

Proof. Let us firstly introduce an algebraic structure on G/H. Consider the projective space $\mathbb{P}(V)$ and a point $[v] \in \mathbb{P}(V)$ corresponding to a pair (V, v) constructed in Theorem 1.4.1 (1). (Since the case $G = H$ is trivial, we may assume that $v \neq 0$.) By Lemma 1.3.14, the induced action $G : \mathbb{P}(V)$ is algebraic, and the stabilizer of $[v]$ coincides with H. The orbit $G[v]$ is open in its closure (Theorem 1.3.4) and thus has a structure of a quasiprojective variety with an algebraic transitive G-action. The orbit map $G \to \mathbb{P}(V)$, $g \to g \cdot [v]$ defines a bijection $G/H \to G[v]$, and induces a structure of a quasiprojective variety on G/H such that the natural action $G : G/H$ is algebraic.

Now assume that there exists another algebraic structure $(G/H)_1$ on G/H. By assumption, there is a commutative diagram

$$\begin{array}{ccc} & G & \\ {}^{\phi}\swarrow & & \searrow^{\psi} \\ G/H & \xrightarrow{\ \tau\ } & (G/H)_1 \end{array},$$

where ϕ and ψ are orbit morphisms and τ is a set-theoretical bijection. Suppose that G is connected. By Theorem 3.0.28, the map τ is a rational morphism, thus there exists an open subset $W \subseteq G/H$ with $\tau : W \to \tau(W)$ being isomorphism. Since G acts transitively on G/H and all maps are G-equivariant, we may produce a covering of G/H (by shifts of W) such that τ extends as an isomorphism to any element of this covering: $\tau(g \cdot w) = g \cdot \tau(g^{-1} \cdot (g \cdot w))$. This shows that G/H and $(G/H)_1$ are isomorphic.

For non-connected G, one again may extend an isomorphism $G^0/(G^0 \cap H) \to (G^0/(G^0 \cap H))_1$ to $G/H \to (G/H)_1$ by translations. □

Corollary 1.4.4. *Let H be a closed normal subgroup of G. Then the quotient group G/H has a unique structure of an affine algebraic group such that the projection $G \to G/H$ is an algebraic homomorphism.*

Proof. Consider the representation $\rho' : G \to \mathrm{GL}(V')$ from Theorem 1.4.1 (2). It defines a bijective homomorphism $G/H \to \rho'(G)$. Since $\rho'(G)$ is closed in $\mathrm{GL}(V)$ (Theorem 1.2.12), it has a structure of an algebraic group, and induces a structure of an algebraic group on G/H.

If G/H admits a structure of an algebraic group such that the projection $G \to G/H$ is algebraic, then we have a sequence of morphisms:

$$G \times G/H \to G/H \times G/H \to G/H, \ (g, g_1H) \to (gH, g_1H) \to gg_1H,$$

which shows that the left action $G : G/H$ is algebraic. Thus the uniqueness follows from Corollary 1.4.3. □

Proposition 1.4.5. *Let G be an algebraic group and H a closed subgroup. Then the projection morphism $p : G \to G/H$ is open.*

Proof. All fibers of the morphism $p: G \to G/H$ are isomorphic to H; in particular, all their components have the same dimension. The variety G/H is smooth (e.g, normal). Thus for connected G Proposition 1.4.5 follows from Theorem 3.0.32. For arbitrary G, one should apply Theorem 3.0.32 to the restriction of p to a connected component of G, where all fibers are isomorphic to $H \cap G^0$. □

Proposition 1.4.6. *Let G be an algebraic group and H a closed subgroup. Then*

$$\mathcal{O}(G/H) = \mathcal{O}(G)^H := \{f \in \mathcal{O}(G) : f(gh) = f(g) \text{ for any } g \in G,\ \ h \in H\}.$$

Proof. The dominant morphism $p : G \to G/H$ corresponds to an embedding $p^* : \mathcal{O}(G/H) \subseteq \mathcal{O}(G)^H$. On the other hand, any function $f \in \mathcal{O}(G)^H$ defines a commutative diagram:

$$\begin{array}{ccc} & G & \\ {}^{p}\swarrow & & \searrow^{f} \\ G/H & \xrightarrow{\ \tau\ } & \mathbb{A}^1. \end{array}$$

If G is connected, then $\tau \in \mathcal{O}(G/H)$ (Corollary 3.0.29). For arbitrary G, these arguments prove that τ is regular on any irreducible (=connected) component of G/H, thus $f \in \mathcal{O}(G/H)$. □

We finish this section with some examples of homogeneous spaces.

Example 1.4.7 (Grassmannians and Flag Varieties)**.** The group $\mathrm{GL}(n)$ acts transitively on the set of k-dimensional subspaces of $V = \mathbb{K}^n$ $(1 \le k \le n)$. The stabilizer of the standard k-subspace $\langle e_1, \ldots, e_k\rangle$ is

$$P(k,n) := \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} :\ A \in \mathrm{GL}(k),\ C \in \mathrm{GL}(n-k),\ B \in \mathrm{Mat}(k \times (n-k)) \right\}.$$

So the homogeneous space $GL(n)/P(k,n)$ is isomorphic to the Grassmannian $\mathrm{Gr}(k,n)$.

Now consider the subgroup $B(n) \subset \mathrm{GL}(n)$. It is the stabilizer of the standard complete flag

$$\{0\} \subset \langle e_1\rangle \subset \langle e_1, e_2\rangle \subset \cdots \subset \langle e_1, \ldots, e_n\rangle = \mathbb{K}^n$$

in $\mathbb{K}^n$. Since $\mathrm{GL}(n)$ acts transitively of the set of complete flags, we again have that $\mathrm{GL}(n)/B(n)$ is isomorphic to the flag variety $\mathcal{F}(n)$. It is well known that the varieties $\mathrm{Gr}(k,n)$ and $\mathcal{F}(n)$ are projective. By Theorem 1.3.4, we have:

$$\dim \mathrm{Gr}(k,n) = n^2 - (k^2 + (n-k)^2 + k(n-k)) = k(n-k),$$

$$\dim \mathcal{F}(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Example 1.4.8 (Homogeneous spaces for $G = \mathrm{SL}(2)$)**.** (1) Take $G = SL(2)$ and $H = B := \{A \in B(2) : \det(A) = 1\}$. In order to apply Chevalley's Theorem, consider the tautological $\mathrm{SL}(2)$-module $V = \mathbb{K}^2$ and the first standard vector $e_1 \in V$. Clearly, $B = \{A \in \mathrm{SL}(2) : A \cdot e_1 \in \langle e_1\rangle\}$. Since $\mathrm{SL}(2)$ acts transitively on the set of lines in V, the homogeneous space $\mathrm{SL}(2)/B$ is isomorphic to the projective line $\mathbb{P}^1$.

(2) Take $G = \mathrm{SL}(2)$ and $H = U := U(2)$. Again consider $V = \mathbb{K}^2$ and $v = e_1$, and note that $U = \{A \in \mathrm{SL}(2) : A \cdot e_1 = e_1\}$. Thus, $\mathrm{SL}(2)/U$ is isomorphism to the orbit of e_1 in V. This is a quasi-affine (non-affine !) variety $\mathbb{K}^2 \setminus \{0\}$.

(3) Finally, take $G = \mathrm{SL}(2)$ and $H = T := \{A \in D(2) : \det(A) = 1\}$. Let V be a three-dimensional space of 2×2-matrices with trace zero, where $\mathrm{SL}(2)$ acts by conjugation: $(A, C) \to ACA^{-1}$. Set

$$v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The stabilizer of v coincides with T, and the orbit Gv consists of matrices with eigenvalues 1 and -1. This orbit is defined in V by the equation $\det(C) = -1$. Thus $\mathrm{SL}(2)/T$ is an affine quadric in $\mathbb{A}^3$.

Exercises to subsection 1.4.

Exercise 1.4.9. Prove Lemma 1.4.2.

Exercise 1.4.10. Find a faithful representation of the group $\mathrm{GL}(n)/Z(\mathrm{GL}(n))$.

Exercise 1.4.11. Prove that any irreducible component of G/H is also a connected component, and G^0 acts transitive on any such component.

Moreover, G/H is connected if and only if H intersects all connected components of G, or, equivalently, $G^0 H = G$.

Exercise 1.4.12 (*)**.** Prove that a homogeneous space G/H is quasi-affine if and only if there are a rational finite-dimensional G-module V and a vector $v \in V$ such that

$$H = \{g \in G : g \cdot v = v\}.$$

Exercise 1.4.13. Prove that a homogeneous space G/H is affine if and only if there are a rational finite-dimensional G-module V and a vector $v \in V$ such that

$$H = \{g \in G : g \cdot v = v\}$$

and the G-orbit Gv is closed in V.

Exercise 1.4.14. Assume that G/H is affine and there are a rational finite-dimensional G-module V and a vector $v \in V$ such that $H = G_v$. Is it true that the orbit Gv is closed in V ?

Exercise 1.4.15. Give an example of a homogeneous space G/H such that $\mathcal{O}(G/H) = \mathbb{K}$, but G/H is not projective.

Exercise 1.4.16. Describe the variety $\mathrm{SL}(3)/U(3)$.

Exercise 1.4.17. Let G be an algebraic group and $H_1 \subseteq H_2 \subseteq G$ closed subgroups. Show that the map $G/H_1 \to G/H_2$, $gH_1 \to gH_2$ is a morphism.

Exercise 1.4.18 (*)**.** Give an example of an algebraic group G and a closed subgroup H such that the fibering $p : G \to G/H$ is not locally trivial.

1.5. **The tangent algebra.** From now on we shall assume that $\operatorname{char}\mathbb{K} = 0$.

Consider the group $G = \mathrm{GL}(n)$. Since $\mathrm{GL}(n) \subset \mathrm{Mat}(n \times n)$ is an open subset, the tangent space $T_E(G)$ at the unit matrix E may be identified with $\mathrm{Mat}(n\times n)$. One may define a bilinear operation in this space: $[A, B] = AB - BA$. It is easy to check that $[\cdot,\cdot]$ possesses the following properties:

$[A, B] = -[B, A]$; $[A, [B, C]]+[B, [C, A]]+[C, [A, B]] = 0$ for any $A, B, C \in \mathrm{Mat}(n\times n)$.

Definition 1.5.1. A $\mathbb{K}$-vector space $\mathfrak{g}$ with a bilinear operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called a *Lie algebra*, if the operation $[\cdot,\cdot]$ satisfies the following conditions:

(i) (*Antisymmetry*) $[x, y] = -[y, x]$ for any $x, y \in \mathfrak{g}$;
(ii) (*The Jacobi Identity*) $[x, [y, z]]+[y, [z, x]]+[z, [x, y]] = 0$ for any $x, y, z \in \mathfrak{g}$.

Remark 1.5.2. By analogy with the Leibniz rule $(fg)' = f'g + fg'$, the Jacobi Identity for Lie algebras may be rewritten as:

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]].$$

The space $\mathrm{Mat}(n\times n)$ with the operation $[A, B] = AB - BA$ is an example of a Lie algebra. The standard notation for this Lie algebra is $\mathfrak{gl}(n)$.

A natural question arises: Why do we define an operation in $T_E(G)$ as $AB - BA$, but not via some other formula ? Below come some motivations for this definition.

(1) Let $A \in \mathrm{GL}(n)$ and $L_A : \mathrm{Mat}(n \times n) \to \mathrm{Mat}(n \times n)$, $L_A(B) = AB$ (resp. $R_A(B) = BA$). Then L_A (resp. R_A) is a linear operator and its differential $dL_A : T_E\mathrm{GL}(n) \to T_E\mathrm{GL}(n)$ is given by $dL_A(X) = AX$ (resp. $dR_A(X) = XA$).
(2) Consider the inverse map $i : \mathrm{GL}(n) \to \mathrm{GL}(n)$, $i(B) = B^{-1}$. The differential $di : T_E\mathrm{GL}(n) \to T_E\mathrm{GL}(n)$ may be calculated using the explicit formula for inverse matrix (Exercise 1.5.23). But now we prefer another arguments. Any vector $X \in T_E\mathrm{GL}(n)$ is a tangent vector to a smooth curve $\Gamma : \mathbb{K} \to \mathrm{GL}(n)$, $\Gamma(0) = E$, i.e., $\frac{d}{dt}|_{t=0}\, \Gamma(t) = X$. The image $di(X)$ is the tangent vector to the curve $i(\Gamma(t)) = \Gamma(t)^{-1}$. One may differentiate the identity $\Gamma(t)\Gamma(t)^{-1} = E$ at $t = 0$ and obtain $X\Gamma(0)^{-1} + \Gamma(0)(di(X)) = 0$, or $di(X) = -X$.
(3) Consider an inner automorphism $a_A : \mathrm{GL}(n) \to \mathrm{GL}(n)$, $a_A(B) = ABA^{-1}$. Here the differential $da_A : T_E\mathrm{GL}(n) \to T_E\mathrm{GL}(n)$ is given as $da_A(X) = AXA^{-1}$, because a_A is the composition of L_A and $R_{A^{-1}}$. We have an (algebraic) representation:
$$\mathrm{Ad} : \mathrm{GL}(n) \to \mathrm{GL}(\mathrm{Mat}(n \times n)), \quad \mathrm{Ad}(A)(X) = AXA^{-1}.$$
(4) Let us calculate the differential
$$\mathrm{ad} := d_E\mathrm{Ad} : \mathrm{Mat}(n \times n) \to \mathrm{Mat}(\mathrm{Mat}(n \times n)).$$
Take again a curve $\Gamma(t)$ with $\Gamma(0) = E$ and $\frac{d}{dt}|_{t=0}\, \Gamma(t) = X$. Then for any $Y \in \mathrm{Mat}(n \times n)$:
$$\frac{d}{dt}|_{t=0}\, \mathrm{Ad}(\Gamma(t))(Y) = \frac{d}{dt}|_{t=0}\, \Gamma(t)Y\Gamma(t)^{-1} =$$
$$= XY\Gamma(0)^{-1} + \Gamma(0)Y(-X) = XY - YX = [X, Y].$$
This proves that $\mathrm{ad}(X)(Y) = [X, Y]$.

Definition 1.5.3. A *representation* of a Lie algebra $\mathfrak{g}$ is a linear map τ from $\mathfrak{g}$ to some $\mathfrak{gl}(m)$ such that

$$\tau([x, y]) = \tau(x)\tau(y) - \tau(y)\tau(x) \text{ for all } x, y \in \mathfrak{g}.$$

For any finite-dimensional Lie algebra $\mathfrak{g}$ define a linear map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$, $\mathrm{ad}(x)(y) = [x, y]$.

Lemma 1.5.4. *The map* $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ *is a representation of the Lie algebra* $\mathfrak{g}$.

Proof. The statement follows from the Jacobi identity:

$$\mathrm{ad}([x,y])(z) = [[x,y],z] = [[x,z],y] + [x,[y,z]] = [x,[y,z]] - [y,[x,z]] =$$
$$= (\mathrm{ad}(x)\mathrm{ad}(y) - \mathrm{ad}(y)\mathrm{ad}(x))(z).$$

□

Proposition 1.5.5. *Let* G *be a closed subgroup of* $\mathrm{GL}(n)$.

(i) *The subspace* $\mathfrak{g} := T_E G \subseteq T_E \mathrm{GL}(n)$ *is a Lie subalgebra of* $\mathfrak{gl}(n)$.
(ii) *The structure of the Lie algebra on* $\mathfrak{g} = T_e G$ *does not depend on a (closed) embedding* $G \subseteq \mathrm{GL}(n)$.

Proof. Since for any $g \in G$ the inner automorphism a_g (of $\mathrm{GL}(n)$) maps G to G, the tangent space $\mathfrak{g} = T_E G$ is an invariant subspace for the representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{gl}(n))$. So the same holds for its differential $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{gl}(n))$, and for any $X, Y \in \mathfrak{g}$ one has $\mathrm{ad}(X)(Y) = [X, Y] \in \mathfrak{g}$. This proves (i).

Further, the maps $a_g : G \to G$, $\mathrm{Ad}(g) : \mathfrak{g} \to \mathfrak{g}$ and $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ are defined in the internal terms of the group G, thus the Lie bracket $[X, Y] := \mathrm{ad}(X)(Y)$ depends only on G itself. □

Since any algebraic group G may be realized as a closed subgroup of some $\mathrm{GL}(n)$ (Theorem 1.3.20), the tangent space $T_e G$ possesses a (canonical) structure of Lie algebra.

Definition 1.5.6. The representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ (resp. $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$) is called the *adjoint* representation of an algebraic group G (resp. of a Lie algebra $\mathfrak{g}$).

More generally, if $H \subseteq G$ is a closed subgroup (resp. $\mathfrak{h} \subseteq \mathfrak{g}$ is a Lie subalgebra) and Ad (resp. ad) is the adjoint representation of G (resp. of $\mathfrak{g}$), then the restriction $\mathrm{Ad}\,|_H$ (resp. $\mathrm{ad}\,|_{\mathfrak{h}}$) defines a linear action $H : \mathfrak{g}$ (resp. $\mathfrak{h} : \mathfrak{g}$). We shall call this action an *adjoint* action of H (resp. of $\mathfrak{h}$).

Remark 1.5.7. Recall that a Lie algebra $\mathfrak{g}$ is *commutative* if $[x, y] = 0$ for any $x, y \in \mathfrak{g}$. If an algebraic group G is commutative, then $a_g = \mathrm{id}$ for all $g \in G \Rightarrow \mathrm{Ad}(g) = E$ for all $g \in G \Rightarrow \mathrm{ad}(x) = 0$ for all $x \in \mathfrak{g}$, thus $\mathrm{Lie}(G)$ is a commutative Lie algebra.

Example 1.5.8. Consider $G = \mathrm{SL}(n)$. In order to find the tangent algebra, one should calculate the differential of the map $\mathrm{GL}(n) \to \mathbb{K}^\times$, $A \to \det(A)$ at the unit. Direct calculations (see Exercise 1.5.22) shows that $d_E(\det)(X) = \mathrm{tr}(X)$, thus the tangent algebra

$$\mathrm{Lie}(\mathrm{SL}(n)) := \mathfrak{sl}(n) = \{X \in \mathfrak{gl}(n) : tr(X) = 0\}.$$

Now take $G = \mathrm{SO}(n)$. For any smooth curve $\Gamma : \mathbb{K} \to \mathrm{SO}(n)$, $\Gamma(0) = E$, we need to find the tangent vector $X = \frac{d}{dt}\,|_{t=0}\, \Gamma(t)$. Differentiating the identity $\Gamma(t)^T \Gamma(t) = E$, we get $X^T + X = 0$. This implies that $\mathrm{Lie}(\mathrm{SO}(n)) := \mathfrak{so}(n)$ is contained in the space of skew-symmetric matrices. On the other hand, in the system $A^T A = E$ of n^2 defining equations for $\mathrm{O}(n)$ only $\frac{n(n+1)}{2}$ are pairwise different, and $\dim \mathfrak{so}(n) = \dim \mathrm{SO}(n) (= \dim \mathrm{O}(n)) \geq n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$ (cf. Example 1.3.5). Hence the algebra $\mathfrak{so}(n)$ is the algebra of skew-symmetric $n \times n$-matrices.

Below for algebraic groups $G, H, \ldots$ we denote their tangent algebras as $\mathrm{Lie}(G) = \mathfrak{g}, \mathrm{Lie}(H) = \mathfrak{h}, \ldots$.

Definition 1.5.9. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras. A linear map $\sigma : \mathfrak{g} \to \mathfrak{h}$ is said to be a homomorphism (of Lie algebras) if

$$\sigma([x, y]) = [\sigma(x), \sigma(y)] \text{ for all } x, y \in \mathfrak{g}.$$

Theorem 1.5.10. *Let G and H be algebraic groups and $\phi : G \to H$ a homomorphism. Then its differential $d\phi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras.*

Proof. Let $L(\mathfrak{g}, \mathfrak{h})$ be the space of linear maps from $\mathfrak{g}$ to $\mathfrak{h}$. Consider the map

$$\Phi : G \to L(\mathfrak{g}, \mathfrak{h}),\ \ \Phi(g)(Y) = (d\phi \circ \mathrm{Ad}(g))(Y) \text{ for all } Y \in \mathfrak{g}.$$

The statement of the theorem follows from the calculation of $d\Phi : \mathfrak{g} \to L(\mathfrak{g}, \mathfrak{h})$ in two ways:

$$\begin{array}{ccc} G & \xrightarrow{\ \phi\ } & H \\ {\scriptstyle a_g}\downarrow & & \downarrow{\scriptstyle a_{\phi(g)}} \\ G & \xrightarrow{\ \phi\ } & H \end{array} \qquad\qquad \begin{array}{ccc} \mathfrak{g} & \xrightarrow{\ d\phi\ } & \mathfrak{h} \\ {\scriptstyle \mathrm{Ad}(g)}\downarrow & \searrow & \downarrow{\scriptstyle \mathrm{Ad}(\phi(g))} \\ \mathfrak{g} & \xrightarrow{\ d\phi\ } & \mathfrak{h}. \end{array}$$

Namely, take a smooth curve $\Gamma(t)$, $\Gamma(0) = E$ in G with $\frac{d}{dt} |_{t=0} \Gamma(t) = X$. Then

$$\Phi(\Gamma(t))(Y) = (d\phi \circ \mathrm{Ad}(\Gamma(t)))(Y) \ \Rightarrow\ d\Phi(X)(Y) = (d\phi \circ \mathrm{ad}(X))(Y) = d\phi([X, Y]).$$

On the other hand,

$$\Phi(\Gamma(t))(Y) = (\mathrm{Ad}(\phi(\Gamma(t))) \circ d\phi(Y)) \ \Rightarrow\ d\Phi(X)(Y) =$$
$$= \mathrm{ad}(d\phi(X))(d\phi(Y)) = [d\phi(X), d\phi(Y)].$$

□

Lemma 1.5.11. *Let G be an algebraic group and H a closed subgroup. Then*

$$T_{eH} G/H \cong \mathfrak{g}/\mathfrak{h}.$$

Proof. Consider the surjection $p : G \to G/H$. By Theorem 3.0.33, for a generic $g \in G$ the kernel of the differential $d_g p : T_g G \to T_{gH} G/H$ equals the tangent space to the fiber. But G acts transitively on G and G/H and the map p is G-equivariant. This implies that all points have equal status, and thus $\mathrm{Ker}(d_g p) = T_g(gH) \cong \mathfrak{h}$ for any $g \in G$. On the other hand, G, H and G/H are smooth, and $\dim(G/H) = \dim G - \dim H$. So $d_g p$ is surjective, and

$$T_{eH} G/H = \mathrm{Im}(d_e p) \cong \mathfrak{g}/\mathrm{Ker}(d_e p) = \mathfrak{g}/\mathfrak{h}.$$

□

Lemma 1.5.12. *Let $\phi : G \to H$ be a homomorphism of algebraic groups. Then*

$$\mathrm{Lie}(\mathrm{Ker}\, \phi) = \mathrm{Ker}\, d\phi, \quad \mathrm{Lie}(\mathrm{Im}\phi) = \mathrm{Im}(d\phi).$$

Proof. We may applies arguments given above to the surjection $G \to \phi(G)$. □

Lemma 1.5.13. *Let $\{H_i : i \in I\}$ be a family of closed subgroups of G. Then*

$$\mathrm{Lie}(\bigcap_{i \in I} H_i) = \bigcap_{i \in I} \mathrm{Lie}(H_i).$$

Proof. Since any variety is Noetherian, there are elements $H_1, \dots, H_k$ of our family with $\bigcap_{j=1}^k H_j = \bigcap_{i\in I} H_i$. Moreover, we may reduce the proof to the case $k = 2$ by induction.

Consider the commutative diagram A of group homomorphisms and the induced commutative diagram B of differentials:

$$\begin{array}{ccc} G & \longrightarrow & G/H_1 \\ \uparrow & & \uparrow \\ H_2 & \longrightarrow & H_2/H_1\cap H_2 \end{array} \qquad \begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathfrak{g}/\mathfrak{h}_1 \\ \uparrow & & \uparrow \\ \mathfrak{h}_2 & \longrightarrow & \mathfrak{h}_2/\mathfrak{h}_1\cap\mathfrak{h}_2. \end{array}$$

By Lemma 1.5.11, the kernel of the lower arrow in diagram B equals $\mathrm{Lie}(H_1 \cap H_2)$. Passing in diagram B by another way and applying Lemma 1.5.11 to the upper arrow, we get that the kernel is $\mathrm{Lie}(H_1) \cap \mathrm{Lie}(H_2)$. □

Definition 1.5.14. Let G be an algebraic group. A Lie subalgebra $\mathfrak{h} \subseteq \mathrm{Lie}(G)$ is called *algebraic* if there exists a closed subgroup $H \subset G$ with $\mathfrak{h} = \mathrm{Lie}(H)$.

Warning: Not any Lie subalgebra $\mathfrak{h} \subseteq \mathrm{Lie}(G)$ is algebraic, see Proposition 1.6.20.

Theorem 1.5.15. *Let G be an algebraic group. The map* $\mathrm{Lie} : H \to \mathrm{Lie}(H)$ *defines a bijection:*

$$\{\textit{closed connected subgroups } H \subseteq G\} \;\leftrightarrow\; \{\textit{algebraic Lie subalgebras } \mathfrak{h} \subseteq \mathrm{Lie}(G)\}.$$

Proof. By definition 1.5.14, the map Lie is surjective. In order to prove injectivity, assume that $\mathrm{Lie}(H_1) = \mathrm{Lie}(H_2)$. In particular, $\dim H_1 = \dim H_2$. If $H_1 \subseteq H_2$, then $H_1 = H_2$ by Theorem 3.0.25. In the general case, one has

$$\mathrm{Lie}(H_1) = \mathrm{Lie}((H_1 \cap H_2)^0) = \mathrm{Lie}(H_2)$$

(Lemma 1.5.13), and $H_1 = (H_1 \cap H_2)^0 = H_2$. □

Now consider a rational representation $\rho : G \to \mathrm{GL}(V)$ and its tangent representation $\tau := d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. For any subspace $U \subseteq V$ define

$$N_G(U) = \{g \in G : \rho(g)U = U\}.$$

Proposition 1.5.16. *For any vector $v \in V$ and any subspace $U \subseteq V$ one has:*

$$\mathrm{Lie}(G_v) = \mathfrak{g}_v := \{x \in \mathfrak{g} : \tau(x)v = 0\}, \;\; \mathrm{Lie}(N_G(U)) = \mathfrak{n}_{\mathfrak{g}}(U) := \{x \in \mathfrak{g} : \tau(x)U \subseteq U\}.$$

Proof. Applying Lemma 1.5.12 to $G \to G/\mathrm{Ker}(\rho)$, we may assume that $\rho : G \to \mathrm{GL}(V)$ is injective. In the case $G = \mathrm{GL}(V)$, the subgroups $\mathrm{GL}(V)_v$ and $N_{\mathrm{GL}(V)}(U)$ and their tangent algebras may be easily described in the matrix form. This description implies that the statements are true. In the general case, $\rho(G_v) = \mathrm{GL}(V)_v \cap \rho(G)$, $\rho(N_G(U)) = N_{\mathrm{GL}(V)}(U) \cap \rho(G)$, and one applies Lemma 1.5.13 to the closed subgroup $\rho(G) \subset \mathrm{GL}(V)$. □

Theorem 1.5.17. *Let $\rho : G \to \mathrm{GL}(V)$ be a rational representation of a connected group G, and $\tau := d\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be the tangent representation. A subspace $U \subseteq V$ is G-invariant if and only if it is $\mathfrak{g}$-invariant. In particular, the representation ρ is irreducible if and only if τ is irreducible.*

Proof. A subspace U is G-invariant (resp. $\mathfrak{g}$-invariant) if and only if $N_G(U) = G$ (resp. $\mathfrak{n}_\mathfrak{g}(U) = \mathfrak{g}$). For a connected G, $N_G(U)$ coincides with G if and only if their tangent algebras coincide (Theorem 1.5.15). By Proposition 1.5.16, $\mathrm{Lie}(N_G(U)) = \mathfrak{g}$ if and only if $\mathfrak{n}_\mathfrak{g}(U) = \mathfrak{g}$. □

Definition 1.5.18. Let $\mathfrak{g}$ be a Lie algebra. A subspace $\mathfrak{a} \subseteq \mathfrak{g}$ is called an *ideal* if $[x, y] \in \mathfrak{a}$ for all $x \in \mathfrak{g}, y \in \mathfrak{a}$.

Remark 1.5.19. A subspace $\mathfrak{a} \subseteq \mathfrak{g}$ is an ideal if and only if it is a $\mathfrak{g}$-invariant subspace with respect to the ad-representation.

Proposition 1.5.20. *Let H be a closed connected subgroup of a connected group G. The subgroup H is normal in G if and only if* $\mathrm{Lie}(H)$ *is an ideal of* $\mathrm{Lie}(G)$.

Proof. A subgroup H is normal in G if and only if $a_g(H) = H$ for all $g \in G$. We claim that this is equivalent to the condition "$\mathrm{Lie}(H)$ is an $\mathrm{Ad}(G)$-invariant subspace of $\mathrm{Lie}(G)$". Indeed, if G preserves H, then it preserves its tangent space. Conversely, assume that $a_g(H) \neq H$ for some $g \in G$. Then $a_g(H)$ is another closed connected subgroup of G, and by Theorem 1.5.15 $\mathrm{Lie}(H) \neq \mathrm{Lie}(a_g(H)) = \mathrm{Ad}(g)(\mathrm{Lie}(H))$.

Finally, the subspace $\mathrm{Lie}(H) \subseteq \mathrm{Lie}(G)$ is G-invariant if and only if it is $\mathfrak{g}$-invariant (Theorem 1.5.17). □

For a Lie algebra $\mathfrak{g}$, the center is defined as

$$\mathfrak{z}(\mathfrak{g}) := \{x \in \mathfrak{g} : [x, y] = 0 \text{ for all } y \in \mathfrak{g}\}.$$

Moreover, set

$$Z(\mathfrak{g}) := \{g \in G : \mathrm{Ad}(g)(x) = x \text{ for all } x \in \mathfrak{g}\},$$

and, for any $x \in \mathfrak{g}$,

$$Z_G(x) = Z(x) := \{g \in G : \mathrm{Ad}(g)x = x\}, \quad \mathfrak{z}_\mathfrak{g}(x) = \mathfrak{z}(x) := \{y \in \mathfrak{g} : [y, x] = 0\}.$$

Proposition 1.5.21. *For any algebraic group G and any $x \in \mathrm{Lie}(G) = \mathfrak{g}$ one has*

$$\mathrm{Lie}(Z(x)) = \mathfrak{z}(x), \quad \mathrm{Lie}(Z(\mathfrak{g})) = \mathfrak{z}(\mathfrak{g}).$$

Proof. The first statement follows directly from Proposition 1.5.16 applied to the adjoint representation Ad with $v = x$.

For the second one, take a basis $x_1, \ldots, x_n$ in $\mathfrak{g}$. Then $Z(\mathfrak{g}) = \cap_{i=1}^n Z(x_i)$, $\mathfrak{z}(\mathfrak{g}) = \cap_{i=1}^n \mathfrak{z}(x_i)$, and we may use Lemma 1.5.13. □

Exercises to subsection 1.5.

Exercise 1.5.22. Consider the map det : $\mathrm{GL}(n) \to \mathbb{K}^{\times}$, $A \to \det(A)$. Prove that $d_E(\det)(X) = \mathrm{tr}(X)$ for any $X \in T_E\mathrm{GL}(n)$.

Exercise 1.5.23. Calculate the differential $di : T_E\mathrm{GL}(n) \to T_E\mathrm{GL}(n)$ using the formula for inverse matrix in terms of minors.

Exercise 1.5.24. Prove that a Lie algebra is commutative if and only if any its subalgebra is an ideal.

Exercise 1.5.25. What is an n-dimensional representation of a commutative Lie algebra $\mathfrak{g}$?

Exercise 1.5.26. Assume that G_1 and G_2 are connected algebraic groups such that the Lie algebras $\mathrm{Lie}(G_1)$ and $\mathrm{Lie}(G_2)$ are isomorphic. Is it true that G_1 and G_2 are isomorphic ?

Exercise 1.5.27. Describe the tangent algebras of the groups $\mathrm{Sp}(2n)$, $D(n)$, $\mathrm{B}(n)$ and $U(n)$.

Exercise 1.5.28. Prove that the following Lie algebras are isomorphic:

(a) $\mathfrak{sl}(2) \cong \mathfrak{so}(3)$;
(b) $\mathfrak{sl}(2) \times \mathfrak{sl}(2) \cong \mathfrak{so}(4)$;
(c) $\mathfrak{sp}(4) \cong \mathfrak{so}(5)$;
(d) $\mathfrak{sl}(4) \cong \mathfrak{so}(6)$.

Exercise 1.5.29. Show that for any rational representation $\rho : G \to \mathrm{GL}(V)$ any G-invariant subspace $U \subseteq V$ is $\mathfrak{g}$-invariant, but for a non-connected G the converse is not true.

Exercise 1.5.30. Let H be a closed subgroup of G. Prove that if H is normal in G, then $\mathrm{Lie}(H)$ is an ideal of $\mathrm{Lie}(G)$. Give an example where the converse is not true.

Exercise 1.5.31. Set $Z_G(g) = \{h \in G : gh = hg\}$. Then

$$\mathrm{Lie}(Z_G(g)) = \{x \in \mathfrak{g} : \mathrm{Ad}(g)x = x\}.$$

Exercise 1.5.32. Prove that a connected algebraic group is commutative if and only if its tangent algebra is commutative.

Exercise 1.5.33. For a connected G, prove that $\mathrm{Lie}(Z(G)) = \mathfrak{z}(\mathfrak{g})$. For any G, one has $\mathrm{Lie}(Z(G)) \subseteq \mathfrak{z}(\mathfrak{g})$, but the inclusion may be strict.

Exercise 1.5.34. Let A be a finite dimensional algebra. By a derivation of A we mean a linear map $D : A \to A$ such that $D(ab) = D(a)b + aD(b)$ for any $a, b \in A$. Denote by $\mathrm{Der}(A)$ the subspace of all derivations in $\mathfrak{gl}(A)$.

(i) Check directly that $\mathrm{Der}(A)$ is a Lie subalgebra in $\mathfrak{gl}(A)$.
(ii) Let $\mathrm{Aut}(A)$ be the automorphism group of A, see Exercise 1.1.23. Prove that

$$\mathrm{Lie}(\mathrm{Aut}(A)) = \mathrm{Der}(A).$$

1.6. **Algebraic tori.** This section is devoted to a special important class of algebraic groups, namely algebraic tori $T^m \cong \mathbb{K}^\times \times \dots \times \mathbb{K}^\times$ (m times). The group T^m is connected, commutative, and has a lot of other important properties that are discussed below.

We begin with the study of characters.

Definition 1.6.1. Let G be an algebraic group. A *character* of G is a homomorphism $\chi : G \to \mathbb{K}^\times$.

The set $\mathbb{X}(G)$ of all characters has a structure of an Abelian group with respect to the operation $(\chi_1 + \chi_2)(g) := \chi_1(g)\chi_2(g)$, with the neutral element χ^0, $\chi^0(g) = 1$ for all g, and with the inverse element $(-\chi)(g) := \chi(g)^{-1}$.

Let G be the torus T^m. Take an integer vector $u = (a_1, \dots, a_m) \in \mathbb{Z}^m$. It defines an (algebraic) homomorphism

$$\chi^u : T^m \to \mathbb{K}^\times, \quad \chi^u((t_1, \dots, t_m)) = t_1^{a_1} \dots t_m^{a_m}.$$

Conversely,

Lemma 1.6.2. *Any character $\chi : T^m \to \mathbb{K}^\times$ has a form χ^u for some $u \in \mathbb{Z}^m$.*

Proof. We start with $m = 1$. By definition,

$$\chi(t) = \sum_{i=-j}^{k} a_i t^i$$

is a Laurent polynomial with $a_k \neq 0$. Since it does not have roots in $\mathbb{K}^\times$, the same is true for $p(t) = t^j \chi(t)$. But any polynomial may be decomposed into linear factors, so $p(t) = a_k t^{k+j}$. The condition $\chi(1) = p(1) = 1$ implies $a_k = 1$ and finally $\chi(t) = t^k$.

For $m > 1$, we have

$$\chi((t_1, \dots, t_m)) = \chi((t_1, 1, \dots, 1)) \dots \chi((1, \dots, 1, t_m)) = t_1^{a_1} \dots t_m^{a_m}$$

for some $a_i \in \mathbb{Z}$. □

Corollary 1.6.3. *The Abelian group $\mathbb{X}(T^m)$ is isomorphic to the lattice $\mathbb{Z}^m$.*

Proof. The map $\sigma : \mathbb{X}(T^m) \to \mathbb{Z}^m$, $\sigma(\chi^u) = u$, is bijective and respects the operations. □

Now we shall describe automorphisms of T^m. Lemma 1.6.2 shows that any homomorphism $\phi : T^m \to T^m$ has a form:

$$\phi((t_1, \dots, t_m)) = (t_1^{a_{11}} \dots t_m^{a_{1m}}, \dots, t_1^{a_{m1}} \dots t_m^{a_{mm}})$$

for some integer a_{ij}. Consider a matrix $A = (a_{ij}) \in \mathrm{Mat}(n \times n, \mathbb{Z})$, and denote the automorphism ϕ as ϕ_A.

Lemma 1.6.4. *$\phi_{BA} = \phi_B \circ \phi_A$ for any $A, B \in \mathrm{Mat}(n \times n, \mathbb{Z})$.*

Proof. For any $t = (t_1, \dots, t_m) \in T^m$, we have

$$\phi_{BA}(t) = (t_1^{b_{11}a_{11} + \dots + b_{1m}a_{m1}} \dots t_m^{b_{11}a_{1m} + \dots + b_{1m}a_{mm}}, \dots, t_1^{b_{m1}a_{11} + \dots + b_{mm}a_{m1}} \dots t_m^{b_{m1}a_{1m} + \dots + b_{mm}a_{mm}}) = \phi_B(\phi_A(t)).$$

□

The lemma implies that a homomorphism ϕ_A is an automorphism if and only if the matrix A is invertible. So the group $\mathrm{Aut}(T^m)$ is isomorphic to $\mathrm{GL}(n,\mathbb{Z})$.

Further, for any algebraic group G, an automorphism $\phi : G \to G$ acts on $\mathbb{X}(G)$ as $(\phi\cdot\chi)(g) := \chi(\phi^{-1}(g))$. In our case, let $e_1,\dots,e_m$ be the standard basis of $\mathbb{X}(T^m)\cong\mathbb{Z}^m$. Since $\phi_A^{-1} = \phi_{A^{-1}}$, we have

$$\phi_A^{-1}((t_1,\dots,t_m)) = (t_1^{c_{11}}\dots t_m^{c_{1m}},\dots,t_1^{c_{m1}}\dots t_m^{c_{mm}}),$$

where $A^{-1} = (c_{ij})$, and thus $\phi_A(e_i) = c_{i1}e_1 + \dots + c_{im}e_m$. This proves that ϕ_A acts in the bases $e_1,\dots,e_m$ via the matrix $(A^T)^{-1}$. In particular, there is an isomorphism between $\mathrm{Aut}(T^m)$ and $\mathrm{Aut}(\mathbb{X}(T^m))$. Let us summarize all these observations.

Proposition 1.6.5. *There are isomorphisms:*

$$\mathrm{Aut}(T^m)\cong\mathrm{Aut}(\mathbb{X}(T^m))\cong\mathrm{GL}(m,\mathbb{Z}).$$

Remark 1.6.6. The group $\mathrm{GL}(m,\mathbb{Z})$ does not have any reasonable structure of an affine algebraic group.

Remark 1.6.7. Any isomorphism $T^m\cong(\mathbb{K}^\times)^m$ defines the standard basis in the lattice $\mathbb{X}(T^m)$. Conversely, any basis $e_1,\dots,e_m$ in $\mathbb{X}(T^m)$ defines an isomorphism $\phi : T^m\cong(\mathbb{K}^\times)^m$, $\phi(t) = (e_1(t),\dots,e_m(t))$.

Our next objective is the classification of rational T^m-modules.

Definition 1.6.8. An (algebraic) *quasitorus* is an algebraic group Q isomorphic to $T^m\times A$, where A is a finite Abelian group.

Lemma 1.6.9. *Elements of finite order are dense in Q.*

Proof. Clearly, an element (t,a), $t\in T^m$, $a\in A$ has a finite order if and only if t does. So it is sufficient to prove the statement for $Q = T^m$.

Let $F(X_1,X_1^{-1},\dots,X_m,X_m^{-1})$ be a Laurent polynomial such that $F(\epsilon_1,\epsilon_1^{-1},\dots,\epsilon_m,\epsilon_m^{-1}) = 0$ for any ϵ_i with $\epsilon_i^N = 1$ for some $N>0$. We claim that F is identically zero. Indeed,

(1) If $m = 1$, then $F(X_1,X_1^{-1})$ has infinitely many roots in $\mathbb{K}$, which is possible only for the zero polynomial;
(2) Assume that $m>1$. By the inductive hypothesis, for any element $\epsilon\in\mathbb{K}^\times$ of finite order, the polynomial

$$F_\epsilon(X_2,X_2^{-1},\dots,X_m,X_m^{-1}) := F(\epsilon,\epsilon^{-1},X_2,X_2^{-1},\dots,X_m,X_m^{-1})$$

is identically zero. This implies that F does not depend on X_1 (Exercise 1.6.21) and F is identically zero.

□

Let us recall a definition and two facts (see Exercises 1.6.25 and 1.6.26) from linear algebra.

Definition 1.6.10. An element $A\in\mathrm{Mat}(n\times n)$ is called *semisimple* (or *diagonalizable*) if it is conjugate to a diagonal matrix.

Lemma 1.6.11. *Any element of finite order in* $\mathrm{GL}(n)$ *is semisimple.*

Lemma 1.6.12. *Let $\{A_i : i\in I\}$ be a family of pairwise commuting semisimple operators on a finite-dimensional vector space. Then the operators A_i may be diagonalized simultaneously.*

Let V be a rational Q-module. For any character $\chi \in \mathbb{X}(Q)$, define a subspace

$$V_\chi = \{v \in V : g \cdot v = \chi(g)v \text{ for all } g \in Q\}.$$

Theorem 1.6.13. *Let Q be a quasitorus. For any rational Q-module V one has:*

$$V = \bigoplus_{\chi \in \mathbb{X}(Q)} V_\chi$$

Proof. We start with a finite-dimensional case.

Lemma 1.6.14. *Any rational finite-dimensional Q-module is a direct sum of a one-dimensional submodules.*

Proof. Let $\rho : Q \to \mathrm{GL}(V)$ be a finite-dimensional rational representation. For any element $g \in Q$ of finite order the operator $\rho(g)$ is semisimple (Lemma 1.6.11). Since Q is commutative, there exists a basis in V such that the operators $\rho(g)$ are represented by diagonal matrices for all elements $g \in Q$ of finite order (Lemma 1.6.12). Thus the image of a dense subset of Q is contained in the subgroup $D(n)$ of diagonal matrices in $\mathrm{GL}(n)$ ($n = \dim V$). But the subgroup $D(n)$ is closed in $\mathrm{GL}(n)$, and this implies $\rho(Q) \subseteq D(n)$. □

Since any one-dimensional submodule of V is contained in some V_χ, we have $V = \sum_{\chi \in \mathbb{X}(Q)} V_\chi$. The same holds for any rational Q-module, because any its element is contained in a finite-dimensional rational submodule.

We claim that the sum $\sum_{\chi \in \mathbb{X}(Q)} V_\chi$ is direct. Indeed, suppose that $v_1 + \cdots + v_k = 0$ is a linear combination with non-zero $v_i \in V_{\chi_i}$, where χ_i are pairwise different, and k is the smallest possible. Take $g \in Q$ with $\chi_1(g) \neq \chi_2(g)$. Then $\chi_1(g)v_1 + \cdots + \chi_k(g)v_k = 0$, and we may find a shorter combination:

$$(\chi_1(g) - \chi_2(g))v_2 + \cdots + (\chi_1(g) - \chi_k(g))v_k = 0,$$

a contradiction. □

Theorem 1.6.15. *Any closed subgroup $H \subseteq T^m$ is a quasitorus. Moreover, there are an isomorphism $T^m \cong (\mathbb{K}^\times)^m$ and positive integers $d_1, \ldots, d_s$ ($s \leq m$) such that*

$$H = \{(t_1, \ldots, t_m) : t_1^{d_1} = \cdots = t_s^{d_s} = 1\}.$$

Proof. The torus T^m is commutative, and H is a normal subgroup of T^m. By Theorem 1.4.1 (2), there is a finite-dimensional rational representation $\rho : T^m \to \mathrm{GL}(k)$ such that $H = \mathrm{Ker}(\rho)$. There are characters $\chi_1, \ldots, \chi_k$ such that the representation ρ is equivalent to $t \to \mathrm{diag}(\chi_1(t), \ldots, \chi_k(t))$ (Theorem 1.6.13). This proves that $H = \{t \in T^m : \chi_1(t) = \cdots = \chi_k(t) = 1\}$.

Now we need one more fact from a course of algebra.

Proposition 1.6.16. *Let B be a finitely generated subgroup of $\mathbb{Z}^m$. Then B is a free Abelian group. Moreover, there are a basis $a_1, \ldots, a_m$ of $\mathbb{Z}^m$, a basis $b_1, \ldots, b_s$ of B ($s \leq n$) and positive integers $d_1, \ldots, d_s$ such that $b_1 = d_1 a_1, \ldots, b_s = d_s a_s$.*

Take a subgroup B generated by $\chi_1, \ldots, \chi_k$ in $\mathbb{X}(T^m) \cong \mathbb{Z}^m$. Clearly,

$$H = \{t \in T^m : \chi(t) = 1 \text{ for all } \chi \in B\}.$$

Let us fix the corresponding bases $a_1, \ldots, a_m$ and $b_1, \ldots, b_s$ of $\mathbb{X}(T^m)$ and B respectively. Applying an automorphism of T^m, one may assume that $a_1, \ldots, a_m$ is a standard basis of $\mathbb{Z}^m$. Then

$$H = \{(t_1, \ldots, t_m) \in T^m : t_1^{d_1} = \cdots = t_s^{d_s} = 1\},$$

so H is isomorphic to

$$\mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_s} \times (\mathbb{K}^\times)^{m-s},$$

where $\mathbb{Z}_r$ is a cyclic group of order r. □

Corollary 1.6.17. *A closed subgroup of a quasitorus is a quasitorus.*

Proof. Since $Q \cong T^m \times A$ and A is isomorphic to the direct product of l finite cyclic groups, the quasitorus Q (as well as any its closed subgroup) is isomorphic to a closed subgroup of the torus $T^m \times T^l$. □

Now we came to the tangent algebra $\mathrm{Lie}(T^m) := \mathfrak{t}$. At the first glance, it is not an interesting object: $\mathfrak{t}$ is just an m-dimensional commutative Lie algebra. But we shall show now that it is naturally equipped with an additional structure, so called $\mathbb{Q}$-form.

The differential of a character $\chi : T^m \to \mathbb{K}^\times$ defines a linear function

$$d_e\chi := d\chi : \mathfrak{t} \to \mathbb{K}$$

that may be considered as an element of the dual space $\mathfrak{t}^*$. The equalities $(\chi_1 + \chi_2)(t) = \chi_1(t)\chi_2(t)$ and $\chi_1(e) = \chi_2(e) = 1$ imply $d(\chi_1 + \chi_2) = d\chi_1 + d\chi_2$. Let us denote the pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$ as $(\cdot, \cdot)$, $(v, l) = l(x)$. Define a sublattice:

$$\mathfrak{t}(\mathbb{Z}) := \{x \in \mathfrak{t} : (x, d\chi) \in \mathbb{Z} \text{ for all } \chi \in \mathbb{X}(T^m)\},$$

and a $\mathbb{Q}$-subspace:

$$\mathfrak{t}(\mathbb{Q}) := \{x \in \mathfrak{t} : (x, d\chi) \in \mathbb{Q} \text{ for all } \chi \in \mathbb{X}(T^m)\}.$$

Definition 1.6.18. A subspace $\mathfrak{h} \subseteq \mathfrak{t}$ is called $\mathbb{Q}$-*defined* if $\mathfrak{h}$ is the $\mathbb{K}$-linear span of $\mathfrak{h} \cap \mathfrak{t}(\mathbb{Q})$, i.e.,

$$\mathfrak{h} = \langle \mathfrak{h} \cap \mathfrak{t}(\mathbb{Q}) \rangle_{\mathbb{K}}.$$

Lemma 1.6.19. *A subspace $\mathfrak{h} \subseteq \mathfrak{t}$ is $\mathbb{Q}$-defined if and only if there are $\chi_1, \ldots, \chi_s \in \mathbb{X}(T^m)$ with*

$$\mathfrak{h} = \bigcap_{i=1}^{s} \mathrm{Ker}(d\chi_i).$$

Proof. There are $\chi_1, \ldots, \chi_s$ such that

$$\mathfrak{h} \cap \mathfrak{t}(\mathbb{Q}) = \{x \in \mathfrak{t}(\mathbb{Q}) : (x, d\chi_i) = 0,\ i = 1, \ldots, s\}.$$

Then

$$\langle \mathfrak{h} \cap \mathfrak{t}(\mathbb{Q}) \rangle_{\mathbb{K}} = \{x \in \mathfrak{t} : (x, d\chi_i) = 0,\ i = 1, \ldots, s\},$$

and $\mathfrak{h}$ coincides with $\langle \mathfrak{h} \cap \mathfrak{t}(\mathbb{Q}) \rangle_{\mathbb{K}}$ if and only if it may be defined by a desired system of linear equations. □

Proposition 1.6.20. *A subspace $\mathfrak{h} \subseteq \mathfrak{t}$ is an algebraic subalgebra of $\mathfrak{t}$ if and only if $\mathfrak{h}$ is $\mathbb{Q}$-defined.*

Proof. We know that any closed subgroup $H \subseteq T^m$ is given as

$$H = \{t \in T^m : \chi_1(t) = \cdots = \chi_s(t) = 1 \text{ for some } \chi_1, \ldots, \chi_s \in \mathbb{X}(T^m)\},$$

or

$$H = \bigcap_{i=1}^{s} \mathrm{Ker}(\chi_i).$$

By Lemma 1.5.12, $\mathrm{Lie}(\mathrm{Ker}(\chi_i)) = \mathrm{Ker}(d\chi_i)$, and by Lemma 1.5.13, $\mathrm{Lie}(H) = \bigcap_{i=1}^{s} \mathrm{Ker}(d\chi_i)$. Now Lemma 1.6.19 completes the proof. □

Exercises to subsection 1.6.

Exercise 1.6.21. Let $F(Y_1,\dots,Y_n)$ be a polynomial and $F_a(Y_2,\dots,Y_n) := F(a,Y_2,\dots,Y_n)$ for $a\in\mathbb{K}$. Prove that if $F_a(Y_2,\dots,Y_n)$ coincides with $F_b(Y_2,\dots,Y_n)$ for infinitely many $b\in\mathbb{K}$, then $F(Y_1,\dots,Y_n)$ does not depend on Y_1.

Exercise 1.6.22. Find the group of characters $\mathbb{X}(G)$ for G equals (a) $\mathrm{GL}(n)$; (b) $\mathrm{SL}(n)$; (c) $\mathrm{O}(2)$; (d) $B(n)$; (e) a quasitorus; (f) a finite group.

Exercise 1.6.23. Describe the set of homomorphisms $\{\phi : T^m \to T^r\}$.

Exercise 1.6.24. Let $A\in\mathrm{Mat}(n\times n)$ be a semisimple element and $U\subseteq V := \mathbb{K}^n$ be an A-invariant subspace. Prove that $A\mid_U$ is a semisimple element of $\mathrm{GL}(U)$.

Exercise 1.6.25. Prove that any element of finite order in $\mathrm{GL}(n)$ is semisimple.

Exercise 1.6.26. Let $\{A_i\}$ be a family of pairwise commuting semisimple operators on a finite-dimensional vector space. Prove that the operators A_i may be diagonalized simultaneously.

Exercise 1.6.27. Do semisimple elements form a subgroup of $\mathrm{GL}(n)$?

Exercise 1.6.28. Prove that semisimple elements form a dense subset of $\mathrm{Mat}(n\times n)$, but for $n>1$ this subset is not open.

Exercise 1.6.29. Let G be a commutative algebraic group such that G^0 is a torus. Prove that G is a quasitorus.

Exercise 1.6.30. Let Q be a quasitorus and $\phi : Q\to G$ be a surjective homomorphism. Prove that G is a quasitorus.

Exercise 1.6.31. Let H be a closed subgroup of T^m. Prove that any character of H may be extended to a character of T^m.

Exercise 1.6.32. In terms of Lemma 1.6.19, prove that

$$\langle \mathfrak{h}\cap\mathfrak{t}(\mathbb{Q})\rangle_{\mathbb{K}} = \{x\in\mathfrak{t} : (x, d\chi_i)=0,\ i=1,\dots,s\}.$$

Exercise 1.6.33. Prove that the differential $d_e : \mathrm{Aut}(T^m)\to \mathrm{GL}(\mathfrak{t})$ defines an isomorphism between $\mathrm{Aut}(T^m)$ and the subgroup

$$H := \{A\in\mathrm{GL}(\mathfrak{t}) : A(\mathfrak{t}(\mathbb{Z}))=\mathfrak{t}(\mathbb{Z})\}.$$

1.7. **Jordan decompositions.**

Definition 1.7.1. An element $A \in \mathrm{Mat}(n \times n)$ is called *nilpotent* if $A^N = 0$ for some N, or, equivalently, all eigenvalues of A are 0.

Definition 1.7.2. An element $A \in \mathrm{GL}(n)$ is called *unipotent* if $A - E$ is nilpotent, or, equivalently, all eigenvalues of A are 1.

Lemma 1.7.3. *The subset* $\mathrm{Nil}(n) \subset \mathrm{Mat}(n \times n)$ *of all nilpotent elements and the subset* $\mathrm{Uni}(n) \subset \mathrm{GL}(n)$ *of all unipotent elements are closed.*

Proof. Let $P_A(x) = x^n + a_1x^{n-1} + \dots + a_n$ be the characteristic polynomial of a matrix A. It is well known that the coefficients $a_1, \dots, a_n$ depend on A polynomially. So the subvariety $\mathrm{Nil}(n)$ is define in $\mathrm{Mat}(n \times n)$ by

$$a_1(A) = \dots = a_n(A) = 0.$$

The subvariety $\mathrm{Uni}(n)$ is $E+\mathrm{Nil}(n)$, thus it is closed in $\mathrm{Mat}(n\times n)$ and in $\mathrm{GL}(n)$. □

For any nilpotent A one may define correctly the exponent:

$$\exp(A) := E + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \dots.$$

Since for a nilpotent $A \in \mathrm{Mat}(n\times n)$ one has $A^n = 0$, the map $\exp : \mathrm{Nil}(n) \to \mathrm{GL}(n)$ is a morphism. Moreover, the matrix $\frac{A}{1!} + \frac{A^2}{2!} + \dots$ is nilpotent, so $\exp(A)$ is unipotent, and exp sends $\mathrm{Nil}(n)$ to $\mathrm{Uni}(n)$.

Conversely, for any $B \in \mathrm{Uni}(n)$ one may define the logarithm:

$$\ln(B) = \ln(E + B_0) := B_0 - \frac{B_0^2}{2} + \frac{B_0^3}{3} - \dots.$$

Lemma 1.7.4. *The map* $\exp : \mathrm{Nil}(n) \to \mathrm{Uni}(n)$ *is an isomorphism of varieties.*

Proof. The fact that exp and ln are inverse to each other follows from well-known identities of formal power series:

$$\exp(\ln(x)) = x; \quad \ln(\exp(x)) = x.$$

□

Let $A \in \mathrm{GL}(n)$. Since the intersection of any family of closed subgroups in $\mathrm{GL}(n)$ is again a closed subgroup, there exists the smallest closed subgroup containing A. Denote it by $G(A)$.

Assume that $A \in \mathrm{Uni}(n)$. For any $t \in \mathbb{K}$ define $A^t := \exp(t \ln(A))$.

Definition 1.7.5. A closed subgroup $G \subset \mathrm{GL}(n)$ is called *unipotent* if any element $A \in G$ is unipotent.

Proposition 1.7.6. *Let $A \neq E$ be a unipotent matrix. Then*

1) $\{A^t : t \in \mathbb{K}\}$ *is a one-dimensional unipotent subgroup in* $\mathrm{GL}(n)$ *isomorphic to* G_a*;*
2) $G(A) = \{A^t : t \in \mathbb{K}\}$.

Proof. Note that

$$A_1, A_2 \in \mathrm{Nil}(n),\ A_1A_2 = A_2A_1 \quad \Rightarrow \quad \exp(A_1 + A_2) = \exp(A_1)\exp(A_2).$$

This implies $\exp((t_1 + t_2)\ln(A)) = \exp(t_1 \ln(A)) \exp(t_2 \ln(A))$, and the map $\exp_A : \mathbb{K} \to \mathrm{GL}(V)$, $\exp_A(t) = A^t$ is a homomorphism of algebraic groups. By Theorem 1.2.12, the subgroup $\exp_A(\mathbb{K}) = \{A^t : t \in \mathbb{K}\}$ is closed in $\mathrm{GL}(n)$.

Lemma 1.7.7. *The group G_a has no proper closed subgroups.*

Proof. Since G_a is one-dimensional and irreducible, any its proper closed subset is finite. But G_a contains no proper finite subgroups, because the only element of finite order in G_a is the unit. □

Thus for 1) it is sufficient to prove that $\{A^t : t \in \mathbb{K}\}$ is non-trivial. But this is clear, since it contains $A \neq E$.

Moreover, $G(A) \subseteq \{A^t : t \in \mathbb{K}\}$ by minimality, and, again by Lemma 1.7.7, $G(A) = \{A^t : t \in \mathbb{K}\}$. □

Now we need some more facts from linear algebra. For any $A \in \mathrm{Mat}(n \times n)$ consider the factorization of the characteristic polynomial:

$$P_A(x) = (x - \lambda_1)^{k_1} \dots (x - \lambda_r)^{k_r}, \quad \lambda_1, \dots, \lambda_r \in \mathbb{K}, \quad k_1 + \dots + k_r = n.$$

For $V = \mathbb{K}^n$, define the root subspaces:

$$V^{\lambda_i} := \{v \in V : (A - \lambda_i E)^{k_i} v = 0\}.$$

It is well known that $V = V^{\lambda_1} \oplus \dots \oplus V^{\lambda_r}$. Consider the semisimple operator A_s that acts on any V^{λ_i} as $\lambda_i E$. The operator $A_n := A - A_s$ is nilpotent. Moreover, if A is invertible, then A_s is invertible too, and $A_u := AA_s^{-1}$ is unipotent. Finally, A_s commutes with A, A_n and A_u.

Definition 1.7.8.
(i) *Additive Jordan decomposition* of a matrix $A \in \mathrm{Mat}(n \times n)$ is a decomposition $A = A_s + A_n$, where A_s is semisimple, A_n is nilpotent, and $A_s A_n = A_n A_s$.
(ii) *Multiplicative Jordan decomposition* of a matrix $A \in \mathrm{GL}(n)$ is a decomposition $A = A_s A_u$, where A_s is semisimple, A_u is unipotent, and $A_s A_u = A_u A_s$.

The arguments given above explain that any $A \in \mathrm{Mat}(n \times n)$ (resp. $A \in \mathrm{GL}(n)$) possesses the additive (resp. multiplicative) Jordan decomposition.

Proposition 1.7.9. *For any $A \in \mathrm{Mat}(n \times n)$ (resp. $A \in \mathrm{GL}(n)$), the additive (resp. multiplicative) Jordan decomposition is unique.*

Proof. Assume that $A = A'_s + A'_n$ is another decomposition. Since $A'_s A = AA'_s$, $A'_n A = AA'_n$, we have

$$A'_s V^{\lambda_i} \subseteq V^{\lambda_i}, \quad A'_n V^{\lambda_i} \subseteq V^{\lambda_i}, \quad i = 1, \dots, r.$$

But A'_s is semisimple, and there is a decomposition

$$V^{\lambda_i} = V^{\lambda_i}_{\mu_1} \oplus \dots \oplus V^{\lambda_i}_{\mu_{l_i}} \quad \text{with} \quad V^{\lambda_i}_{\mu_j} := \{v \in V^{\lambda_i} : A'_s v = \mu_j v\}.$$

The operators A and A'_n preserve all $V^{\lambda_i}_{\mu_j}$. Let v_{ij} be an eigenvector for A'_n in $V^{\lambda_i}_{\mu_j}$. Then

$$Av_{ij} = (A'_s + A'_n)v_{ij} = \mu_j v_{ij} + 0 v_{ij} = \mu_j v_{ij}$$

Thus $\mu_j = \lambda_i$ for any i, j, and $A'_s = A_s$, $A'_n = A - A'_s = A - A_s = A_n$.

The same arguments work in the multiplicative case. □

Corollary 1.7.10. *An element $A \in \mathrm{Mat}(n \times n)$ is semisimple (resp. nilpotent, unipotent) if and only if $A = A_s$ (resp. $A = A_n$, $A = A_u$).*

Lemma 1.7.11. *For any $A \in \mathrm{Mat}(n \times n)$ there is a polynomial $F(x) \in \mathbb{K}[x]$ such that $A_s = F(A)$.*

Proof. By the Chinese remainder theorem there exists $F(x) \in \mathbb{K}[x]$ with

$$F(x) \equiv \lambda_i \ (\mathrm{mod}(x - \lambda_i)^{k_i}).$$

Then $F(A)\mid_{V^{\lambda_i}} \equiv \lambda_i E$, thus $F(A) = A_s$. □

Corollary 1.7.12. *For any $B \in \mathrm{Mat}(n \times n)$ the condition $AB = BA$ implies $A_sB = BA_s$, $A_nB = BA_n$, $A_uB = BA_u$.*

Theorem 1.7.13. *Let $G \subseteq \mathrm{GL}(n)$ be a closed subgroup and $A \in G$. Then $A_s, A_u \in G$.*

Proof. By definition, $A \in G$ implies $G(A) \subseteq G$. One may suppose that $A_s, A_u \neq E$.

Lemma 1.7.14. *$G(A_s)$ is a quasitorus.*

Proof. The element A_s is contained in a closed subgroup H conjugate to $D(n) \subset \mathrm{GL}(n)$. Thus $G(A_s) \subseteq H$ and $G(A_s)$ is a quasitorus (Theorem 1.6.15). □

The intersection of $G(A_s)$ and $G(A_u)$ consists of e, because all elements of $G(A_s)$ are semisimple, and all elements of $G(A_u)$ are unipotent.

Lemma 1.7.15. *$BC = CB$ for any $B \in G(A_s), C \in G(A_u)$.*

Proof. Note that $A_s, A_u \in Z_{\mathrm{GL}(n)}(A_s) \cap Z_{\mathrm{GL}(n)}(A_u)$, thus $G(A_s), G(A_u) \subseteq Z_{\mathrm{GL}(n)}(A_s) \cap Z_{\mathrm{GL}(n)}(A_u)$ by minimality. This proves that

$$A_s \in H_s := \{B \in G(A_s) : BC = CB \text{ for all } C \in G(A_u)\},$$

$$A_u \in H_u := \{C \in G(A_u) : BC = CB \text{ for all } B \in G(A_s)\},$$

Again by minimality we have $G(A_s) = H_s$, $G(A_u) = H_u$. □

Denote by $G(A_s, A_u)$ the subgroup of $\mathrm{GL}(n)$ generated by $G(A_s)$ and $G(A_u)$. We have proved that the map $G(A_s) \times G(A_u) \to \mathrm{GL}(n)$, $(B, C) \to BC$ is an injective homomorphism. Thus $G(A_s, A_u)$ is a closed subgroup of $\mathrm{GL}(n)$ isomorphic to $G(A_s) \times G(A_u)$, and $A = A_sA_u$ implies $G(A) \subseteq G(A_s, A_u)$.

Consider the projections $\pi_s : G(A) \to G(A_s)$, $\pi_u : G(A) \to G(A_u)$. Since $\mathrm{Ker}(\pi_s) \subseteq G(A_u)$, and $G(A_u)$ has no proper closed subgroups, there are two possibilities:

(i) $\mathrm{Ker}(\pi_s) = G(A_u) \Rightarrow A_u \in G(A) \Rightarrow A_u \in G \Rightarrow A_s \in G$;
(ii) $\mathrm{Ker}(\pi_s) = \{e\}$. Then $G(A)$ is a quasitorus and $\pi_u(g) = E$ for any $g \in G(A)$ (Proposition 1.7.16). But $\pi_u(A) = A_u$, a contradiction.

The proof of Theorem 1.7.13 is completed. □

Proposition 1.7.16. *Let Q be a quasitorus. Then all homomorphisms $\phi : Q \to G_a$ and $\psi : G_a \to Q$ are trivial.*

Proof. Since the set F of elements of finite order is dense in Q (Lemma 1.6.9), and the only element of finite order in G_a is e, we have $\phi(F) = \{e\}$. The preimage $\phi^{-1}(e)$ is a closed subset of Q containing F, thus $\phi^{-1}(e) = Q$.

Conversely, $\psi(G_a)$ is contained in Q^0, and by Proposition 1.6.15 $\psi(G_a)$ is a torus. Let $g \neq e$, $g \in \psi(G_a)$ be an element of order N. Then $g = \psi(u)$, and $\psi(Nu) = e$. But G_a has no proper closed subgroups, and $\mathrm{Ker}(\psi) = G_a$. □

Corollary 1.7.17. $G(A) = G(A_s, A_u)$.

Proof. By Theorem 1.7.13 $A_s, A_u \in G(A)$, thus $G(A_s), G(A_u) \subseteq G(A)$, and $G(A_s, A_u) \subseteq G(A)$. On the other hand, $A = A_s A_u \in G(A_s, A_u)$, and $G(A) \subseteq G(A_s, A_u)$. □

Theorem 1.7.18. *Let $G \subseteq \mathrm{GL}(n)$, $G_1 \subseteq \mathrm{GL}(n_1)$ be closed subgroups and $\phi : G \to G_1$ be a homomorphism of algebraic groups. Then for any $A \in G$ one has $\phi(A_s) = \phi(A)_s$, $\phi(A_u) = \phi(A)_u$.*

Proof. The condition $\phi(A) \in \phi(G(A)) \subset \mathrm{GL}(n_1)$ implies $G(\phi(A)) \subseteq \phi(G(A))$. On the other hand, the closed subgroup $\phi^{-1}(G(\phi(A)))$ contains A, thus $G(\phi(A)) = \phi(G(A))$. One knows that

$$G(A) \cong G(A_s) \times G(A_u), \quad G(\phi(A)) \cong G(\phi(A)_s) \times G(\phi(A)_u),$$

and may consider homomorphisms

$$\pi'_s \circ \phi : G(A_u) \to G(\phi(A)_s), \quad \pi'_u \circ \phi : G(A_s) \to G(\phi(A)_u).$$

By Proposition 1.7.16, they are trivial, and $\phi(G(A_s)) \subseteq G(\phi(A)_s)$, $\phi(G(A_u)) \subseteq G(\phi(A)_u)$. Hence $\phi(A_s)$ is semisimple and $\phi(A_u)$ is unipotent. Moreover, these elements commute, and $\phi(A) = \phi(A_s)\phi(A_u)$ is the multiplicative Jordan decomposition of A. By the uniqueness property, $\phi(A_s) = \phi(A)_s$, $\phi(A_u) = \phi(A)_u$. □

Corollary 1.7.19. *Let G be an algebraic group and $g \in G$. The Jordan decomposition $g = g_s g_u$ is well-defined, i.e., does not depend of a closed embedding $G \subseteq \mathrm{GL}(n)$.*

Now we come to Jordan decomposition in the tangent algebra.

Let $G \subseteq \mathrm{GL}(n)$ be a closed subgroup and $x \in \mathfrak{g} \subseteq \mathfrak{gl}(n)$. Define $G(x)$ as the smallest closed subgroup in $\mathrm{GL}(n)$ such that x belongs to its tangent algebra. Clearly, $G(x) \subseteq G$, $G(x)$ is connected and $\dim G(x) \geq 1$ for any $x \neq 0$. If x is a nilpotent element of $\mathfrak{gl}(n)$, then $G(x)$ coincides with $\{\exp(tx)\}$; if x is semisimple, then $G(x)$ is a torus.

Theorem 1.7.20. *1) $x \in \mathfrak{g} \Rightarrow x_s, x_n \in \mathfrak{g}$;*

2) if $\phi : G \to G_1$ is a homomorphism, then $d\phi(x_s) = d\phi(x)_s$, $d\phi(x_n) = d\phi(x)_n$.

Proof. The proof is parallel to the proofs of Theorems 1.7.13 and 1.7.18, so we just sketch it. Let us take the stabilizer (in $\mathrm{GL}(n)$) of the element x_s under the adjoint action. Its tangent algebra is the centralizer of x_s in $\mathfrak{gl}(n)$, and thus contains x_n. This shows that $G(x_n)$ stabilizes x_s. We claim that $G(x_n)$ normalizes $G(x_s)$. Indeed, if $gG(x_s)g^{-1} \neq G(x_s)$ for some $g \in G(x_n)$, then $\mathrm{Lie}(gG(x_s)g^{-1}) \neq \mathrm{Lie}(G(x_s))$ and

$$x_s \in \mathrm{Lie}(gG(x_s)g^{-1} \cap G(x_s)) = \mathrm{Lie}(gG(x_s)g^{-1}) \cap \mathrm{Lie}(G(x_s)),$$

a contradiction with minimality of $G(x_s)$.

The same arguments show that $G(x_s)$ normalizes $G(x_n)$. Since $G(x_s) \cap G(X_n) = \{e\}$, we again have that $G(x_s, x_n) := \langle G(x_s), G(x_n) \rangle$ is a closed subgroup isomorphic to $G(x_s) \times G(x_n)$. Moreover, $G(x) \subseteq G(x_s, x_n)$. Using projections we get $G(x_n) \subseteq G(x) \subseteq G$, thus $x_n \in \mathfrak{g}$ and $x_s = x - x_n \in \mathfrak{g}$.

Concerning 2), one easily checks that

$$\phi(G(x)) = G(\phi(x)), \ \phi(G(x_s)) \subseteq G(d\phi(x)_s), \quad \phi(G(x_n)) \subseteq G(d\phi(x)_n).$$

This implies that $d\phi(x_s)$ is semisimple, $d\phi(x_n)$ is nilpotent and by Lemma 1.5.10 they commute, thus form additive Jordan decomposition of $d\phi(x)$. □

Corollary 1.7.21. *Additive Jordan decomposition $x = x_s + x_n$ for an element $x \in \mathfrak{g} = \mathrm{Lie}(G)$ is well-defined.*

Theorem 1.7.22. *Let $G \subset \mathrm{GL}(n)$ be a unipotent subgroup. Then any element $x \in \mathfrak{g}$ is nilpotent and the map $\exp : \mathfrak{g} \to G$ is an isomorphism of algebraic varieties.*

Proof. If $x = x_s + x_n$ with $x_s \neq 0$, then $G(x_s) \subset G$, where $G(x_s)$ is a torus, a contradiction. Thus the map $\exp : \mathfrak{g} \to \mathrm{GL}(n)$ is defined. Moreover, for any $x \in \mathfrak{g}$ the subgroup $G(x) = \{\exp(tx) : t \in \mathbb{K}\}$ is contained in G, hence $\exp : \mathfrak{g} \to G$.

Conversely, for any $A \in G$ the subgroup $G(A) = \{A^t : t \in \mathbb{K}\}$ is contained in G. Set $B = \ln(A)$. The tangent vector to the curve

$$A^t = E + \frac{tB}{1!} + \frac{t^2B^2}{2!} + \dots$$

at $t = 0$ is B. This shows that $B \in \mathfrak{g}$ and ln sends G to $\mathfrak{g}$. Since G is a closed subset of $\mathrm{GL}(n)$, Lemma 1.7.4 completes the proof. □

Corollary 1.7.23. *Any unipotent group G is isomorphic (as a variety) to an affine space. In particular, G is connected.*

Proposition 1.7.24. *Let G be a commutative unipotent group. Then $\exp : \mathfrak{g} \to G$ is an isomorphism of algebraic groups, where $\mathfrak{g}$ is considered as an additive group of the underlying vector space.*

Proof. Since $\mathfrak{g}$ is a commutative Lie algebra (Remark 1.5.7), any linear operators $B, C \in \mathfrak{g}$ commute, and $\exp(B + C) = \exp(B)\exp(C)$. □

Corollary 1.7.25. *Any commutative unipotent group of dimension m is isomorphic to $(G_a)^m$.*

Exercises to subsection 1.7.

Exercise 1.7.26. Prove that any connected one-dimensional algebraic group is isomorphic either to G_a or to G_m.

Exercise 1.7.27. Prove that the product and the sum of two commuting semisimple (resp. nilpotent) operators is again semisimple (resp. nilpotent). Prove that the product of two commuting unipotent operators is again unipotent.

Exercise 1.7.28. Is the variety $\mathrm{Nil}(n)$ irreducible ?

Exercise 1.7.29. Assume that A is a degenerate matrix and define a multiplicative Jordan decomposition of A as in Definition 1.7.8 (ii). Does such a decomposition always exist and is it unique ?

Exercise 1.7.30. Prove that for any $A \in \mathrm{Mat}(n \times n)$ there is a polynomial $f(x)$ such that $f(A) = A_s$ and $f(0) = 0$.

Exercise 1.7.31. Find an element $A \in \mathrm{GL}(n)$ with $G(A) = D(n)$.

Exercise 1.7.32. What is the maximal possible dimension of a subgroup of the form $G(A)$, where $A \in \mathrm{GL}(n)$?

Exercise 1.7.33. Assume that all elements of $\mathrm{Lie}(G)$ are nilpotent. Prove that G^0 is a unipotent group.

Exercise 1.7.34. Let G_1 and G_2 be connected algebraic groups. Show that for a homomorphism of Lie algebras $\tau : \mathrm{Lie}(G_1) \to \mathrm{Lie}(G_2)$, the equalities $\tau(x_s) = \tau(x)_s$ and $\tau(x_n) = \tau(x)_n$ do not hold in general. In particular, not any homomorphism of tangent algebras may be "integrated" to a homomorphism of corresponding connected groups.

Exercise 1.7.35 (*). Does there exist an algebraic group G such that all elements of G^0 are unipotent, but there is a connected component gG^0 of G consisting of semisimple elements ?

Exercise 1.7.36 (*). Prove that any commutative algebraic group G is isomorphic to $G_s \times G_u$, where G_s is a quasitorus and G_u is a commutative unipotent group.

Exercise 1.7.37 (*). Give an example of a Lie subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$ and an element $A \in G$ such that $A_s, A_u \notin G$.

1.8. **Solvable groups. Maximal tori and Borel subgroups.**

Theorem 1.8.1 (Borel's Fixed Point Theorem (1956)). *Assume that a connected solvable algebraic group G acts on a complete (e.g., projective) variety. Then $X^G \neq \emptyset$.*

Proof. We proceed by induction on the length of the derived series of G. For the basis, assume that G is commutative. Then a closed orbit Y in X (see Corollary 1.3.6) is isomorphic to G/H, where H is a closed (normal) subgroup of G, thus Y is affine (Corollary 1.4.4). On the other hand, Y is irreducible (G is connected) and complete (as a closed subset of a complete variety). This proves that Y is a G-fixed point.

For a non-commutative G, consider the commutant $[G, G]$. We know that $[G, G]$ is a proper closed connected subgroup of G (Corollary 1.2.9). Since $[G, G]$ is normal in G, the group $G/[G, G]$ acts on $X^{[G,G]}$. By inductive hypothesis, the set $X^{[G,G]}$ is non-empty. By Proposition 1.3.9, $X^{[G,G]}$ is closed in X, thus complete. Again by inductive hypothesis, the set $(X^{[G,G]})^{G/[G,G]} = X^G$ is non-empty. □

Theorem 1.8.2 (The Lie-Kolchin Theorem (1948)). *Let G be a connected solvable algebraic group and $\rho : G \to \mathrm{GL}(V)$ be a rational representation. Then there is a non-zero vector $v \in V$ such that $\rho(g)v = \chi(g)v$ for some $\chi \in \mathbb{X}(G)$ and any $g \in G$.*

Proof. On may assume that V is finite-dimensional. The action $G : \mathbb{P}(V)$ (see Lemma 1.3.14) has a G-fixed point. This point corresponds to a G-invariant line L in V, and G acts on L via some character. □

The Lie-Kolchin Theorem admits the following useful reformulation.

Theorem 1.8.3. *Let G be a connected solvable algebraic group and $\rho : G \to \mathrm{GL}(V)$ be a rational finite-dimensional representation. Then there is an element $A \in \mathrm{GL}(V)$ such that*

$$A\rho(G)A^{-1} \subseteq B(n) = \left\{ \begin{pmatrix} * & * & \dots & * \\ 0 & * & \dots & * \\ \dots & \dots & \dots & * \\ 0 & 0 & \dots & * \end{pmatrix} \right\}.$$

Proof. Since the variety $\mathcal{F}(V)$ of complete flags in V is projective, the natural action $G : \mathcal{F}(V)$ has a fixed point. Now we should take a basis in V compatible with a G-fixed flag. □

Corollary 1.8.4. *Let G be a connected solvable algebraic group and $U(G) \subset G$ be the set of all unipotent elements in G. Then $U(G)$ is a closed normal subgroup of G and $G/U(G)$ is a torus.*

Proof. We may assume that $G \subset \mathrm{GL}(n)$ and even $G \subseteq B(n)$. Here $U(G) = G \cap U(n)$ is a closed normal subgroup in G and the natural homomorphism $G \to B(n)/U(n) \cong T^n$ identifies $G/U(G)$ with a connected subgroup of T^n. □

Theorem 1.8.5. *Let $G \subset \mathrm{GL}(n)$ be a unipotent subgroup. Then there is $A \in \mathrm{GL}(n)$ such that*

$$AGA^{-1} \subseteq U(n) = \left\{ \begin{pmatrix} 1 & * & \dots & * \\ 0 & 1 & \dots & * \\ \dots & \dots & \dots & * \\ 0 & 0 & \dots & 1 \end{pmatrix} \right\}.$$

Proof. We claim that G is solvable. In order to prove it, we use induction on $\dim G$. If $\dim G = 1$, then $G = G(A)$ for any $A \in G$, $A \neq e$, and thus is commutative. If $\dim G > 1$, take a maximal proper closed subgroup $H \subset G$. It is connected (as a unipotent group) and solvable by inductive hypothesis. The Ad-representation defines a linear action $H : \mathfrak{g}/\mathfrak{h}$. By the Lie-Kolchin Theorem, there is a non-zero eigenvector $x + \mathfrak{h} \in \mathfrak{g}/\mathfrak{h}$. But all elements of H are unipotent, and $x + \mathfrak{h}$ is H-fixed. The condition $\mathrm{Ad}(H)x \subseteq x + \mathfrak{h}$ implies $\mathrm{ad}(\mathfrak{h})x \subseteq \mathfrak{h}$, or $[x, \mathfrak{h}] \subseteq \mathfrak{h}$. The one-dimensional subgroup $G(x)$ is not contained in H ($x \notin \mathfrak{h}$) and normalizes H. By maximality of H, the semidirect product $G(x) \leftthreetimes H$ coincides with G. We have proved that H is normal in G and $\dim G/H = 1$. Hence H and G/H are solvable, and so is G.

Applying Theorem 1.8.3, we find $A \in \mathrm{GL}(n)$ such that $AGA^{-1} \subseteq B(n)$. But all element of G are unipotent, so $AGA^{-1} \subseteq U(n)$. □

Corollary 1.8.6. *Any unipotent group is solvable.*

In fact, we may say more.

Corollary 1.8.7 (of the proof of Theorem 1.8.5)**.** *If G is a unipotent group, then there is a sequence of subgroups*

$$\{e\} = G_0 \lhd G_1 \lhd \cdots \lhd G_k = G,$$

such that $G_i/G_{i-1} \cong G_a$.

Corollary 1.8.8. *The subgroup $U(n)$ is a maximal (with respect to inclusion) unipotent subgroup of* $\mathrm{GL}(n)$*.*

Corollary 1.8.9. $\mathbb{X}(G) = 0$ *for any unipotent group G.*

Proof. Assume that $\chi : G \to \mathbb{K}^\times$ is a non-trivial character. Then there is a number i with $\chi(G_{i-1}) = 1$ and $\chi\mid_{G_i}$ being non-trivial, so χ is a non-trivial character of the group G_i/G_{i-1}, a contradiction with Proposition 1.7.16. □

Now we are ready to prove the structural theorem for connected solvable groups.

Theorem 1.8.10. *Let G be a connected solvable algebraic group. Then there is a subtorus $T \subseteq G$ such that $G = T \leftthreetimes U(G)$.*

Proof. By Corollary 1.8.4, the quotient $G/U(G)$ is a torus H.

Lemma 1.8.11. *For any torus T^m, there is $t_0 \in T^m$ such that $G(t_0) = T^m$.*

Proof. By Theorem 1.6.15, it is sufficient to find an element $t_0 \in T^m$ such that any character of T^m is not equal to 1 at t_0. On may take $t_0 = (p_1, \ldots, p_n)$, where $p_1, \ldots, p_n$ are pairwise different (positive) prime integers. □

Now take $t_0 \in H$ with $G(t_0) = H$, consider the projection $\pi : G \to H$ and fix $g \in G$ with $\pi(g) = t_0$. If $g = g_s g_u$ is Jordan decomposition, then $t_0 = \pi(g_s)\pi(g_u)$. Hence $\pi(g_u) = e$, and one may assume that $g = g_s$ is semisimple. The subgroup $G(g)^0$ is a torus of dimension $\geq \dim H$, because $\pi(G(g)^0)$ can not be a proper subgroup of H. By dimension reasons, $G(g)^0 \leftthreetimes U(G)$ coincides with G, and $G(g)^0$ is the desired subtorus T. □

Definition 1.8.12. Let G be an algebraic group. A subtorus $T \subseteq G$ is called a *maximal torus* of G if it is not contained in any other subtorus of G.

Example 1.8.13. The subgroup $D(n)$ is a maximal torus of $\mathrm{GL}(n)$.

Proposition 1.8.14. *Any two maximal tori of a connected solvable algebraic group G are conjugate.*

Proof. We know that $G = T \leftthreetimes U(G)$.

Lemma 1.8.15. *Any semisimple element $s \in G$ is conjugate to an element of T.*

Proof. Consider the homogeneous G-space G/T. An element $s \in G$ is conjugate to an element of T if and only if s has a fixed point on G/T. The projection

$$\psi := \pi\mid_U (G) : U(G) \to G/T$$

is bijective, and thus is an isomorphism of varieties. By Theorem 1.7.22, the map

$$\phi := \psi \circ \exp : \mathrm{Lie}(U(G)) \to G/T$$

is also an isomorphism. Moreover, the torus T acts on $\mathrm{Lie}(U(G))$ via adjoint action, $t \exp(A) t^{-1} = \exp(tAt^{-1})$, and thus the morphism ϕ is T-equivariant with respect to the action $T : G/T$, $t \cdot uT = tuT = tut^{-1}T$. This shows that G/T may be identified with an affine space $\mathbb{A}^m$ with a linear T-action.

Suppose that $U(G)$ is commutative. Then $\exp : \mathrm{Lie}(U(G)) \to U(G)$ is an isomorphism of groups (Proposition 1.7.24) and, via our identification $G/T \cong \mathbb{A}^m$, the $U(G)$-action on G/T is the action by parallel translations on $\mathbb{A}^m$. Finally, the group G acts on $\mathbb{A}^m$ via affine transformations, and thus preserves the finite-dimensional subspace A of affine functions in $\mathcal{O}(\mathbb{A}^m)$. By assumption, the element s is semisimple, and there is a basis $\{f_1, \ldots, f_{m+1}\}$ in A consisting of s-eigenvectors. Renumbering, one may suppose that the linear parts of $f_1, \ldots, f_m$ are linearly independent. The hyperplanes $f_1 = 0, \ldots, f_m = 0$ are s-invariant and their intersection is an s-fixed point.

If $U(G)$ is not commutative, we proceed by induction. The commutant $[U(G), U(G)]$ is a closed normal subgroup of G, and for the group

$$G/[U(G), U(G)] \cong T \leftthreetimes U(G)/[U(G), U(G)]$$

the element $s[U(G), U(G)]$ is conjugate to an element of T. Thus for some $g \in G$, one has $gsg^{-1} \in T \leftthreetimes [U(G), U(G)]$. Again by inductive hypothesis, this element is conjugate to an element of T. □

The condition $s \in gTg^{-1}$ implies $G(s) \subseteq gTg^{-1}$. This proves that any subgroup $G(s)$ is conjugate to a subgroup of T. By Lemma 1.8.11, any subtorus in G is conjugate to a subtorus in T. (In particular, T is a maximal torus in G.) □

Definition 1.8.16. Let G be an algebraic group. A maximal connected solvable subgroup of G is called a *Borel* subgroup.

Example 1.8.17. Theorem 1.8.3 implies that $B(n)$ is a Borel subgroup of $\mathrm{GL}(n)$.

Lemma 1.8.18. *Let G be an algebraic group and $B \subseteq G$ be a Borel subgroup of maximal dimension. Then the homogeneous space G/B is projective.*

Proof. By Theorem 1.4.1, there exist a rational G-module V and a non-zero $v \in V$ such that

$$B = \{g \in G : g \cdot v \in \langle v \rangle\}.$$

Let $\mathcal{F}_0$ be a closed subvariety of the flag variety $\mathcal{F}(V)$ consisting of complete flags with the first element $\langle v \rangle$. The subvariety $\mathcal{F}_0$ is B-invariant, and by Borel's Fixed Point Theorem B has a fixed point $F \in \mathcal{F}_0$. The stabilizer G_F coincides with B, because B is the stabilizer of F's first element. We claim that the G-orbit of F is closed in $\mathcal{F}(V)$. Indeed, any G-orbit in the closure of GF has smaller dimension.

On the other hand, the stabilizer of any point on $\mathcal{F}(V)$ is solvable, and B has a maximal dimension among (closed) solvable subgroups of G.

We have proved that $GF \cong G/B$ is closed in the projective variety $\mathcal{F}(V)$, thus is projective too. □

Theorem 1.8.19. *Let G be an algebraic group. Then*

(1) *any two Borel subgroups of G are conjugate;*
(2) *any two maximal tori of G are conjugate.*

Proof. Let B be a Borel subgroup of G and B_0 be a Borel subgroup of maximal dimension. By Borel's Fixed Point Theorem, B has a fixed point on G/B_0, or, equivalently, there is $g \in G$ with $gBg^{-1} \subseteq B_0$. By maximality of B, $gBg^{-1} = B_0$, and the first statement is proved.

Now take two maximal tori T_1 and T_2 in G. Since T_1 and T_2 are connected and solvable, there are Borel subgroups B_1 and B_2 with $T_1 \subseteq B_1$, $T_2 \subseteq B_2$. We know that $gB_1g^{-1} = B_2$ for some $g \in G$. By Proposition 1.8.14, the torus T_2 and gT_1g^{-1} are conjugate. □

Corollary 1.8.20. *Let G be an algebraic group and B a Borel subgroup of G. Then the homogeneous space G/B is projective.*

Exercises to subsection 1.8.

Exercise 1.8.21. Let B and B_1 be Borel subgroups in SL(2) and SO(3) respectively. Are B and B_1 isomorphic as algebraic groups ?

Exercise 1.8.22. Show by examples that the following conditions:

a) G is connected;
b) G is solvable;
c) X is complete

are essential in Borel's Fixed Point Theorem.

Exercise 1.8.23. Check that $N_{\mathrm{GL}(n)}B(n) = B(n)$.

Exercise 1.8.24. Let G be a solvable algebraic group and $U(G) \subset G$ be the set of all unipotent elements in G. Prove that $U(G)$ is a closed normal subgroup of G. Is it true that $G/U(G)$ is a quasitorus ?

Exercise 1.8.25. Let G be a unipotent group and X an affine G-variety. Prove that any G-orbit on X is closed.

Exercise 1.8.26. A subgroup P of an algebraic group G is said to be *parabolic* if P contains a Borel subgroup of G. Prove that the homogeneous space G/H is projective if and only if H is parabolic.

Exercise 1.8.27 (*). Classify up to conjugation parabolic subgroups of GL(n).

Exercise 1.8.28 (*). Let G be an algebraic group acting on a variety X and B be a Borel subgroup of G. Assume that $Y \subseteq X$ is a closed B-invariant subset. Prove that $GY := \{g \cdot y : g \in G, y \in Y\}$ is closed in X.

Exercise 1.8.29. Prove that any two maximal unipotent subgroups of an algebraic group G are conjugate.

Exercise 1.8.30. Show that two maximal connected commutative subgroups of G need not be conjugated.

Exercise 1.8.31 (*). Describe Borel subgroups and maximal tori in SO(n) and Sp($2n$).

Exercise 1.8.32 (**). Let G be a solvable algebraic group and H a closed subgroup of G. Prove that the homogeneous space G/H is affine.

1.9. **Reductive groups.** In the present section we first introduce two concepts of the radical for algebraic groups.

Lemma 1.9.1. *Let F be a solvable (resp. unipotent) algebraic group and $\phi : G \to F$ be a surjective homomorphism. The group G is solvable (resp. unipotent) if and only if so is* $\mathrm{Ker}\phi$.

Proof. The statement concerning solvability is standard. If $s \in G$ is a semisimple element, then $\phi(s)$ is semisimple, so $\phi(s) = e$ and $s \in \mathrm{Ker}\phi$. This proves that G contains no semisimple elements (i.e., is unipotent) if and only if so does $\mathrm{Ker}\phi$. □

Lemma 1.9.2. *Let G be an algebraic group and H_1, H_2 two normal closed connected solvable (resp. unipotent) subgroups of G. Then the subgroup H_1H_2 possesses the same properties.*

Proof. The image of the map $H_1 \times H_2 \to G$, $(h_1, h_2) \to h_1h_2$ satisfies the conditions of Proposition 1.2.6, so H_1H_2 is a closed connected subgroup of G. Clearly, it is normal. Applying Lemma 1.9.1 to the homomorphism

$$H_1H_2 \to H_1H_2/H_2 \cong H_1/(H_1 \cap H_2),$$

we get that H_1H_2 is solvable (resp. unipotent). □

Definition 1.9.3. The *radical* $R(G)$ of an algebraic group G is the largest (closed) connected normal solvable subgroup of G.

Remark 1.9.4. By Lemma 1.9.2, the subgroup $R(G)$ is well-defined. Moreover, $R(G) = R(G^0)$ because $R(G) \subseteq R(G^0)$ and $R(G^0)$ is a normal subgroup of G.

Definition 1.9.5. An algebraic group G is *semisimple* if $R(G) = \{e\}$.

Lemma 1.9.6. *If G is an algebraic group, then $G/R(G)$ is semisimple.*

Proof. Let $\pi : G \to G/R(G)$ be the projection. Assume that $R(G/R(G)) \neq \{e\}$. Then it has positive dimension. By Lemma 1.9.1, the preimage $H := \pi^{-1}(R(G/R(G)))$ is a normal solvable subgroup of G of positive dimension, thus $H^0 \subseteq R(G)$ and $\pi(H)$ is finite, a contradiction. □

Proposition 1.9.7. *An algebraic group G is semisimple if and only if its tangent algebra $\mathfrak{g}$ is semisimple.*

Proof. If $\mathfrak{r}(\mathfrak{g}) \neq 0$, then $\mathfrak{g}$ contains a non-zero commutative ideal $\mathfrak{a}$: take the last non-zero element in the sequence of commutants of $\mathfrak{r}(\mathfrak{g})$. Then $\mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}))$. On the other hand, the ideal $\mathfrak{z}(\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}))$ is the tangent algebra to the normal algebraic subgroup

$$H = \{g \in G : \mathrm{Ad}(g)y = y \text{ for all } y \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})\}.$$

By Proposition 1.9.8, the group H^0 is commutative, and thus $H^0 \subseteq R(G)$.

Conversely, if $R(G) \neq \{e\}$, then it contains a connected commutative subgroup that is normal in G, and its tangent algebra is a commutative ideal of $\mathfrak{g}$. □

Proposition 1.9.8. *If G is connected and $\mathfrak{g} = \mathrm{Lie}(G)$ is a commutative Lie algebra, then the group G is commutative.*

Proof. Let $\mathfrak{g}_s$ (resp. $\mathfrak{g}_n$) be the set on semisimple (resp. nilpotent) elements in $\mathfrak{g}$. Since $\mathfrak{g}$ is commutative, $\mathfrak{g}_s$ and $\mathfrak{g}_n$ are ideals in $\mathfrak{g}$. Let us fix a closed embedding $G \subseteq \mathrm{GL}(n)$. The map $\exp : \mathfrak{g}_n \to G$ defines an isomorphism between $\mathfrak{g}_n$ and some commutative unipotent subgroup G_u of G. Moreover, G_u is normal in G (because $\mathfrak{g}_n$ is an ideal).

One may assume that $\mathfrak{g}_s$ is the set of diagonal matrices in $\mathfrak{g}$ (Lemma 1.6.12). Then $\mathfrak{g}_s = \mathrm{Lie}(G \cap D(n)) = \mathrm{Lie}(G_s)$, where $G_s := (G \cap D(n))^0$ is a torus. Again, the subgroup G_s is normal in G. Since $G_u \cap G_s = \{e\}$, the map $\phi : G_u \times G_s \to G$, $\phi(g_1, g_2) = g_1 g_2$ is a injective homomorphism. Since any element of G admits multiplicative Jordan decomposition, ϕ is an isomorphism. □

Now we introduce a product of normal subgroups of G which is "almost" direct.

Definition 1.9.9. Let G be a connected algebraic group and $H_1, \ldots, H_k$ its closed connected subgroups. The group G is said to be an *almost direct* product of $H_1, \ldots, H_k$ (notation: $G = H_1 \cdot H_2 \cdot \ldots \cdot H_k$), if

(i) $h_i h_j = h_j h_i$ for any $h_i \in H_i$, $h_j \in H_j$, $i \neq j$;
(ii) the homomorphism
$$\phi : H_1 \times \cdots \times H_k \to G, \quad \phi(h_1, \ldots, h_k) = h_1 \ldots h_k,$$
is surjective and $\mathrm{Ker}\phi$ is a finite (central) subgroup of $H_1 \times \cdots \times H_k$.

Example 1.9.10. The group $\mathrm{GL}(n)$ is an almost direct product of $H_1 = \{\lambda E : \lambda \in \mathbb{K}^\times\}$ and $H_2 = \mathrm{SL}(n)$ with $\mathrm{Ker}\phi = \{(\epsilon E, \epsilon^{-1}E) : \epsilon^n = 1\}$.

Lemma 1.9.11. *Let G be a connected algebraic group and $H_1, \ldots, H_k$ be closed connected subgroups. Then G is an almost direct product of $H_1, \ldots, H_k$ if and only if $\mathfrak{g} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_k$ (a direct sum of Lie algebras).*

Proof. Assume that $k = 2$. If $G = H_1 \cdot H_2$, then $\mathfrak{h}_1 \cap \mathfrak{h}_2 = 0$, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are ideals of $\mathfrak{g}$ and the differential of $\phi : H_1 \times H_2 \to G$ defines an isomorphism $\mathfrak{h}_1 \oplus \mathfrak{h}_2 \cong \mathfrak{g}$.

Conversely, if $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, then H_1 and H_2 are normal subgroups of G with $H_1 \cap H_2$ being finite. A commutator $h_1 h_2 h_1^{-1} h_2^{-1}$ belongs to $H_1 \cap H_2$, and irreducibility of $H_1 H_2 H_1^{-1} H_2^{-1}$ implies that $h_1 h_2 = h_2 h_1$. Finally, $\phi : H_1 \times H_2 \to G$ is surjective, because of Theorem 1.2.12.

For $k > 2$ we use induction. If $G = H_1 \cdot \ldots \cdot H_k$, then
$$\mathfrak{g} = \mathrm{Lie}(H_1 \cdot \ldots \cdot H_{k-1}) \oplus \mathfrak{h}_k = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_k.$$

Conversely, if $\mathfrak{g} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_k$, then the subgroups H_i are normal in G, and, looking at
$$H_1 \times H_2 \times \cdots \times H_{k-1} \to H_1 H_2 \ldots H_{k-1},$$
one checks that $\mathrm{Lie}(H_1 H_2 \ldots H_{k-1}) = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \cdots \oplus \mathfrak{h}_{k-1}$. Hence we obtain
$$G = (H_1 \cdot \ldots . \cdot H_{k_1}) \cdot H_k = H_1 \cdot \ldots \cdot H_{k-1} \cdot H_k.$$
□

Definition 1.9.12. The *unipotent radical* $R^u(G)$ of an algebraic group G is the largest normal unipotent subgroup of G.

Remark 1.9.13. By Lemma 1.9.2, the subgroup $R^u(G)$ is well-defined and again $R^u(G) = R^u(G^0)$.

Definition 1.9.14. An algebraic group G is *reductive* if $R^u(G) = \{e\}$.

Example 1.9.15. Any finite group and any quasitorus are reductive.

Theorem 1.9.16. *Let G be an algebraic group. The following conditions are equivalent:*

(1) *G is reductive;*
(2) *$R(G)$ is a torus;*
(3) *$G^0 = T \cdot S$, where T is a torus and S is a connected semisimple subgroup;*
(4) *any finite-dimensional rational representation of G is completely reducible;*
(5) *G admits a faithful finite-dimensional completely reducible rational representation.*

Proof. (1)⇒(2) Since $R(G) = T \leftthreetimes U(R(G))$, where $U(R(G))$ is the set of all unipotent elements in $R(G)$, the subgroup $U(R(G))$ is invariant under any automorphism of G, and thus $U(R(G)) \subseteq R^u(G)$.

(2)⇒(3) Set $T = R(G) = R(G^0)$ and $S = [G^0, G^0]$. Let V be a finite-dimensional G-module. Since T is a torus, there is a decomposition

$$V = \bigoplus_{\lambda \in \mathbb{X}(T)} V_\lambda, \quad V_\lambda := \{v \in V : t \cdot v = \lambda(t)v\}.$$

The subgroup T is normal in G, thus G permutes the summands V_λ. On the other hand, the stabilizer of V_λ is closed in G, so G^0 preserves any V_λ. In particular, taking a faithful representation ρ, we see that T is contained in the center of G^0.

Any element of S is a product of commutators, so it acts on any V_λ with determinant 1. This implies that $S \cap T$ is finite.

By Lemma 1.9.6, the group G/T is semisimple. Then $\mathrm{Lie}(G/T)$ is semisimple and $[\mathrm{Lie}(G/T), \mathrm{Lie}(G/T)] = \mathrm{Lie}(G/T)$ (Corollary 4.0.40).

Lemma 1.9.17. *Let F be a connected algebraic group with* $[\mathrm{Lie}(F), \mathrm{Lie}(F)] = \mathrm{Lie}(F)$. *Then $[F, F] = F$.*

Proof. If $[F, F]$ is a proper subgroup, then $F/[F, F]$ is an Abelian group of a positive dimension, $\mathrm{Lie}(F/[F, F]) = \mathrm{Lie}(F)/\mathrm{Lie}([F, F])$ is commutative, and thus $[\mathrm{Lie}(F), \mathrm{Lie}(F)] \subseteq \mathrm{Lie}([F, F])$. □

We have proved that the group G^0/T coincides with its commutant, and the natural projection $\pi : S \to G^0/T$ is surjective, with $\mathrm{Ker}(\pi)$ being finite. This proves that S is semisimple ($\mathrm{Lie}(S) = \mathrm{Lie}(G^0/T)$, use Proposition 1.9.7) and $G^0 = T \cdot S$.

(3)⇒(4) Take any finite-dimensional G-module V and again consider the decomposition $V = \oplus_{\lambda \in \mathbb{X}(T)} V_\lambda$ with respect to T. Any subspace V_λ is G^0-invariant. We claim that V_λ is a completely reducible S-module. Indeed, Theorem 1.5.17 reduces the problem to representations of Lie algebras, where the statement is know as Weyl's Theorem (see Theorem 4.0.41).

We have obtained that V is completely reducible G^0-module. Now let U be a G-invariant subspace of V and $p : V \to U$ be a G^0-equivariant projection. For any $g \in G$ define an operator $p_g(v) := g \cdot p(g^{-1} \cdot v)$. Fix a representative g_i in any connected component of G.

Lemma 1.9.18. *The operator*

$$P := \frac{1}{|G/G^0|} \sum_i p_{g_i} : V \to V$$

is a G-equivariant projection on U.

Proof. Clearly, $P : V \to U$, and it is sufficient to check that $P(u) = u$ for any $u \in U$. But

$$P(u) = \frac{1}{|G/G^0|} \sum_i g_i \cdot p(g_i^{-1} \cdot u) = \frac{1}{|G/G^0|} \sum_i g_i \cdot (g_i^{-1} \cdot u) = u.$$

□

Finally, the kernel of P is a G-invariant subspace complementary to U.

(4)⇒(5) Follows from Theorem 1.3.20.

(5)⇒(1) Let $\rho : G \to \mathrm{GL}(V)$ be a faithful completely reducible representation . Suppose that $R^u(G) \neq \{e\}$. By Theorem 1.8.5, the subspace $V^{R^u(G)} \neq 0$. But $R^u(G)$ is normal in G, and $V^{R^u(G)}$ is G-invariant. By complete reducibility, there is a complementary G-invariant subspace W:

$$V = V^{R^u(G)} \oplus W.$$

The subspace W is non-zero (ρ is faithful), and, again by Theorem 1.8.5, $W^{R^u(G)} \neq 0$, a contradiction with $V^{R^u(G)} \cap W = 0$.

□

Corollary 1.9.19. *For a reductive G, $R(G)$ is a central subtorus of G^0.*

Corollary 1.9.20. *All classical groups are reductive.*

Proof. Since the tautological representation of a classical group is completely reducible, we may apply (5). (In fact, the tautological representation is almost always irreducible, see Exercise 1.9.34.) □

Definition 1.9.21. A connected algebraic group G is said to be *simple*, if $\mathrm{Lie}(G)$ is a simple Lie algebra.

Warning: A simple algebraic group G may contain a proper normal subgroup: take $G = \mathrm{SL}(2)$.

Proposition 1.9.22. *For any reductive group G, one has*

$$G^0 = T \cdot G_1 \cdot \ldots \cdot G_k,$$

where T is a torus and G_i are simple subgroups.

Proof. For the semisimple part $S \subseteq G^0$, one has $\mathrm{Lie}(S) = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$, where $\mathfrak{g}_i$ are simple ideals (Theorem 4.0.39). We claim that $\mathfrak{g}_i$ is an algebraic Lie subalgebra of $\mathrm{Lie}(S)$. Indeed, the center of a simple Lie algebra is trivial, and

$$\mathfrak{g}_i = \{x \in \mathrm{Lie}(S) : [x, y] = 0 \text{ for all } y \in \mathfrak{g}_j, \quad j \neq i\}.$$

Thus $\mathfrak{g}_i = \mathrm{Lie}(S_i)$, where

$$S_i := \{g \in S : \mathrm{Ad}(g)y = y \text{ for all } y \in \mathfrak{g}_j, \quad j \neq i\}.$$

Lemma 1.9.11 implies that $S = G_1 \cdot \ldots \cdot G_k$, where $G_i := S_i^0$, and, finally, $G^0 = T \cdot G_1 \cdot \ldots \cdot G_k$. □

We finish this section with complete reducibility of any rational G-module for a reductive G.

Let $\Lambda = \Lambda(G)$ be the set of isomorphism classes of rational finite-dimensional simple G-modules. For any rational G-module V one can define a submodule V_λ as the sum of all simple submodules of V whose class is λ.

Definition 1.9.23. The submodule V_λ is called the *isotypic component* of type λ of the module V.

Theorem 1.9.24. *Let G be a reductive group and V a rational G-module. Then*

$$V = \oplus_{\lambda \in \Lambda} V_\lambda.$$

Proof. Since any vector $v \in V$ lies in a finite-dimensional submodule, Theorem 1.9.16 (4) implies that $v \in \sum_{\lambda \in \Lambda} V_\lambda$.

Suppose that $v_1 + \cdots + v_k = 0$, where non-zero v_i belong to different V_{λ_i}. There is a finite-dimensional G-submodule W that contains $v_1, \ldots, v_k$. Consider a decomposition

$$W = W_1 \oplus \cdots \oplus W_s,$$

where W_i are simple G-modules. We claim that $V_\lambda \cap W$ is a sum of some W_i. Indeed, if U is a simple G-submodule in W, then, by the Schur Lemma, U has zero projection on any W_i that is not isomorphic to U. This shows that the sum $W = \sum_\lambda V_\lambda \cap W$ is direct, a contradiction. □

Remark 1.9.25. The decomposition $V = \oplus_{\lambda \in \Lambda} V_\lambda$ is called the *isotypic decomposition* of a rational G-module V.

Exercises to subsection 1.9.

Exercise 1.9.26. Classify all closed subgroups of positive dimension in SL(2) up to conjugation.

Exercise 1.9.27. Fix a sequence $1 \le k_1 < k_2 < \cdots < k_s = n$ of positive integers and the standard basis $e_1, \ldots, e_n$ in $\mathbb{K}^n$. Let $P_{k_1,\ldots,k_s}$ be the stabilizer in $\mathrm{GL}(n)$ of the flag

$$\{0\} \subset \langle e_1, \ldots, e_{k_1} \rangle \subset \langle e_1, \ldots, e_{k_2} \rangle \subset \cdots \subset \langle e_1, \ldots, e_{k_s} \rangle = \mathbb{K}^n.$$

Calculate $R(P_{k_1,\ldots,k_s})$ and $R^u(P_{k_1,\ldots,k_s})$.

Exercise 1.9.28. Prove that SO(4) is an almost direct product of two closed subgroups isomorphic to SL(2).

Exercise 1.9.29. Let G be an algebraic group. Prove that $G/R^u(G)$ is reductive.

Exercise 1.9.30. Do there exist connected algebraic groups G_1 and G_2 such that G_1 is reductive, G_2 is not, and $\mathrm{Lie}(G_1) \cong \mathrm{Lie}(G_2)$?

Exercise 1.9.31. Show that a direct product, a normal subgroup and a quotient group of a reductive group is reductive.

Exercise 1.9.32. Let G be a reductive group and $\phi : F \to G$ be a surjective homomorphism with finite kernel. Show that F is reductive.

Exercise 1.9.33. Prove that $\mathrm{Lie}(R(G)) = \mathfrak{r}(\mathfrak{g})$.

Exercise 1.9.34. Let V be the tautological module of a classical group G (recall that $G = \mathrm{GL}(n), \mathrm{SL}(n), \mathrm{O}(n), \mathrm{SO}(n), \mathrm{Sp}(2n)$). Prove that V is irreducible with the only exception $G = \mathrm{SO}(2)$.

Exercise 1.9.35 (*). List all classical groups that are semisimple. Which of them are simple ?

Exercise 1.9.36. Let H and N be closed connected subgroups of an algebraic group G. Assume that H normalizes N. Then the subgroup HN is closed in G and $\mathrm{Lie}(HN) = \mathrm{Lie}(H) + \mathrm{Lie}(N)$.

Exercise 1.9.37 (*). Assume that all elements of an algebraic group G are semisimple. Prove that G^0 is a torus.

2. Invariant Theory

2.1. **Finite generation.** Let G be an algebraic group and X a G-variety. Recall that the algebra of regular functions $\mathcal{O}(X)$ is a rational G-module with respect to the action:

$$(g \cdot f)(x) := f(g^{-1} \cdot x).$$

Define $\mathcal{O}(X)^G := \{f \in \mathcal{O}(X) : g \cdot f = f \text{ for all } g \in G\}$. Clearly, $\mathcal{O}(X)^G$ is a subalgebra in $\mathcal{O}(X)$ and the restriction of an element $f \in \mathcal{O}(X)^G$ to a G-orbit in X is a constant. The algebra $\mathcal{O}(X)^G$ is called the *algebra of invariants* of the action $G : X$, and an element $f \in \mathcal{O}(X)^G$ is called an *invariant*.

Theorem 2.1.1 (Hilbert's Theorem on Invariants (1890))**.** *Let G be a reductive algebraic group and X an affine G-variety. Then the algebra of invariants $\mathcal{O}(X)^G$ is finitely generated.*

Proof. Consider the isotypic decomposition of the G-module $\mathcal{O}(X)$:

$$\mathcal{O}(X) = \oplus_{\lambda \in \Lambda(G)} \mathcal{O}(X)_\lambda.$$

If $\lambda = 0$ is a class of the one-dimensional G-modules with trivial G-action, then $\mathcal{O}(X)_0 = \mathcal{O}(X)^G$. Consider a G-invariant subspace

$$\mathcal{O}(X)_G := \oplus_{\lambda \neq 0} \mathcal{O}(X)_\lambda.$$

Clearly, $\mathcal{O}(X) = \mathcal{O}(X)^G \oplus \mathcal{O}(X)_G$.

Definition 2.1.2. The projection $R : \mathcal{O}(X) \to \mathcal{O}(X)^G$ along $\mathcal{O}(X)_G$ is called the *Reynolds operator* on $\mathcal{O}(X)$.

Let us collect some properties of the Reynolds operator.

Lemma 2.1.3. (i) *The map R is G-invariant;*
(ii) *$R(ff_1) = fR(f_1)$ for any $f \in \mathcal{O}(X)^G$, $f_1 \in \mathcal{O}(X)$;*
(iii) *any G-invariant subspace of $\mathcal{O}(X)$ is R-invariant.*

Proof. (i) Since the subspaces $\mathcal{O}(X)^G$ and $\mathcal{O}(X)_G$ are G-invariant, we have $R(g\cdot f) = R(f)$.

(ii) Set $f_1 = f_1^+ + f_1^-$, where $f_1^+ \in \mathcal{O}(X)^G$, $f_1^- \in \mathcal{O}(X)_G$. Then $ff_1 = ff_1^+ + ff_1^-$, $ff_1^+ \in \mathcal{O}(X)^G$. Since for any simple G-submodule $U \subset \mathcal{O}(X)$ of type λ the subspace $fU = \{fu : u \in U\}$ is also a simple G-submodule of the same type, one has

$$\mathcal{O}(X)^G \mathcal{O}(X)_G \subseteq \mathcal{O}(X)_G,$$

thus $ff_1^- \in \mathcal{O}(X)_G$, and $R(ff_1) = ff_1^+ = fR(f_1)$.

(iii) Let $W \subseteq \mathcal{O}(X)$ be a G-invariant subspace. The isotypic decomposition of W leads to $W = W^G \oplus W_G$, there $W^G \subseteq \mathcal{O}(X)^G$, $W_G \subseteq \mathcal{O}(X)_G$. Thus $f = f' + f''$ with $f' \in W^G$, $f'' \in W_G$, and $R(f) = f' \in W$. □

We come to the proof of Theorem 2.1.1 in the particular case, where X is a rational G-module V. Here the algebra $\mathcal{O}(V)$ is a polynomial algebra that has a natural grading $\mathcal{O}(V) = \oplus_{n \geq 0} \mathcal{O}(V)_n$, where $\mathcal{O}(V)_n$ is the span of monomials of degree n. Since G acts linearly, it preserves homogeneous components. In particular,

$$\mathcal{O}(V)^G = \oplus_{n \geq 0} \mathcal{O}(V)_n^G.$$

Put $I_1 := \oplus_{n>0} \mathcal{O}(V)_n^G$. Then $I := I_1 \mathcal{O}(V) \lhd \mathcal{O}(V)$ is the ideal of $\mathcal{O}(V)$ generated by invariants of positive degree. By the Hilbert Basis Theorem, any ideal of $\mathcal{O}(V)$ is finitely generated, thus $I = (f_1, \ldots, f_k)$. One may assume that the elements $f_1, \ldots, f_k$ are homogeneous and belong to I_1.

Lemma 2.1.4. *$f_1, \dots, f_k$ generate the algebra $\mathcal{O}(V)^G$.*

Proof. Take a homogeneous element $F \in \mathcal{O}(V)^G$. In order to prove that $F \in \mathbb{K}[f_1, \dots, f_k]$, we apply induction on $\deg F$. The case $\deg F = 0$ is obvious. If $\deg F > 0$, then $F \in I$, thus $F = a_1 f_1 + \dots + a_k f_k$ for some $a_i \in \mathcal{O}(V)$. By Lemma 2.1.3, one has

$$F = R(F) = R(a_1 f_1) + \dots + R(a_k f_k) = R(a_1) f_1 + \dots + R(a_k) f_k.$$

Since $\deg f_i > 0$, we may suppose that $R(a_i)$ are homogeneous invariants with $\deg R(a_i) < \deg F$. By inductive hypothesis, $R(a_i) \in \mathbb{K}[f_1, \dots, f_k]$, so $F \in \mathbb{K}[f_1, \dots, f_k]$. □

Now we are ready to prove Theorem 2.1.1 for arbitrary affine G-variety X. By Theorem 1.3.19, X may be realized as a closed G-invariant subvariety of some G-module V. The embedding $X \hookrightarrow V$ corresponds to the surjective G-equivariant restriction homomorphism $p : \mathcal{O}(V) \to \mathcal{O}(X)$. Clearly, the restriction of p to $\mathcal{O}(V)^G$ defines a homomorphism $p' : \mathcal{O}(V)^G \to \mathcal{O}(X)^G$.

Lemma 2.1.5. *Let V be an affine G-variety and X its closed G-invariant subvariety. Then the restriction homomorphism $p' : \mathcal{O}(V)^G \to \mathcal{O}(X)^G$ is surjective.*

Proof. For any non-zero $f \in \mathcal{O}(X)^G$, the preimage of the line $p^{-1}(\langle f \rangle)$ is a G-submodule $W \subset \mathcal{O}(V)$ that is mapped surjectively on $\langle f \rangle$. By the Schur Lemma, p sends any isotypic component of non-zero type in W to zero. Thus there is a G-invariant $F \in W$ such that $p(F) = f$. □

Since $\mathcal{O}(V)^G$ is finitely generated, so is its homomorphic image $\mathcal{O}(X)^G$, and Theorem 2.1.1 is proved. □

Unfortunately, the proof given above does not provide any algorithm for finding a generating set for $\mathcal{O}(X)^G$. It turns out to be a very difficult problem. We finish this section this a method that allows to find a generating set in many important cases. This method is called the *method of sections*.

Assume that $S \subset X$ is a closed subvariety. Define

$$Z(S) := \{g \in G : g \cdot s = s \text{ for all } s \in S\}, \quad N(S) := \{g \in G : g \cdot s \in S \text{ for all } s \in S\}.$$

Clearly, $Z(S)$ is a normal subgroup of the group $N(S)$, and the quotient group $W = W(S) := N(S)/Z(S)$ acts (algebraically) on S. The surjection $\mathcal{O}(X) \to \mathcal{O}(S)$ defines a homomorphism $\phi : \mathcal{O}(X)^G \to \mathcal{O}(S)^W$.

Lemma 2.1.6. *Suppose that there is an open dense subset $U \subseteq X$ such that for any $x \in U$ the orbit Gx intersects S. Then ϕ is injective.*

Proof.

$$f \in \mathcal{O}(X)^G, \phi(f) = 0 \ \Rightarrow \ f|_S \equiv 0 \ \Rightarrow \ f|_U \equiv 0 \ \Rightarrow f = 0.$$

□

Moreover, if $f_1, \dots, f_k \in \mathcal{O}(X)^G$ and $\phi(f_1), \dots, \phi(f_k)$ generate $\mathcal{O}(S)^W$, then ϕ is an isomorphism. In particular, $f_1, \dots, f_k$ generate $\mathcal{O}(X)^G$.

Example 2.1.7. Consider $G = \mathrm{GL}(n)$ and $X = \mathrm{Mat}(n \times n)$ with the adjoint G-action: $A \cdot M := AMA^{-1}$. Take S to be the subspace of diagonal matrices. Here $Z(S) = D(n)$ and $N(S)$ is the group of monomial matrices. Indeed, the standard basis in $\mathbb{K}^n$ is the only basis proper for any operator from S, and $N(S)$ may only permutes basis vectors. This shows that W is isomorphic to the permutation group Σ_n and it acts on S permuting the diagonal entries. Let $\sigma_1, \ldots, \sigma_n$ be elementary symmetric polynomials in standard coordinates on S. It is well known that $\sigma_1, \ldots, \sigma_n$ is a generating set for $\mathcal{O}(S)^W$.

In order to verify the condition of Lemma 2.1.6, one may take the open subset U of matrices with pairwise different eigenvalues. It is well known that any matrix from U is diagonalizable.

Now let $f_1, \ldots, f_n$ be the coefficients of the characteristic polynomial

$$P_A(x) = x^n + f_1(A)x^{n-1} + \cdots + f_n(A),$$

considered as polynomial functions on the space $\mathrm{Mat}(n \times n)$. The restriction of f_i to the subspace of diagonal matrices equals $(-1)^i \sigma_i$. This proves that

$$\mathcal{O}(\mathrm{Mat}(n \times n))^{\mathrm{GL}(n)} = \mathbb{K}[f_1, \ldots, f_n].$$

Moreover, we know that $\sigma_1, \ldots, \sigma_n$ generate $\mathcal{O}(S)^W$ freely, and so do $f_1, \ldots, f_n$.

Example 2.1.8. Let $G = \mathrm{SL}(n)$ and X be the space $\mathrm{Sym}(n)$ of symmetric $n \times n$-matrices with the action: $A \cdot M := AMA^T$. It is natural to consider the line aE, $a \in \mathbb{K}$ as the subvariety S. Here $Z(S) = \mathrm{SO}(n)$. If $A \in N(S)$, then $AEA^T = aE$. Taking determinant, we get $a^n = 1$. Thus $N(S) = C_n Z(S)$, where

$$C_n = \{\mathrm{diag}(-\delta, \delta, \ldots, \delta) : \delta^n = -1\},$$

and $W \cong \mathbb{Z}_n$ acts on S via multiplications by $\epsilon = \delta^2$, $\epsilon^n = 1$. This proves that $\mathcal{O}(S)^W = \mathbb{K}[F]$, $F(aE) = a^n$.

Since any non-degenerate symmetric matrix is $\mathrm{GL}(n)$-equivalent to E, its $\mathrm{SL}(n)$-orbit intersects S, and we may put U to be the set of non-degenerate symmetric matrices.

The determinant det defines a G-invariant function on X. Its restriction to S coincides with F, so

$$\mathcal{O}(\mathrm{Sym}(n))^{\mathrm{SL}(n)} = \mathbb{K}[\det].$$

Exercises to subsection 2.1.

Exercise 2.1.9. Give an example of a non-finitely generated subalgebra in $\mathbb{K}[x_1, x_2]$.

Exercise 2.1.10. Check that R is the only G-invariant projection of $\mathcal{O}(X)$ to $\mathcal{O}(X)^G$.

Exercise 2.1.11. Assume that G is finite and $f \in \mathcal{O}(X)$. Show that

$$R(F) = \frac{1}{|G|} \sum_{g \in G} g \cdot f.$$

Exercise 2.1.12. For the tautological action $\mathrm{GL}(n) : \mathbb{K}^n$ and $f \in \mathcal{O}(\mathbb{K}^n)$, show that $R(f) = f((0, \ldots, 0))$.

Exercise 2.1.13. Let $f(x_1, \ldots, x_n) \in \mathbb{K}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree k. Prove that there exist linear forms $l_1(x_1, \ldots, x_n), \ldots, l_N(x_1, \ldots, x_n)$ such that $f = l_1^k + \cdots + l_N^k$.

Exercise 2.1.14. Let G be a finite group and V a finite-dimensional G-module. Prove that $\mathcal{O}(V)^G$ is generated by homogeneous invariants of degree $\leq |G|$ (Noether's Theorem).

Exercise 2.1.15. Using the method of sections, find a generating set of the algebra of invariants for

(a) the tautological action $\mathrm{SO}(n) : \mathbb{K}^n$;

(b) the tautological action $\mathrm{Sp}(2n) : \mathbb{K}^{2n}$;

(c) the diagonal action $\mathrm{SL}(n) : \mathbb{K}^n \oplus \cdots \oplus \mathbb{K}^n$ (s times, $s \leq n$).

Exercise 2.1.16. Find a generating set of the algebra of invariants for the linear actions:

(a) $\mathbb{Z}_n : \mathbb{K}^2$, $\epsilon \cdot (x_1, x_2) = (\epsilon x_1, \epsilon x_2)$, $\epsilon^n = 1$;

(b) $\mathbb{Z}_n : \mathbb{K}^2$, $\epsilon \cdot (x_1, x_2) = (\epsilon x_1, \epsilon^{-1} x_2)$, $\epsilon^n = 1$.

Exercise 2.1.17. Find a generating set of the algebra of invariants for the linear action

$$\mathbb{K}^\times : \mathbb{K}^4, \quad t \cdot (x_1, x_2, x_3, x_4) = (t^3 x_1, t x_2, t^{-1} x_3, t^{-2} x_4).$$

2.2. **The quotient morphism and categorical quotients.** Let G be a reductive group and X an affine G-variety. Assume that $f_1, \dots, f_k$ generate the algebra $\mathcal{O}(X)^G$. Consider a morphism:

$$\phi : X \to \mathbb{K}^k, \quad \phi(x) = (f_1(x), \dots, f_k(x)).$$

Clearly, ϕ is constant on G-orbits. Moreover, the morphism $\pi : X \to \overline{\phi(X)}$ does not depend on the choice of generators $f_1, \dots, f_k$, because it may be realized as the morphism $X \to \operatorname{Spec}(\mathcal{O}(X)^G)$ corresponding to the inclusion $\mathcal{O}(X)^G \subset \mathcal{O}(X)$.

Definition 2.2.1. The morphism $\pi : X \to Y := \operatorname{Spec}(\mathcal{O}(X)^G)$ corresponding to the inclusion $\mathcal{O}(X)^G \subset \mathcal{O}(X)$ is called the *quotient morphism* for an affine G-variety X.

Let us summarize basic properties of the quotient morphism.

Theorem 2.2.2. (1) *π is surjective;*
(2) *if $Z \subseteq X$ is a closed G-invariant subvariety, then $\pi(Z)$ is closed in Y;*
(3) *if $\{Z_\alpha\}$ is a family of closed G-invariant subvarieties of X, then*

$$\bigcap_\alpha \pi(Z_\alpha) = \pi(\bigcap_\alpha Z_\alpha);$$

(4) *if $U \subseteq Y$ is an open subset, then $\pi_U^* : \mathcal{O}(U) \to \mathcal{O}(\pi^{-1}(U))^G$ is an isomorphism.*

Proof. (1) For any $y \in Y$ consider the ideal

$$I_y := \{f \in \mathcal{O}(Y) : f(y) = 0\} \lhd \mathcal{O}(Y) = \mathcal{O}(X)^G.$$

Applying the Reynolds operator, we get $(\mathcal{O}(X)I_y)^G \subset I_y$. In particular, $\mathcal{O}(X)I_y \neq \mathcal{O}(X)$. Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{O}(X)$ with $\mathcal{O}(X)I_y \subseteq \mathfrak{m}$. Then $\pi(x) = y$, where $x \in X$ is a point corresponding to the maximal ideal $\mathfrak{m}$.

(2) Let $\phi : Z \hookrightarrow X$ be the closed G-equivariant embedding. The restriction homomorphisms $\phi^* : \mathcal{O}(X) \to \mathcal{O}(Z)$ and $\phi^{*'} : \mathcal{O}(X)^G \to \mathcal{O}(Z)^G$ are surjective (Lemma 2.1.5). This proves that the morphism $\psi : Y_Z \to Y$, where $Y_Z := \operatorname{Spec}(\mathcal{O}(Z)^G)$, is a closed embedding,

$$\begin{array}{ccc} \mathcal{O}(Z) & \xleftarrow{\phi^*} & \mathcal{O}(X) \\ \uparrow & & \uparrow \\ \mathcal{O}(Z)^G & \xleftarrow{\phi^{*'}} & \mathcal{O}(X)^G \end{array} \qquad \begin{array}{ccc} Z & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle \pi_Z} & & \downarrow{\scriptstyle \pi} \\ Y_Z & \xrightarrow{\psi} & Y \end{array}$$

and $\pi(Z) = \psi(Y_Z)$, because the quotient morphism π_Z for the affine G-variety Z is surjective.

(3) Let $I_\alpha \lhd \mathcal{O}(X)$ be the ideal corresponding to Z_α. We claim that $(\sum_\alpha I_\alpha)^G = \sum_\alpha I_\alpha^G$. Indeed, if $F = f_1 + \dots + f_k \in (\sum_\alpha I_\alpha)^G$ with $f_i \in I_{\alpha_i}$, then $F = R(F) = R(f_1) + \dots + R(f_k)$ with $R(f_i) \in I_{\alpha_i}^G$ (Lemma 2.1.3 (iii)), and $F \in \sum_\alpha I_\alpha^G$. The opposite inclusion is obvious.

Suppose that $f \in \mathcal{O}(X)^G$ is identically zero on $\bigcap_\alpha Z_\alpha$. Then there is $m \in \mathbb{N}$ such that

$$f^m \in (\sum_\alpha I_\alpha)^G = \sum_\alpha I_\alpha^G,$$

and thus f vanishes on $\bigcap_\alpha \pi(Z_\alpha)$. Since both $\bigcap_\alpha \pi(Z_\alpha)$ and $\pi(\bigcap_\alpha Z_\alpha)$ are closed in Y, this proves the inclusion

$$\bigcap_\alpha \pi(Z_\alpha) \subseteq \pi(\bigcap_\alpha Z_\alpha).$$

The opposite inclusion holds set-theoretically.

(4) First assume that $U = Y_f \subseteq Y$ is a principal open subset. Here

$$\mathcal{O}(U) = (\mathcal{O}(X)^G)_f = \{\frac{f_1}{f^m} : f_1 \in \mathcal{O}(X)^G\}$$

and

$$\mathcal{O}(\pi^{-1}(U))^G = (\mathcal{O}(X)_f)^G = \{\frac{f_2}{f^k} : \frac{f_2}{f^k} \text{ is } G-\text{invariant}\}.$$

But then $f_2 \in \mathcal{O}(X)^G$ and $\mathcal{O}(U) = \mathcal{O}(\pi^{-1}(U))^G$.

Since any open U is a finite union of principal open subsets, the statement follows from

$$\mathcal{O}(U_1 \cup U_2) = \mathcal{O}(U_1) \cap \mathcal{O}(U_2) = \mathcal{O}(\pi^{-1}(U_1))^G \cap \mathcal{O}(\pi^{-1}(U_2))^G =$$
$$= (\mathcal{O}(\pi^{-1}(U_1)) \cap \mathcal{O}(\pi^{-1}(U_2)))^G = \mathcal{O}(\pi^{-1}(U_1 \cup U_2))^G.$$

□

In general, invariants do not separate all G-orbits in X. The following corollary shows that they do separate closed orbits.

Corollary 2.2.3. *Let $\pi : X \to Y$ be the quotient morphism. For any $y \in Y$, the fiber $\pi^{-1}(y)$ contains a unique closed G-orbit O_y. This orbit is contained in the closure of any orbit from $\pi^{-1}(y)$.*

Proof. Since $\pi^{-1}(y)$ is closed and G-invariant, it contains a closed G-orbit (Corollary 1.3.6). Let O_1 and O_2 be two closed G-orbits in $\pi^{-1}(y)$. Then $O_1 \cap O_2 = \emptyset$, and by Theorem 2.2.2 (3), $\pi(O_1) \cap \pi(O_2) = \emptyset$. But $\pi(O_1) = \pi(O_2) = y$, a contradiction. Finally, Corollary 1.3.6 also implies that the closure of any orbit from $\pi{-}1(y)$ contains a closed G-orbit, and it should be O_y.

□

Example 2.2.4. Consider $G = \mathrm{GL}(n)$ and $X = \mathrm{Mat}(n \times n)$ with the adjoint G-action: $A \cdot M := AMA^{-1}$. It was shown in Example 2.1.7 that here $Y = \mathbb{K}^n$ and the map π sends a matrix to the coefficients of its characteristic polynomial. A fiber of π is the set of matrices with fixed eigenvalues (with multiplicities). Any fiber contains a finite number of G-orbits which are parametrized by Jordan normal forms with the prescribed diagonal.

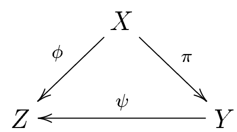
In particular, the fiber $\pi^{-1}(0, \ldots, 0)$ is the set $\mathrm{Nil}(n)$ of nilpotent matrices and

$$\pi^{-1}((-\binom{n}{1}, \binom{n}{2}, \ldots, (-1)^n\binom{n}{n}))$$

is the set $\mathrm{Uni}(u)$ of unipotent matrices. Moreover, if the discriminant of the polynomial $x^n + a_1x^{n-1} + \cdots + a_n$ is non-zero, then the fiber $\pi^{-1}(a_1, \ldots, a_n)$ consists of one G-orbit.

Now we are going to prove an important universal property of the quotient morphism.

Definition 2.2.5. Let G be an algebraic group and X a G-variety. A G-invariant morphism $\pi : X \to Y$ is said to be a *categorical quotient*, if for any G-invariant morphism $\phi : X \to Z$ there exists a unique morphism $\psi : Y \to Z$ such that the following diagram is commutative:



Remark 2.2.6. It follows from the definition that, if the categorical quotient exists, then it is unique (up to isomorphism), see Exercise 2.2.16.

The usual notation for the categorical quotient is $\pi : X \to X//G$.

Theorem 2.2.7. *Let G be a reductive group and X an affine G-variety. Then the quotient morphism $\pi : X \to Y$ is the categorical quotient.*

Proof. Let $\phi : X \to Z$ be a G-invariant morphism. We define firstly the desired map ψ as a map of sets: for any $y \in Y$ put $\psi(y) = \phi(\pi^{-1}(y))$. Clearly, it is the only way to make the above diagram commutative. But we need to explain that $\phi(\pi^{-1}(y))$ is one point. Indeed, the fiber $\pi^{-1}(y)$ contains a unique closed G-orbit O_y, and any other orbit $O \subseteq \pi^{-1}(y)$ contains O_y in its closure, thus $\phi(O) = \phi(O_y) = z$ for some $z \in Z$.

Now we prove that ψ is a continuous map.

Definition 2.2.8. A G-invariant subset $W \subseteq X$ is said to be *saturated* if

$$x \in X,\ w \in W,\ \overline{Gx} \cap \overline{Gw} \neq \emptyset \ \Rightarrow\ x \in W.$$

Lemma 2.2.9. *A subset $W \subseteq X$ is saturated if and only if there is a subset $W_1 \subseteq Y$ such that $W = \pi^{-1}(W_1)$.*

Proof. The condition $\overline{Gx_1} \cap \overline{Gx_2} \neq \emptyset$ is equivalent to "x_1 and x_2 are in the same fiber of π". So, W is saturated if and only if it consists of fibers of the quotient morphism. □

Take an open subset $U \subseteq Z$. Then $\phi^{-1}(U)$ is open and saturated in X. By Theorem 2.2.2 (2), the set $D := \pi(X \setminus \phi^{-1}(U))$ is closed in Y, thus $\psi^{-1}(U) = \pi(\phi^{-1}(U)) = Y \setminus D$ is open.

Finally we check that ψ is a morphism, i.e., for any open $U \subseteq Z$ and $f \in \mathcal{O}(U)$ the function $\psi^*(f)$ lies in $\mathcal{O}(\psi^{-1}(U))$. Theorem 2.2.2 (4) implies

$$\phi^*(f) \in \mathcal{O}(\phi^{-1}(U))^G = \mathcal{O}(\pi^{-1}(\psi^{-1}(U)))^G = \pi^*(\mathcal{O}(\psi^{-1}(U))) \ \Rightarrow\ \psi^*(f) \in \mathcal{O}(\psi^{-1}(U)).$$

□

We finish this section with the following unexpected alternative: for a G-module V the quotient space $V//G$ is either an affine space or a singular variety.

Proposition 2.2.10. *Let V be a finite-dimensional rational G-module. Then $\mathcal{O}(V)^G$ is a polynomial algebra if and only if $\pi(0)$ is a smooth point on $V//G$.*

Proof. If $\mathcal{O}(V)^G$ is a polynomial algebra, then $V//G = \mathrm{Spec}(\mathcal{O}(V)^G)$ is an affine space, and any its point is smooth. Conversely, suppose that $\pi(0) \in V//G$ is smooth. Let $\mathfrak{m} \lhd \mathcal{O}(V)^G$ be the maximal ideal corresponding to $\pi(0)$. We know that $T_{\pi(0)} V//G \cong (\mathfrak{m}/\mathfrak{m}^2)^*$. Hence, if $n = \dim V//G$, then $\dim \mathfrak{m}/\mathfrak{m}^2 = n$, and there are homogeneous elements $f_1, \dots, f_n \in \mathfrak{m}$ whose images form a basis of $\mathfrak{m}/\mathfrak{m}^2$.

Using induction on degree, one easily checks that $f_1, \ldots, f_n$ generate the ideal $\mathfrak{m}$. By Lemma 2.1.4, the elements $f_1, \ldots, f_n$ generate the algebra $\mathcal{O}(V)^G$. If there exists a non-zero polynomial $F(X_1, \ldots, X_n)$ such that $F(f_1, \ldots, f_n) \equiv 0$, then the transcendency degree of the field of quotients $Q\mathcal{O}(V)^G$ is less then n, a contradiction with $n = \dim V//G$. □

Exercises to subsection 2.2.

Exercise 2.2.11. Let G be an algebraic group and X an affine G variety. Suppose that the algebra $\mathcal{O}(X)^G$ is finitely generated. Give an example where the morphism $\pi : X \to \operatorname{Spec}(\mathcal{O}(X)^G)$ defined by the embedding $\mathcal{O}(X)^G \subset \mathcal{O}(X)$ is not surjective.

Exercise 2.2.12. Let $\pi : X \to Y$ be the quotient morphism from Example 2.2.4. Prove that the set of semisimple elements in any fiber $\pi^{-1}(y)$ is the unique closed $\mathrm{GL}(n)$-orbit O_y.

Exercise 2.2.13. Let G be a finite group, X a G-variety and $\pi : X \to X//G$ the quotient morphism. Prove that π is a finite morphism and any fiber of π is a G-orbit.

Exercise 2.2.14. Let G be a reductive group and X an irreducible affine G-variety. Show by an example that the components of a fiber of the quotient morphism $\pi : X \to X//G$ may have different dimension.

Exercise 2.2.15. Let $\pi : X \to X//G$ be the quotient morphism and $U \subseteq X//G$ be an open subset. Prove that

$$\pi\,|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$$

is the categorical quotient for the G-variety $\pi^{-1}(U)$.

Exercise 2.2.16. Prove that the categorical quotient is unique (up to isomorphism).

Exercise 2.2.17. Let G be an algebraic group, X a G-variety and $\pi : X \to X//G$ the categorical quotient. Prove that the morphism π is surjective.

Exercise 2.2.18. Consider an action $T^m : \mathbb{K}^n$, $t \cdot (x_1, \dots, x_n) = (\chi_1(t)x_1, \dots, \chi_n(t)x_n)$ with some $\chi_1, \dots, \chi_n \in \mathbb{X}(T^m)$. Prove that $\mathbb{K}^n//T^m$ is a point if and only if all χ_i are non-zero and the cone generated by $\chi_1, \dots, \chi_n$ in the space $\mathbb{X}(T^m) \otimes_{\mathbb{Z}} \mathbb{Q}$ is strictly convex.

Exercise 2.2.19 (Igusa's Criterion)**.** Let G be a reductive group, X an irreducible affine G-variety, and Y a normal irreducible affine variety. Assume that there is a dominant G-invariant morphism $\phi : X \to Y$ such that $\operatorname{codim}_Y(Y \setminus \phi(X)^0) \geq 2$ and there exists open $U \subseteq Y$ such that $\phi^{-1}(y)$ contains a dense G-orbit for any $y \in U$. (Here $\phi(X)^0$ denotes the maximal open subset of the image $\phi(X)$, see Theorem 3.0.24.) Prove that $\phi : X \to Y$ is the categorical quotient.

Exercise 2.2.20. Set $G = \mathrm{O}(n)$ and $X = \mathbb{K}^n \oplus \cdots \oplus \mathbb{K}^n$ (s times, $s \leq n$) with the diagonal G-action. Define $F_{ij}(v_1, \dots, v_s) := q(v_i, v_j)$. Prove that $\mathcal{O}(X)^G$ is a polynomial algebra generated by F_{ij}. Formulate and prove the corresponding statement for $G = \mathrm{Sp}(2n)$.

Exercise 2.2.21. Let G be an algebraic group and $H \subseteq G$ a reductive subgroup. Prove that the homogeneous space G/H is affine.

Exercise 2.2.22. Let G be an algebraic group, X a G-variety, and $\phi : X \to Z$ a G-invariant morphism. We say that two points $x, x' \in X$ are equivalent if there is a sequence of points on X: $x = x_1, x_2, \dots, x_k = x'$ such that $\overline{Gx_i} \cap \overline{Gx_{i+1}} \neq \emptyset$ for $i = 1, \dots, k-1$. Prove that $\phi(x) = \phi(x')$.

Exercise 2.2.23 (*)**.** Consider an action $T^m : \mathbb{K}^{n+1}$,

$$t \cdot (x_1, \dots, x_{n+1}) = (\chi_1(t)x_1, \dots, \chi_{n+1}(t)x_{n+1})$$

with some $\chi_1, \dots, \chi_{n+1} \in \mathbb{X}(T^m)$, $\chi_1 \neq \chi_2$. Prove that the corresponding action $T^m : \mathbb{P}^n$ admits the categorical quotient with $\mathbb{P}^n//T^m$ being a point.

2.3. **Rational invariants. Rosenlicht's Theorem.** Let G be an algebraic group and X an irreducible G-variety. There is a natural G-action on the field of rational functions $\mathbb{K}(X)$: $(g \cdot f)(x) = f(g^{-1} \cdot x)$. Consider the *field of rational invariants*:

$$\mathbb{K}(X)^G := \{f \in \mathbb{K}(X) : g \cdot f = f \text{ for all } g \in G\}.$$

In this situation, there is no problem with finite generation.

Proposition 2.3.1. *Let $\mathbb{K} \subset L$ be a finitely generated field extension. Then for any $\mathbb{K} \subset E \subseteq L$, the extension $\mathbb{K} \subset E$ is finitely generated.*

Proof. Let $Y_1, \dots, Y_r$ be a transcendency basis in E. It may be extended to a transcendency basis $Y_1, \dots, Y_r, Y_{r+1}, \dots, Y_k$ in L. Then $\mathbb{K}(Y_1, \dots, Y_k) \subseteq L$ is a finite algebraic extension of some degree N. Any element $\alpha \in E$ is algebraic over $\mathbb{K}(Y_1, \dots, Y_r)$. Let $F_\alpha(T)$ (resp. $H_\alpha(T)$) be the minimal polynomial of α over $\mathbb{K}(Y_1, \dots, Y_r)$ (resp. $\mathbb{K}(Y_1, \dots, Y_k)$). Then $H_\alpha(T)$ divides $F_\alpha(T)$, thus $H_\alpha(T)$ does not depend on $Y_{r+1}, \dots, Y_k$, and $F_\alpha(T) \equiv H_\alpha(T)$. In particular, the degree of α over $\mathbb{K}(Y_1, \dots, Y_r)$ is bounded by N. Applying the Primitive Element Theorem, we get that the degree of E over $\mathbb{K}(Y_1, \dots, Y_r)$ is bounded by N. □

Corollary 2.3.2. *For any algebraic group G and any irreducible G-variety X, the field $\mathbb{K}(X)^G$ is finitely generated over $\mathbb{K}$.*

The quotient of two regular invariants is a rational invariant, thus $Q\mathcal{O}(X)^G \subseteq \mathbb{K}(X)^G$. In general, we do not have an equality here.

Example 2.3.3. Consider an action $\mathbb{K}^\times : \mathbb{K}^2$, $t \cdot (x_1, x_2) = (tx_1, tx_2)$. In this case $\mathcal{O}(X)^G = \mathbb{K}$, but the function $\frac{x_1}{x_2}$ is a non-constant rational invariant.

Proposition 2.3.4. *Assume that X is an irreducible affine G-variety and one of the following conditions holds:*

(a) *G is finite;*
(b) *G is unipotent;*
(c) *G is connected, $\mathbb{X}(G) = 0$, $\mathcal{O}(X)$ is factorial, and $\mathcal{O}(X)^\times = \mathbb{K}^\times$.*

Then $Q\mathcal{O}(X)^G = \mathbb{K}(X)^G$.

Proof. (a) If $F \in \mathbb{K}(X)^G$, then

$$F = \frac{f_1}{f_2} = \frac{f_1(\prod_{g \neq e} g \cdot f_2)}{\prod_{g \in G} g \cdot f_2} \in Q\mathcal{O}(X)^G.$$

(b) Take $F = \frac{f_1}{f_2} \in \mathbb{K}(X)^G$. By Theorem 1.8.5, the linear span of the orbit Gf_2 contains a non-zero G-invariant $h = \sum_i \lambda_i (g_i \cdot f_2)$, $\lambda_i \in \mathbb{K}^\times$. Then

$$F = \frac{f_1}{f_2} = \frac{g_i \cdot f_1}{g_i \cdot f_2} = \frac{\lambda_i(g_i \cdot f_1)}{\lambda_i(g_i \cdot f_2)} = \frac{\sum_i \lambda_i(g_i \cdot f_1)}{\sum_i \lambda_i(g_i \cdot f_2)} \in Q\mathcal{O}(X)^G.$$

(c) Assume that

$$F = \frac{p_1^{a_1} \dots p_k^{a_k}}{q_1^{b_1} \dots q_s^{b_s}} \in \mathbb{K}(X)^G,$$

where $p_1, \dots, p_k, q_1, \dots, q_s$ are pairwise different primes. The G-action preserves this decomposition. On the other hand, G can not permute the factors (G is connected) and can not send a factor to an associated element ($\mathcal{O}(X)^\times = \mathbb{K}^\times$ and $\mathbb{X}(G) = 0$). Hence all $p_1, \dots, p_k, q_1, \dots, q_s$ are G-invariants. □

Now we came to the "best possible" notion of a quotient for a G-variety.

Definition 2.3.5. Let G be an algebraic group and X an irreducible G-variety. A G-invariant morphism $\pi : X \to Y$ is said to be a *geometric quotient*, if the following conditions hold:

(G1) π is surjective;
(G2) $\pi^{-1}(y)$ is a G-orbit for any $y \in Y$;
(G3) π is open;
(G4) $\pi_U^* : \mathcal{O}(U) \to \mathcal{O}(\pi^{-1}(U))^G$ is an isomorphism for any open $U \subseteq Y$.

Notation: $\pi : X \to X/G$.

By Proposition 1.3.8 and Theorem 3.0.26, if there is a G-invariant morphism $\pi : X \to Y$ satisfying condition (G2), then all orbits in X are closed and have the same dimension. Unfortunately, the later conditions are not sufficient for the existence of geometric quotient (Exercise 2.3.23).

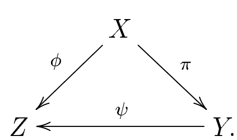
Proposition 2.3.6. *Let G be an algebraic group and X an irreducible G-variety. If there is a surjective G-invariant morphism $\pi : X \to Y$, where Y is a normal G-variety and $\pi^{-1}(y)$ is a G-orbit for any $y \in Y$, then π is a geometric quotient.*

Proof. Conditions (G1) and (G2) are included into assumptions. Condition (G3) follows from Theorem 3.0.32. Finally, (G4) follows from Corollary 3.0.29. □

Example 2.3.7. Let $G = \mathbb{K}^\times$ and $X = \mathbb{K}^n \setminus \{0\}$ with $t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$. By Proposition 2.3.6, the morphism $\pi : X \to \mathbb{P}^{n-1}$, $\pi((x_1, \ldots, x_n)) = [x_1 : ... : x_n]$ is a geometric quotient. In particular, for reductive G and quasiaffine X the variety X/G need not be quasiaffine.

Proposition 2.3.8. *A geometric quotient $\pi : X \to X/G$ is the categorical one.*

Proof. Let $\phi : X \to Z$ be a G-invariant morphism:



Define $\psi(y) = \phi(\pi^{-1}(y))$. The map ψ is continuous, because for any open $U \subseteq Z$ the subset $\psi^{-1}(U) = \pi(\phi^{-1}(U))$ is open in Y (use (G3)). Finally, for any $f \in \mathcal{O}(U)$ one has $\phi^*(f) \in \mathcal{O}(\phi^{-1}(U))^G = \pi^*(\mathcal{O}(\psi^{-1}(U)))$ (use (G4)). Since π^* is injective and $\phi^* = \pi^* \circ \psi^*$, the function $\psi^*(f)$ is contained in $\mathcal{O}(\psi^{-1}(U))$. □

Corollary 2.3.9. *If a geometric quotient $\pi : X \to X/G$ exists, then it is unique (up to isomorphism).*

We come to the main result of this section.

Theorem 2.3.10 (Rosenlicht's Theorem (1956))**.** *Let G be an algebraic group and X an irreducible G-variety. Then there is a non-empty open G-invariant subset $U \subseteq X$ which admits a geometric quotient $\pi : U \to U/G$.*

Proof. By Proposition 1.3.8, there is an open G-invariant subset $U \subseteq X$ such that all G-orbits on U are of the same dimension. Fix a generating set $f_1, \ldots, f_m$ of $\mathbb{K}(X)^G$. Reducing U, one may assume that $f_1, \ldots, f_m$ are regular on U. Moreover, $f_1, \ldots, f_m$ are regular on $\bigcup_{g \in G} gU$, so U may be supposed to be invariant.

Consider a subalgebra $\mathbb{K}[f_1, \ldots, f_m] \subseteq \mathbb{K}(X)^G$ and set $Y := \mathrm{Spec}(\mathbb{K}[f_1, \ldots, f_m])$, The inclusion $\mathbb{K}[f_1, \ldots, f_m] \subseteq \mathcal{O}(U)$ defines a dominant morphism $\pi : U \to Y$.

Reducing Y to a principal open subset, one may assume that Y is normal and π is surjective.

Consider a morphism

$$\phi : G \times U \to U \times U, \quad \phi(g, x) = (x, g \cdot x)$$

and two sets

$$A := \mathrm{Im}(\phi) = \{(x, x') \in U \times U : x' \in Gx\},$$
$$B := U \times_Y U = \{(x, x') \in U \times U : \pi(x) = \pi(x')\}.$$

Since π is G-invariant, we get $A \subseteq B$.

Lemma 2.3.11. *A is dense in B.*

Proof. Let V and V' be open affine subsets of U. It is sufficient to prove that

$$\psi : W := \phi^{-1}(V \times_Y V') \to V \times_Y V', \quad (g, x) \to (x, g \cdot x)$$

is dominant. Since U is irreducible, the set $V \times V'$ meets the diagonal Δ_U, and W is non-empty.

The variety $V \times_Y V'$ is affine, and we shall check that

$$\psi^* : \mathcal{O}(V \times_Y V') = \mathcal{O}(V) \otimes_{\mathcal{O}(Y)} \mathcal{O}(V') \to \mathcal{O}(W),$$
$$\psi^*(\sum_{i=1}^{s} u_i \otimes v_i) \to \left[(g, x) \to \sum_{i=1}^{s} u_i(x) v_i(g \cdot x)\right]$$

is injective. Suppose that $\sum_{i=1}^{s} u_i \otimes v_i$ lies in $\mathrm{Ker}(\psi^*)$. For any $g \in G$ the function

$$h_g(x) = \sum_{i=1}^{s} u_i(x) v_i(g \cdot x)$$

is a rational function on U. Since it vanishes on $V \cap g^{-1}V'$, it is identically zero. We claim that this condition implies that $F = \sum_{i=1}^{s} u_i \otimes v_i$ is a zero (rational) function on $U \times_Y U$. Firstly, one may assume that $v_1, \ldots, v_s$ are linearly independent over $\mathbb{K}(U)^G = \pi^*(\mathbb{K}(Y))$. Indeed, if, say, $v_1 = c_2 v_2 + \cdots + c_s v_s$, $c_i \in \mathbb{K}(U)^G$, then F coincides with $\sum_{i=2}^{s} (u_i + c_i u_1) \otimes v_i$ on $U \times_Y U$. Now the statement follows from

Lemma 2.3.12. *Let u_i, v_i, $i = 1, \ldots, s$ be rational functions on U such that $v_1, \ldots, v_s$ are linearly independent over $\mathbb{K}(U)^G$. If $\sum_{i=1}^{s} u_i (g \cdot v_i) = 0$ for all $g \in G$, then $u_1 = \cdots = u_s = 0$.*

Proof. We argue by induction on s. The case $s = 1$ is obvious. Suppose that $s > 1$. If $u_1 \neq 0$, then

$$\sum_{i=1}^{s} h \cdot (u_i u_1^{-1})(g \cdot v_i) = 0$$

for all $g, h \in G$. Thus

$$\sum_{i=2}^{s} (h \cdot (u_i u_1^{-1}) - u_i u_1^{-1})(g \cdot v_i) = 0.$$

By inductive hypothesis, $h \cdot (u_i u_1^{-1}) = u_i u_1^{-1}$ for all $h \in G$, so $u_i u_1^{-1} \in \mathbb{K}(U)^G$. The linear dependence

$$v_1 + u_2 u_1^{-1} v_2 + \cdot + u_s u_1^{-1} v_s = 0$$

leads to a contradiction. □

Lemma 2.3.11 is proved. □

Further, the variety $B = U \times_Y U$ contains a dense irreducible subset $A = \mathrm{Im}(\phi)$, thus B is irreducible.

Lemma 2.3.13. *Let B be an irreducible variety, $W \subseteq B$ a non-empty open subset and $\tau : B \to Z$ a dominant morphism. Then there is a non-empty open subset $V \subseteq Z$ such that $\tau^{-1}(z) \cap W$ is dense in $\tau^{-1}(z)$ for any $z \in V$.*

Proof. Let $C_1, \dots, C_k$ be irreducible components of $B \setminus W$. Renumbering, one may assume that $\tau(C_1), \dots, \tau(C_r)$ are dense in Z, and $\tau(C_{r+1}), \dots, \tau(C_k)$ are not. Set

$$V := Z \setminus \bigcup_{i=r+1}^{k} \overline{\tau(C_i)}.$$

Reducing V, we may assume that

(i) the dimension of any component of any fiber of the morphism $\tau' : \tau^{-1}(V) \to V$ equals $\dim B - \dim Z$;
(ii) for any $i = 1, \dots, r$ the dimension of any component of any fiber of the morphism $\tau \mid_{C_i} : C_i \to V$ equals $\dim C_i - \dim Z$

(Theorem 3.0.26). Since $\dim C_i < \dim B$, now any component of a fiber $\tau^{-1}(z)$ is not contained in $C_1 \cup \dots \cup C_r$, and hence meets the open subset W. □

We are going to apply Lemma 2.3.13 to $W = A^0$, where A^0 is the maximal open subset of $A = \mathrm{Im}(\phi)$, and to the projection $\tau : B \to U$, $\tau(x, x') = x$. Reducing Y, we may suppose that any fiber of τ has a dense intersection with A. But

$$\tau^{-1}(x) = \{x\} \times \pi^{-1}(\pi(x)), \quad A \cap \tau^{-1}(x) = \{x\} \times Gx,$$

and all G-orbits in U are closed. This proves that for any $y \in Y$ the fiber $\pi^{-1}(y)$ is a G-orbit. Now Proposition 2.3.6 shows that $\pi : U \to Y$ is a geometric quotient. □

Corollary 2.3.14 (of the proof). *In notation of Theorem 2.3.10, for any generating set $f_1, \dots, f_m$ of $\mathbb{K}(X)^G$ there is a non-empty open G-invariant subset $U \subseteq X$ such that all f_i are regular on U and for any two points $x, x' \in U$ the condition $Gx = Gx'$ is equivalent to $f_i(x) = f_i(x')$, $i = 1, \dots, m$.*

Corollary 2.3.15. *An action $G : X$ has an open orbit if and only if $\mathbb{K}(X)^G = \mathbb{K}$.*

Proposition 2.3.16. *Assume that there is a non-empty open G-invariant subset $W \subseteq X$ and rational invariants $f_1, \dots, f_k$ that are regular on W, such that for any $x_1, x_2 \in W$ the condition $f_i(x_1) = f_j(x_2)$ for all $i, j = 1, \dots, k$ implies $Gx_1 = Gx_2$. Then $f_1, \dots, f_k$ generate $\mathbb{K}(X)^G$.*

Proof. Extend $f_1, \dots, f_k$ to a generating set $f_1, \dots, f_k, f_{k+1}, \dots, f_m$ of $\mathbb{K}(X)^G$. By assumptions, there is a dominant morphism $\phi : W \to \mathrm{Spec}(\mathbb{K}[f_1, \dots, f_k])$ whose fibers are G-orbits. Reducing W, we may assume that there is a dominant morphism $\psi : W \to \mathrm{Spec}(\mathbb{K}[f_1, \dots, f_m])$ with the same fibers. By Theorem 3.0.28, the varieties $\mathrm{Spec}(\mathbb{K}[f_1, \dots, f_k])$ and $\mathrm{Spec}(\mathbb{K}[f_1, \dots, f_m])$ are birationally isomorphic. □

Exercises to subsection 2.3.

Exercise 2.3.17. Give an example of an SL(2)-action on an irreducible affine variety X with $Q\mathcal{O}(X)^{\mathrm{SL}(2)} \neq \mathbb{K}(X)^{\mathrm{SL}(2)}$.

Exercise 2.3.18. Let G be a connected algebraic group and $H \subseteq G$ a closed subgroup. Check that the projection $p : G \to G/H$ is a geometric quotient with respect to the right H-action on G.

Exercise 2.3.19. For a reductive G and an irreducible affine X, the quotient morphism $\pi : X \to X//G$ is a geometric quotient if and only if all G-orbits in X are of the same dimension.

Exercise 2.3.20. For a finite G and an irreducible affine X, the quotient morphism $\pi : X \to X//G$ is a geometric quotient.

Exercise 2.3.21. Let G be reductive and X be irreducible affine G-variety. Prove that the following conditions are equivalent:

(i) $Q\mathcal{O}(X)^G = \mathbb{K}(X)^G$;
(ii) there is a non-empty open $V \subseteq X//G$ such that for any $y \in V$ the fiber $\pi^{-1}(y)$ of the quotient morphism contains a dense G-orbit.

Exercise 2.3.22. Let $\pi : X \to X/G$ be a geometric quotient. Prove that $\mathbb{K}(X)^G = \pi^*(\mathbb{K}(X/G))$. Is the same true for a categorical quotient ?

Exercise 2.3.23 (*). Consider an action $\mathbb{K}^\times : \mathbb{K}^2 \setminus \{0\}$, $t \cdot (x_1, x_2) = (tx_1, t^{-1}x_2)$. Prove that here all orbits are closed and of dimension one, but the action does not admit geometric quotient.

3. Appendix One: Some Facts from Algebraic Geometry

Theorem 3.0.24. *Let $\phi : X \to Y$ be a dominant morphism of irreducible algebraic varieties. Then $\phi(X)$ contains an open subset of Y.*

Theorem 3.0.25. *Let X be an irreducible variety and $Y \subseteq X$ a closed subvariety such that $\dim X = \dim Y$. Then $X = Y$.*

Theorem 3.0.26. *Let $\phi : X \to Y$ be a morphism of irreducible algebraic varieties. Then there is a non-empty open subset $U \subset Y$ such that for any $y_0 \in U \cap \phi(X)$ any irreducible component of the fiber $\phi^{-1}(y_0)$ has dimension $\dim X - \dim Y$. Moreover, any irreducible component of any non-empty fiber $\phi^{-1}(y)$, $y \in Y$, has dimension $\geq \dim X - \dim Y$.*

Theorem 3.0.27. *Assume that $\operatorname{char} \mathbb{K} = 0$ and $\phi : X \to Y$ is a bijective morphism between normal (e.g., smooth) varieties. Then ϕ is an isomorphism.*

Theorem 3.0.28 (Factorization of a morphism)**.** *Assume that $\operatorname{char} \mathbb{K} = 0$, and X, Y, Z are irreducible varieties with given dominant morphisms $\phi : X \to Y$, $\psi : X \to Z$ such that there is a non-empty open subset $U \subset X$ with the following property: $\phi(x) = \phi(x')$ implies $\psi(x) = \psi(x')$ for any $x, x' \in U$. Then there exists a rational (dominant) morphism $\tau : Y \to Z$ such that the diagram*

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{\phi} & & \searrow^{\psi} & \\ Y & & \xrightarrow{\ \tau\ } & & Z \end{array}$$

is commutative.

Corollary 3.0.29. *In notations of Theorem 3.0.28, assume that $Z = \mathbb{A}^1$, Y is normal and $\operatorname{codim}_Y (Y \setminus \phi(X)^0) \geq 2$ (here $\phi(X)^0$ is the maximal open subset in $\phi(X)$):*

$$\begin{array}{ccccc} & & X & & \\ & \swarrow^{\phi} & & \searrow^{\psi} & \\ Y & & \xrightarrow{\ \tau\ } & & \mathbb{A}^1. \end{array}$$

If $\psi \in \mathcal{O}(X)$, then there exists $\tau \in \mathcal{O}(Y)$ with $\phi^*(\tau) = \psi$.

Theorem 3.0.30. [Hu75, Th. 4.5] *Let $\phi : X \to Y$ be a dominant morphism of irreducible varieties, $r = \dim X - \dim Y$. Assume that for each closed irreducible subset $Z \subseteq Y$ all components of $\phi^{-1}(Z)$ have dimension $r + \dim Z$. Then the morphism ϕ is open.*

Theorem 3.0.31. [Hu75, Th. 4.3] *Let $\phi : X \to Y$ be a dominant morphism of irreducible varieties, $r = \dim X - \dim Y$. Then Y has a non-empty open subset U such that:*

(i) *$U \subseteq \phi(X)$;*
(ii) *if $W \subseteq Y$ is an irreducible closed set which meets U, and if Z is a component of $\phi^{-1}(W)$ which meets $\phi^{-1}(U)$, then $\dim Z = \dim W + r$.*

Theorem 3.0.32. *(cf.*[PV94, p. 187]*) Let $\phi : X \to Y$ be a dominant morphism of irreducible varieties such that the dimension of any component of any fiber equals $\dim X - \dim Y$. Assume that Y is normal. Then ϕ is an open morphism.*

Theorem 3.0.33. *Let $\phi : X \to Y$ be a dominant morphism between irreducible varieties. Then, for any $x \in X$,*

$$T_x(\phi^{-1} \circ \phi)(x) \subseteq \operatorname{Ker} d_x\phi,$$

and there is a non-empty open subset $U \subseteq X$ such that the equality takes place for any $x \in U$.

4. Appendix Two: Some Facts on Lie Algebras

In this Appendix all Lie algebras are supposed to be finite-dimensional. We refer to [Hu72] for a detailed exposition of the subject.

Definition 4.0.34. A Lie algebra $\mathfrak{g}$ is called *simple* if it is not commutative and has no proper ideals.

Definition 4.0.35. The *commutant* $[\mathfrak{g}, \mathfrak{g}]$ of a Lie algebra $\mathfrak{g}$ is the linear span of $[x, y]$, $x, y \in \mathfrak{g}$.

Clearly, $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$. This definition leads to the notion of a solvable Lie algebra.

Remark 4.0.36. If $\mathfrak{g}$ is simple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Definition 4.0.37. The radical $\mathfrak{r}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the largest solvable ideal of $\mathfrak{g}$.

Definition 4.0.38. A Lie algebra $\mathfrak{g}$ is called *semisimple* if $\mathfrak{r}(\mathfrak{g}) = 0$.

Theorem 4.0.39. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if it is isomorphic to a direct sum of simple Lie algebras:*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k.$$

Corollary 4.0.40. *If $\mathfrak{g}$ is semisimple, then* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Theorem 4.0.41 (G. Weyl)**.** *Any finite-dimensional representation of a semisimple Lie algebra is completely reducible.*

5. Hints and Solutions to Exercises

1.1.10. $a_{ij} = e_i^T A e_j = e_i^T B e_j = b_{ij}$.

1.1.11. Let $e_1, \dots, e_n$ be an orthogonal basis. Set $A_1 = E$ and $A_2(e_1) = -e_1$, $A_2(e_i) = e_i$, $i > 1$.

1.1.12. $SO(2) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a^2 + b^2 = 1$. For O(2), take $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

1.1.13. $q(v_1, v_2) = v_1^T Q v_2$, where Q is symmetric non-degenerate, so there is a non-degenerate S with $S^T Q S = E$, or $Q = (S^{-1})^T S^{-1}$. Then

$$A \in \mathrm{O}(q) \iff A^T Q A = Q \iff A^T (S^{-1})^T S^{-1} A = (S^{-1})^T S^{-1} \iff$$
$$\iff S^T A^T (S^{-1})^T S^{-1} A S = E \iff S^{-1} A S \in \mathrm{O}(n).$$

Similarly, any bilinear skew-symmetric non-degenerate form is equivalent to the standard one.

1.1.14. $(\operatorname{char} \mathbb{K} = 0)$ $A \in \mathrm{Sp}(2n)$ if and only if A preserves $\omega \in \bigwedge^2 V^*$. Then A preserves $\wedge^n \omega \in \bigwedge^{2n} V^* = \langle \det \rangle$. Since ω is non-degenerate, $\wedge^n \omega$ is a non-zero multiple of det. This proves that $\det(Av_1, \dots, Av_{2n}) = \det(v_1, \dots, v_{2n})$ for any $v_1, \dots, v_{2n} \in V$, so $\det A = 1$.

In arbitrary characteristic, one may prove that $\mathrm{Sp}(2n)$ is generated by symplectic transections:

$$\tau_{u,a} : V \to V, \ \tau_{u,a}(v) = v + a\omega(v, u)v, \quad u \in V, \ a \in \mathbb{K},$$

see (L.C. Grove, Classical Groups and Geometric Algebras, Grad. Studies in Math. **39**, AMS, 2002) for details.

1.1.15. For $\mathbb{K}^2$, det coincides with the standard bilinear skew-symmetric non-degenerate form. For $n > 1$, $E + E_{13} \in \mathrm{SL}(2n) \setminus \mathrm{Sp}(2n)$.

1.1.16. The intersection of $\mathrm{GL}(n, \mathbb{R})$ with the subvariety of scalar matrices is not closed in Zariski topology.

1.1.17. For $n > 1$ $\langle \mathrm{SL}(n) \rangle = \mathrm{Mat}(n \times n)$, because $E, E + E_{ij}$ $(i \neq j)$, and $E - E_{ii} + E_{ij} - E_{ji}$ belong to $\mathrm{SL}(n)$.

1.1.18. If A commutes with all matrices of a linear group G, then A commutes with all matrices of the linear span of G. But if A commutes with all E_{ij}, then it is scalar, so $Z(\mathrm{GL}(n)) = \{\lambda E : \lambda \in \mathbb{K}^\times\}$, $Z(\mathrm{SL}(n)) = \{\lambda E : \lambda^n = 1\}$. Similarly, $Z(B(n)) = \{\lambda E : \lambda \in \mathbb{K}^\times\}$ and $Z(U(n)) = \{E + aE_{1n} : a \in \mathbb{K}\}$.

1.1.19. $\dim D(n) = n$, $\dim B(n) = \frac{n(n+1)}{2}$, $\dim U(n) = \frac{n(n-1)}{2}$.

1.1.20. There are infinitely many elements $\epsilon \in \mathbb{K}^\times$ with $\epsilon^N = 1$ for some N. If $\operatorname{char} \mathbb{K} = 0$, then 0 is the only element of finite order in G_a. If $\operatorname{char} \mathbb{K} = p > 0$, then any non-zero element of G_a has order p.

1.1.21. No, if $n > 1$. For $n = 2$ consider the reflections $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$.

1.1.22.

$$\mathrm{diag}(t_1, \dots, t_k, \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & a_s \\ 0 & 1 \end{pmatrix}).$$

1.1.23. Let $e_1, \dots, e_n$ be a basis of A and $\circ$ be the (bilinear) multiplication. Clearly, an invertible linear map $\phi : A \to A$ is an isomorphism if and only if $\phi(e_i) \circ \phi(e_j) = \phi(e_i \circ e_j)$ for any $i, j = 1, \dots, n$. Set $\phi(e_i) = \sum_l a_{li} e_l$ and $e_i \circ e_j = \sum_k c_{ij}^k e_k$, where c_{ij}^k are structural constants of the algebra A. Then

$$(\sum_l a_{li} e_l) \circ (\sum_s a_{sj} e_s) = \sum_k c_{ij}^k (\sum_r a_{rk} e_r)$$

is a system on polynomial equations on a_{ij}.

1.1.24. $(\operatorname{char} \mathbb{K} = 0)$ Consider $\tau : G \times G \to G \times G$, $\tau(g_1, g_2) = (g_1, g_1 g_2)$. This morphism is bijective, thus it is an isomorphism, and $\tau^{-1}(h_1, h_2) = (h_1, h_1^{-1} h_2)$. Then $i(g) = p_2(\tau^{-1}(g, e))$, where $p_2 : G \times G \to G$, $p_2(g_1, g_2) = g_2$.

1.1.25. I have only topological arguments: $\mathrm{SL}(2, \mathbb{C})$ is homotopy-equivalent to the real sphere S^2.

1.2.15. $(G_a)^3$ and $U(3)$.

1.2.16. Denote $X = \mathbb{A}^1 \setminus \{0,1\}$. It is sufficient to prove that the automorphism group $\mathrm{Aut}(X)$ of the variety X does not act transitively on X. Since $\mathcal{O}(X) = \mathbb{K}[T, \frac{1}{T}, \frac{1}{T-1}]$, any automorphism $\phi \in \mathrm{Aut}(X)$ defines an automorphism of the field $\mathbb{K}(T)$. But the automorphism group of $\mathbb{K}(T)$ is isomorphic to $\mathrm{PGL}(2)$, or $\mathrm{Aut}(\mathbb{P}^1)$. Our ϕ should preserve $\{0,1,\infty\} \subset \mathbb{P}^1$. But any automorphism of $\mathbb{P}^1$ that fixes 0, 1 and ∞ is the identity, so $\mathrm{Aut}(X)$ is finite. (In fact, $\mathrm{Aut}(X) \cong \Sigma_3$.)

1.2.17. $\mathrm{SO}(2) \to G_m$, $\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \to a + bi$, $i^2 = -1$. For the second component, $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}^2 = E$.

1.2.18. If $H \cap G^0$ is a proper (closed) subgroup, then

$$\dim H = \dim H^0 = \dim H \cap G^0 < \dim G^0 = \dim G.$$

1.2.19. $G = \mathrm{O}(2)$.

1.2.20. $G = \{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \cup \begin{pmatrix} 0 & -s \\ s^{-1} & 0 \end{pmatrix} : t, s \in \mathbb{K}^{\times}\}$: there are no elements of order two in the second component.

1.2.21. One may assume that $\mathbb{K}$ is algebraically closed. Corollary 1.2.7 implies that the hypersurface defined by $\det(a_{ij})-1$ is irreducible. In order to see that $\det(a_{ij})-1$ is not a proper power, consider the leading term of $\det(a_{ij}) - 1$ with respect to a monomial order.

1.2.22. Clearly, $[\mathrm{GL}(n), \mathrm{GL}(n)] \subseteq \mathrm{SL}(n)$ and $[B(n), B(n)] \subseteq U(n)$. On the other hand,

$$(E+E_{ij})(E+(t-1)E_{ii}+(t^{-1}-1)E_{jj})(E-E_{ij})(E+(t^{-1}-1)E_{ii}+(t-1)E_{jj}) = E+(1-t^2)E_{ij}.$$

Thus $[\mathrm{GL}(n), \mathrm{GL}(n)] = [\mathrm{SL}(n), \mathrm{SL}(n)] = \mathrm{SL}(n)$ and $[B(n), B(n)] = U(n)$.

If $j > i + 1$, then

$$(E + E_{ii+1})(E + E_{i+1j})(E - E_{ii+1})(E - E_{i+1j}) = E + E_{ij}.$$

This implies that $[U(n), U(n)]$ is defined in $U(n)$ by $a_{12} = a_{23} = \cdots = a_{n-1n} = 0$.

1.2.23. Let $g \in G \setminus Z(G)$. The map $\phi_g : G \to G$, $\phi_g(h) = ghg^{-1}h^{-1}$ has an irreducible image which is not equal to $\{e\}$. Thus $\phi_g(G)$ is not contained in $Z(G)$, and $gZ(G) \notin Z(G/Z(G))$.

In the disconnected case the statement is not true: one may take a finite group G, say, a non-commutative group of order 8.

1.2.24. Consider the subgroup H generated by $xyx^{-1}y^{-1}, x \in G^0, y \in G$. It is generated by subsets $\phi_y(G^0)$, where $\phi_y(x) = xyx^{-1}y^{-1}$, thus is closed and connected (Proposition 1.2.6). So it is sufficient to check that H has a finite index in $[G, G]$. Since $g_1[g,h]g_1^{-1} = [g_1gg_1^{-1}, g_1hg_1^{-1}]$ for any $g, g_1 \in G$, $h \in G^0$, the subgroup H is normal in G. Further, $[gH, hH] = [g,h]H = H$, and G^0/H is contained in the center $Z(G/H)$. This shows that $Z(G/H)$ has a finite index in G/H, and the group $[G/H, G/H]$ is finite. But $[G,G]/H \subseteq [G/H, G/H]$, and H has a finite index in $[G, G]$.

1.2.25. Let G_1 be $(\mathbb{R}, +)$ and G_2 a compact torus $(\mathbb{R}^2, +)/\mathbb{Z}^2$. Consider a homomorphism $\phi(a) = (a, \sqrt{2}a)+\mathbb{Z}^2$. Check that the intersection of $\mathrm{Im}(\phi)$ with the circle $x_1 \in \mathbb{Z}$ in G_2 is an infinite countable set.

1.3.24. Note that $\{B \in \mathrm{Mat}(n \times n) : AB = BA\}$ is a subspace, and its intersection with an open subset $\mathrm{GL}(n)$ is irreducible. For $\mathrm{SL}(2)$, take $A = \begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}$.

1.3.25. For any $x \in A$, $g \in N_G(A)$ if and only if $f(gxg^{-1}) = 0$ for all $f \in \mathbb{I}(A)$.

1.3.26. If V is a rational G-module, then $G \times V \to V$, $(g, v) \to \rho(g)v$ is a morphism. Conversely, suppose that $G \times V \to V$ is a morphism. By assumption, $(g, v) \to \rho(g)v$, where $\rho(g)$ is a linear operator on V. Fixing a basis $\{e_i\}$ in V and restricting the action to $G \times \{e_i\} \to V$, we get that $\rho(g)$ depends on g algebraically.

1.3.27. (a) $n > 1$: $\{0\}$ and $\mathbb{K}^n \setminus \{0\}$ (any non-zero vector may be included into a unimodular basis); (b) 2^n orbits (fix zero coordinates); (c) $n+1$ orbits (fix a place of the last non-zero coordinate); (d) $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are in the same orbits $\iff \exists\, k$: $x_k = \cdots = x_n = y_k = \cdots = y_n = 0$, $x_{k-1} = y_{k-1} \neq 0$ (set formally $x_{n+1} = y_{n+1} = 0$, $x_0 = y_0 = 1$); (e) and (f) $n > 2$: non-zero vectors v_1 and v_2 are in the same orbit $\iff q(v_1, v_1) = q(v_2, v_2)$. If $q(v, v) \neq 0$, then v may be included in an orthogonal basis. If $q(v, v) = 0$ then there is $e \in \mathbb{K}^n$ with $q(e, e) = 1$, $q(e, v) = 1$, so $q(e, v - e) = 0$, $q(v - e, v - e) = -1$, and we may include e and $v - e$ in an orthogonal basis. For $n = 2$, the conditions $q(v, v) = 0, v \neq 0$ define two $\mathrm{SO}(2)$-orbits that are permuted by $\mathrm{O}(2)$. (g) $\{0\}$ and $\mathbb{K}^{2n} \setminus \{0\}$ (any non-zero vector may be included into a symplectic basis).

1.3.28. $\dim \mathrm{Sp}(2n) = (2n)^2 - \frac{2n(2n-1)}{2} = 2n^2 + n$.

1.3.29. We know that $\mathrm{SO}(2) \cong \mathbb{K}^\times$, $\mathrm{Sp}(2) = \mathrm{SL}(2)$ are connected. Further, $\phi : \mathrm{SO}(n) \to S^{n-1}$, where $S^{n-1} = \{(x_1, \ldots, x_n) : x_1^2 + \cdots + x_n^2 = 1\}$ is irreducible, and $\phi(g) = g \cdot e_1$. Any fiber of ϕ is isomorphic to $\mathrm{SO}(n-1)$, thus is connected. This shows that $\mathrm{SO}(n)$ is connected. Similarly, $\psi : \mathrm{Sp}(2n) \to \mathbb{K}^{2n} \setminus \{0\}$, $\psi(g) = g \cdot e_1$, and any fiber of ψ is isomorphic to $\mathrm{Sp}(2n-2) \times \mathbb{K}^{2n-1}$.

Remark. If an orbit Gx and the stabilizer G_x are connected, then G is connected. Indeed, the stabilizer G_x is contained in G^0 and different connected components of G correspond to different G^0-orbits in Gx. Such orbits are closed and have empty intersections, a contradiction with connectedness of Gx.

1.3.30. In all cases it is sufficient to construct a homomorphism with kernel of order two (use Theorem 1.2.12 and compare dimensions). (a) $\mathrm{SL}(2) : \{M \in \mathrm{Mat}(2 \times 2) : \mathrm{tr}(M) = 0\}$, $A \cdot M := AMA^{-1}$, the action preserves det; (b) $\mathrm{SL}(2) \times \mathrm{SL}(2) : \mathrm{Mat}(2 \times 2)$, $(A, B) \cdot M := AMB^{-1}$, the action preserves det; (d) $\mathrm{SL}(4) : \bigwedge^2 \mathbb{K}^4$, the action preserves $q(w_1, w_2) = w_1 \wedge w_2$; (c) $\mathrm{Sp}(4) : V$, where V is a 5-dimensional complement to an invariant line in $\bigwedge^2 \mathbb{K}^4$.

1.3.31. It is sufficient to prove that the hypersurface $\det(a_{ij}) = 0$ is irreducible. But this hypersurface contains a dense $\mathrm{GL}(n) \times \mathrm{GL}(n)$-orbit, where $\mathrm{GL}(n) \times \mathrm{GL}(n) : \mathrm{Mat}(n \times n)$, $(A, B) \cdot M = AMB^{-1}$.

1.3.32. Note that $\mathcal{F}(V)$ is a closed G-invariant subvariety in

$$\mathbb{P}(V) \times \mathbb{P}(\wedge^2 V) \times \cdots \times \mathbb{P}(\wedge^{n-1} V),$$

where $n = \dim V$.

1.3.33. Take a line with a double point and the $\mathbb{Z}_2$-action permuting the glueing lines.

1.3.34. $G_a : \mathbb{K}^n$, $a \cdot v = v + a(1, 1, \ldots, 1)$ – no fixed points.

1.3.35. The linear span of any orbit is finite-dimensional and any representation of a finite group is algebraic.

1.3.36. Take $G = G_m$, $X = \mathbb{K}^1$. Prove that $\dim\langle \frac{1}{tx+1} : t \in G_m \rangle = \infty$.

1.3.37. Suppose $A' \subset \mathcal{O}(Y)$ is a finitely generated subalgebra defining an embedding of Y as a dense open subvariety of an affine variety $X' := \mathrm{Spec}(A')$. By Theorem 1.3.16, a generating set of A' is contained in a finite-dimensional G-invariant subspace. Hence the subalgebra A' is contained in a finitely generated G-invariant subalgebra $A \subset \mathcal{O}(Y)$. Put $X := \mathrm{Spec}(A)$. There is a commutative diagram of dominant morphisms:

$$\begin{array}{ccc} & Y & \\ {\scriptstyle J}\swarrow & & \searrow \\ X & \longrightarrow & X'. \end{array}$$

One may find $f_1, \ldots, f_k \in A'$ such that $Y = \bigcup_i X'_{f_i}$. Since $A' \subseteq A \subseteq \mathcal{O}(Y) \subseteq \mathcal{O}(X'_{f_i})$, all morphisms in the diagram become isomorphisms after localization at f_i. Thus $J(Y) = \bigcup_i X_{f_i}$ is open in X.

1.4.9. It is sufficient to check that the subspace $U = \langle u_1, \dots, u_k \rangle \subseteq W$ is uniquely defined by $\omega := u_1 \wedge \dots \wedge u_k$. We claim that $U = \{w \in W : \omega \wedge w = 0\}$. Indeed, $w \in U$ if and only if $u_1, \dots, u_k, w$ are linearly dependent.

1.4.10. $\rho : \mathrm{GL}(n) \to \mathrm{GL}(\mathrm{Mat}(n \times n))$, $A \cdot M := AMA^{-1}$.

1.4.11. The group G/G^0 acts transitively on the set of G^0-orbits in G/H, so these orbits are closed. Hence they are irreducible components with empty intersections. It implies that G/H is connected if such an orbit is unique, or $G^0eH = G$.

1.4.12. If G/H is quasiaffine, use Exercise 1.3.37 and Theorem 1.3.19. Conversely, the orbit Gv is open in the affine variety $\overline{Gv}$.

1.4.13. Follows from Theorem 1.3.19.

1.4.14. No, consider $G_m : \mathbb{K}^1$, or $\mathrm{SL}(2) : \mathrm{S}^3\mathbb{K}^2$, $v = e_1^2 e_2$.

1.4.15. Take $G = \mathrm{SL}(3)$ and $H = \left\{ \begin{pmatrix} t_1 & 0 & a \\ 0 & t_2 & b \\ 0 & 0 & t_1^{-1}t_2^{-1} \end{pmatrix} \right\}$. Here $G/H \cong \mathbb{P}^2 \times \mathbb{P}^2 \setminus \Delta$.

1.4.16. $\mathrm{SL}(3)/U(3) \cong X \setminus (Y \cup Z)$, where $X, Y, Z \subset \mathbb{K}^6$ defined by $x_1y_1 + x_2y_2 + x_3y_3 = 0$ $x_1 = x_2 = x_3 = 0$ and $y_1 = y_2 = y_3 = 0$ respectively : consider $V = \mathbb{K}^3 \oplus (\mathbb{K}^3)^*$, $v = (e_1, e_3^*)$.

1.4.17. Use Theorem 3.0.28 with $X = G$.

1.4.18. Let G be a connected group and H a non-trivial finite subgroup. If there is an open $U \subseteq G/H$ such that $p^{-1}(U) \cong U \times H$, then G contains an open reducible subset.

1.5.22.
$$\frac{\partial \det}{\partial a_{ij}} \mid_{A=E} = \sum_{\sigma \in \Sigma_n, \sigma(i)=j} \operatorname{sgn}(\sigma)\, a_{1\sigma(1)} \dots \overline{a_{i\sigma(i)}} \dots a_{n\sigma(n)} \mid_{A=E} = \delta_{ij},$$
and $d_E(\det)(X) = \sum_{i,j} \delta_{ij} x_{ij} = \mathrm{tr} X$.

1.5.23.
$$\frac{\partial \frac{A_{ji}}{\det A}}{\partial a_{km}} \mid_{A=E} = \frac{\frac{\partial A_{ji}}{\partial a_{km}} \det A - A_{ji} \frac{\partial \det A}{\partial a_{km}}}{(\det A)^2} \mid_{A=E} =$$
$$= -\delta_{ki}\delta_{mj}(1-\delta_{ij}) + \delta_{ij}\delta_{km}(1-\delta_{ik}) - \delta_{ij}\delta_{km} = -\delta_{ik}\delta_{jm}.$$

1.5.24. In order to prove that $[x, y] = 0$, consider the one-dimensional subalgebras $\langle x \rangle$ and $\langle y \rangle$.
One may be interesting in a group-theoretical analog of this statement: A group G is commutative if and only if any its subgroup is normal. This statements is wrong: take $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, where $ghg^{-1} = \pm h$ for any $g, h \in G$.

1.5.25. It corresponds to a subspace U in $\mathrm{Mat}(n \times n)$ consisting of pairwise commuting matrices together with a surjective linear map $\mathfrak{g} \to U$.

1.5.26. Take $G_1 = G_a$, $G_2 = G_m$. They are not isomorphic, because G_m contains infinitely many elements of finite order, but G_a – only one.

1.5.27. $\mathfrak{sp}(2n) = \{X \in \mathrm{Mat}(2n \times 2n) : X^T \Omega + \Omega X = 0\}$, where Ω is the skew-symmetric matrix, corresponding to the standard form (follow Example 1.5.8);

$$Lie(D(n)) = \left\{ \begin{pmatrix} * & 0 & \dots & 0 \\ 0 & * & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & \dots & * \end{pmatrix} \right\}, \ \mathrm{Lie}(B(n)) = \left\{ \begin{pmatrix} * & * & \dots & * \\ 0 & * & \dots & * \\ \dots & \dots & \dots & * \\ 0 & 0 & \dots & * \end{pmatrix} \right\}, \ \mathrm{Lie}(U(n)) = \left\{ \begin{pmatrix} 0 & * & \dots & * \\ 0 & 0 & \dots & * \\ \dots & \dots & \dots & * \\ 0 & 0 & \dots & 0 \end{pmatrix} \right\}$$

($D(n)$ and $B(n)$ are open in subspaces of $\mathrm{Mat}(n \times n)$, and $U(n)$ is a shift of a subspace).

1.5.28. Follows from Lemma 1.5.12 and Exercise 1.3.30.
For (a), one may get an isomorphism directly. Set $e = \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}$, $h = \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}$, $f = \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}$. Then $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. On the other hand, in $\mathfrak{so}(3)$ we have $x_{12} = E_{12} - E_{21}$, $x_{23} = E_{23} - E_{32}$, $x_{13} = E_{13} - E_{31}$ with $[x_{12}, x_{23}] = x_{13}$, $[x_{23}, x_{13}] = x_{12}$, $[x_{13}, x_{12}] = x_{23}$. Calculating the eigenvalues of $\mathrm{ad}(x_{12})$, we get an isomorphism $e \to ix_{23} - x_{13}$, $h \to 2ix_{12}$, $f \to ix_{23} + x_{13}$.

1.5.29. Any G-invariant subspace is G^0-invariant. For converse, take a finite group.

1.5.30. If $H \lhd G$, then $H^0 \lhd G$. For converse, take a finite group.

1.5.31. We may assume that $G \subset \mathrm{GL}(n)$ is a closed subgroup. There is a linear action $G : \mathrm{Mat}(n \times n)$, $h \cdot A = hAh^{-1}$, which is the restriction of the adjoint representation for $\mathrm{GL}(n)$. Thus the tangent action is $x \cdot A = [x, A]$ for all $x \in \mathfrak{g}$. By Proposition 1.5.16,
$$\mathrm{Lie}(Z_G(g)) = \mathfrak{g}_g = \{x \in \mathfrak{g} : [x, g] = 0\} = \{x \in \mathfrak{g} : gxg^{-1} = x\} = \{x \in \mathfrak{g} : \mathrm{Ad}(g)x = x\}.$$

1.5.32. If $\mathfrak{g}$ is commutative, then $Z(x) = G$ for any $x \in \mathfrak{g}$, because $\mathrm{Lie}(Z(x)) = \mathfrak{z}(x) = \mathfrak{g}$ (Proposition 1.5.21). This means that for any $g \in G$ $\mathrm{Lie}(Z_G(g)) = \mathfrak{g}$ (Exercise 1.5.31), and $Z_G(g) = G$.

1.5.33. By definition, $Z(G) = \cap_{g \in G} Z_G(g)$. Using the Noether property, one may assume that this is an intersection of finitely many subgroups, and Lemma 1.5.13 implies
$$\mathrm{Lie}(Z(G)) = \{x \in \mathfrak{g} : \mathrm{Ad}(g)x = x \text{ for all } g \in \mathfrak{g}\}.$$
If G is connected, then, by Proposition 1.5.16,
$$\mathrm{Lie}(Z(G)) = \{x \in \mathfrak{g} : \mathrm{ad}(y)x = 0 \text{ for all } y \in \mathfrak{g}\} = \mathfrak{z}(\mathfrak{g}).$$
For non-connected G, consider $G = \mathrm{O}(2)$.

1.5.34(ii). Assume that $D \in \mathrm{Lie}(\mathrm{Aut}(A))$. Then $D = \frac{d}{dt} \mid_{t=0} \Gamma(t)$, where $\Gamma(t)$ is a smooth curve in $\mathrm{Aut}(A)$ with $\Gamma(0) = \mathrm{id}$. Fix a basis $e_1, \dots, e_n$ in A. Let $(\gamma_{ij}(t))$ be the matrix representing $\Gamma(t)$, $\gamma_{ij}(0) = \delta_{ij}$. Then $\frac{d}{dt} \mid_{t=0} \gamma_{ij}(t) = d_{ij}$, where

$D=(d_{ij})$. It is sufficient to check that $D(e_ie_j)=D(e_i)e_j+e_iD(e_j)$ for any i,j. So,

$$D(e_ie_j)=\frac{d}{dt}\,|_{t=0}\,\Gamma(t)(e_ie_j)=\frac{d}{dt}\,|_{t=0}\,(\Gamma(t)(e_i))(\Gamma(t)(e_j))=$$

$$=\frac{d}{dt}\,|_{t=0}\,(\sum_k\gamma_{ki}(t)e_k)(\sum_l\gamma_{lj}(t)e_l)=\frac{d}{dt}\,|_{t=0}\,(\sum_{k,l}\gamma_{ki}(t)\gamma_{lj}(t)e_ke_l)=$$

$$=\sum_k d_{ki}e_ke_j+\sum_l d_{lj}e_ie_l=D(e_i)e_j+e_iD(e_j).$$

Conversely, suppose that $D\in\mathrm{Der}(A)$ ($\mathbb{K}=\mathbb{C}$). Consider the (non-algebraic) curve

$$\Gamma(t):=\exp(tD)=E+tD+\frac{t^2D^2}{2!}+\dots$$

Clearly, $\frac{d}{dt}\,|_{t=0}\,\Gamma(t)=D$. We have to show that $\exp(D)\in\mathrm{Aut}(A)$, or

$$(E+D+\frac{D^2}{2!}+\dots)(ab)=(E+D+\frac{D^2}{2!}+\dots)(a)(E+D+\frac{D^2}{2!}+\dots)(b).$$

It is easy to prove by induction the "higher Leibniz rule":

$$D^n(ab)=\sum_{k=0}^{n}\binom{n}{k}D^k(a)D^{n-k}(b).$$

Then the desired equality follows from $\frac{1}{k!}\frac{1}{(n-k)!}=\binom{n}{k}\frac{1}{n!}$.

1.6.21. Consider $F(Y_1, \dots, Y_n)$ as polynomial in $Y_2, \dots, Y_n$. Then all its coefficients are constants.

1.6.22. Note that $\mathbb{X}(G) = \mathbb{X}(G/[G,G])$. This implies $\mathbb{X}(\mathrm{GL}(n)) = \mathbb{X}(T^1) \cong \mathbb{Z}$; $\mathbb{X}(\mathrm{SL}(n)) = 0$; $\mathbb{X}(B(n)) = \mathbb{X}(T^n) \cong \mathbb{Z}^n$ (Exercise 1.2.22). Moreover, $[\mathrm{O}(2), \mathrm{O}(2)] = \mathrm{SO}(2)$ and $\mathbb{X}(\mathrm{O}(2)) \cong \mathbb{Z}_2$. For a quasitorus $Q \cong T^m \times A$, we have $\mathbb{X}(Q) \cong \mathbb{Z}^m \times A$. Finally, $\mathbb{X}(G) \cong G/[G,G]$ for a finite G.

1.6.23. $\mathrm{Hom}(T^m, T^n) = \mathbb{X}(T^m)^n$.

1.6.24. Recall that A is semisimple if and only if its minimal polynomial $\mu_A(T)$ has no multiple roots. Note that $\mu_{A|_U}(T)$ divides $\mu_A(T)$.

1.6.25. $\mu_A(T)$ divides $T^n - 1$.

1.6.26. Restrict all A_i to eigenspaces V_{λ_j} of A_1 and use induction.

1.6.27. Consider $\begin{matrix} 2 & 1 \\ 0 & 1 \end{matrix} \quad \begin{matrix} \frac{1}{2} & 1 \\ 0 & 1 \end{matrix}$.

1.6.28. The set U of matrices with pairwise different eigenvalues is defined as $\mathrm{Disc}(P_A(T)) \neq 0$, where $P_A(T)$ is the characteristic polynomial of A, and thus U is open. On the other hand, $E + tE_{11}$ is semisimple only if $t = 0$.

1.6.29. The finite group G/G^0 is commutative, thus $G/G^0 \cong \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_s}$. Let t_i be a representative of the component, corresponding to a fixed generator $a_i \in \mathbb{Z}_{n_i}$. Since $t_i^{n_i} \in G^0$ and the group G^0 is divisible (i.e., for any $t \in G^0$ and $N \in \mathbb{N}$ there is $h \in G^0$ with $h^N = t$), one may assume that $t_i^{n_i} = e$. Let F be the subgroup of G generated by t_i. Check that $G \cong G^0 \times F$.

1.6.30. Theorem 1.6.15 implies that $\phi(Q^0)$ is a torus. Use Exercise 1.6.29.

1.6.31. Use Theorem 1.6.15.

1.6.32. Let $Ax = 0$ be a system of linear equations in $\mathbb{K}^m$. Assume that $F \subseteq \mathbb{K}$ is a subfield and $A \in \mathrm{Mat}(m \times m, F)$. Since $\mathrm{rk}_{\mathbb{K}} A = \mathrm{rk}_F A$, $\mathrm{Ker}_{\mathbb{K}}(A)$ and $\mathrm{Ker}_F(A)$ have the same dimension over $\mathbb{K}$ and F respectively. This proves that there is a basis of $\mathrm{Ker}_{\mathbb{K}}(A)$ with elements in F^n .

1.6.33. Let $\mu_1, \dots, \mu_m$ be the standard basis of $\mathbb{X}(T^m)$: $\mu_i((t_1, \dots, t_m)) = t_i$. Then $d\mu_i((c_1, \dots, c_m)) = c_i$, where $(c_1, \dots, c_m) \in \mathfrak{t}$ and $\mathfrak{t}(\mathbb{Z}) = \{(c_1, \dots, c_m) : c_i \in \mathbb{Z}\}$. On the other hand, if $\phi((t_1, \dots, t_m)) = (t_1^{a_{11}} \dots t_m^{a_{1m}}, \dots, t_1^{a_{m1}} \dots t_m^{a_{mm}})$, then $d\phi$ acts in $\mathfrak{t}$ via matrix $A = (a_{ij})$.

1.7.26. Note that $\mathrm{Lie}(G) = \langle x \rangle$, where x is either semisimple or nilpotent, and $G = G(x)$.

1.7.27. For semisimple elements, see Lemma 1.6.12; for nilpotents, use the binomial formula; reduce the unipotent case to the nilpotent one.

1.7.28. Yes, it contains a dense $\mathrm{GL}(n)$-orbit.

1.7.29. One may reduce the questions to the case where A is nilpotent. The operator A_u restricted to any proper space of A_s defines a non-degenerate operator. Thus $A_s = 0$, and $A = 0$. In the last case a decomposition is not unique.

1.7.30. By the Chinese remainder theorem there exists $f(x) \in \mathbb{K}[x]$ with

$$f(x) \equiv \lambda_i \ (\mathrm{mod}(x - \lambda_i)^{k_i}), \quad f(x) \equiv 0 \ (\mathrm{mod} x).$$

1.7.31. See Lemma 1.8.11.

1.7.32. Any $G(A)$ is a direct product of the quasitorus $G(A_s)$ and the unipotent subgroup $G(A_u)$. Elements of $G(A_s)$ are diagonalizable simultaneously (Lemma 1.6.12), so $\dim G(A_s) \leq n$. Moreover, $\dim G(A_u) \leq 1$. But all elements of $G(A_s)$ commute with A_u, and if $A_u \neq E$, then $G(A_s)$ is a proper subgroup of $D(n)$. This proves that $\dim G(A) \leq n$, and the value n is attained by Exercise 1.7.31.

1.7.33. If $x \in \mathrm{Lie}(G)$, then $G(x) \subseteq G^0$ and thus $\exp(\mathrm{Lie}(G)) \subseteq G^0$. By Lemma 1.7.4, $\exp(\mathrm{Lie}(G))$ is a closed irreducible subvariety of G^0 of the same dimension, so $\exp(\mathrm{Lie}(G)) = G^0$.

1.7.34. Consider a non-zero map $\mathrm{Lie}(G_a) \to \mathrm{Lie}(G_m)$.

1.7.35. Take $G = \left\{ \begin{pmatrix} \epsilon & a \\ 0 & \epsilon^{-1} \end{pmatrix} : \epsilon^3 = 1, \ a \in \mathbb{K} \right\}$.

1.7.36. When G is connected, see the proof of Proposition 1.9.8. For arbitrary G, check that G^0 is divisible and repeat arguments of Exercise 1.6.29.

1.7.37.

$$\left\{ \exp \begin{pmatrix} t & t \\ 0 & t \end{pmatrix} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} : t \in \mathbb{C} \right\}$$

or

$$\left\{ \exp \begin{pmatrix} t & 0 & 0 \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} e^t & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{C} \right\}.$$

1.8.21. No, $Z(B) \cong \mathbb{Z}_2$, but $Z(B_1) = \{e\}$, see Exercise 1.2.23.

1.8.22. a) $\mathbb{Z}_2 \times \mathbb{Z}_2$ acts on $\mathbb{P}^1$ by $[x_1 : x_2] \to [-x_1 : x_2]$ and $[x_1 : x_2] \to [x_2 : x_1]$;
b) SL(2) acts on $\mathbb{P}^1$ transitively;
c) G acts on G by left translations transitively.

1.8.23. Clearly, $\langle e_1 \rangle$ is the only line fixed by $B(n)$. Taking $V/\langle e_1 \rangle$ and applying induction, one checks that the standard complete flag is a unique $B(n)$-fixed point on $\mathcal{F}(V)$. Hence this point is also fixed by $N_{\mathrm{GL}(n)} B(n)$.

1.8.24. For any unipotent $A \in G$ the irreducible curve $\{A^t : t \in \mathbb{K}\}$ connects A and E, thus any unipotent element is contained in G^0. We know that U is a closed normal subgroup of G^0. Since a conjugate to a unipotent element is again unipotent, U is normal in G and G/U is a finite extension of a torus. But it need not be commutative !

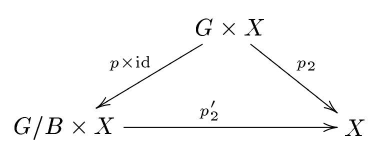
1.8.25. Assume that for some $x \in X$ the boundary $Y := \overline{Gx} \setminus Gx$ is non-empty. By Theorem 1.8.5, there is a non-zero G-fixed vector f in the G-invariant ideal $\mathbb{I}(Y) \lhd \mathcal{O}(\overline{Gx})$. But then f is a constant on $\overline{Gx}$, a contradiction.

1.8.26. If $B \subseteq P$, then $G/B \to G/P$ is surjective, thus G/P is complete and by Corollary 1.4.3 is projective. Conversely, if G/P is projective, then by Borel's Fixed Point Theorem, B has a fixed point on $G^0/(P \cap G^0)$.

1.8.27. Take the set $I = \{(12), (23), \dots, (n-1n)\}$, a subset $J \subseteq I$, and consider a parabolic subgroup $P_J \subseteq \mathrm{GL}(n)$ defined by $a_{ij} = 0$, where $i > j$ and $\{(jj+1), \dots, (i-1i)\}$ is not contained in J. In particular, $P_\emptyset = B(n)$ and $P_I = \mathrm{GL}(n)$. Prove that any connected parabolic is conjugate to some P_J.

First, prove that any Lie subalgebra in $\mathfrak{gl}(n)$ containing all diagonal matrices is spanned by some E_{ij}. Then describe subalgebras which contain $\mathrm{Lie}(B(n))$. Check that different P_J are not conjugate (use Exercise 1.8.23 and Theorem 1.8.19). Again, Exercise 1.8.23 and Theorem 1.8.19 imply that any P_J coincides with its normalizer in $\mathrm{GL}(n)$, thus any parabolic subgroup is connected.

1.8.28. Consider a morphism $\phi : G \times X \to X$, $\phi(g, x) = g^{-1} \cdot x$. Then $Z := \phi^{-1}(Y) = \{(g, x) : g^{-1}x \in Y\}$ is a closed subset. Consider the projections $p : G \to G/B$, $p_2 : G \times X \to X$ and $p_2' : G/B \times X \to X$. There is a commutative diagram:



Since Y is B-invariant, one has $Z = (p \times \mathrm{id})^{-1}((p \times \mathrm{id})(Z))$. Taking normalization, one may assume that X in normal. By Theorem 3.0.32, the image of $(G \times X) \setminus Z$ is open in $G/B \times X$, so $(p \times \mathrm{id})(Z)$ is closed. Further $Y' := p_2'((p \times \mathrm{id})(Z))$ is closed in X, because G/B is complete. On the other hand, $Y' = p_2(Z) = GY$.

1.8.29. Any maximal unipotent group is solvable and coincides with the set of unipotent elements of some Borel subgroup.

1.8.30. Check that $D(n)$ is a maximal connected commutative subgroup of $\mathrm{GL}(n)$. On the other hand, a commutative unipotent group is not conjugated to a subgroup of $D(n)$.

1.8.31. Prove that for $q((x_1, \dots, x_n), (y_1, \dots, y_n)) = x_1 y_n + \dots + x_n y_1$ the set of diagonal matrices in $\mathrm{SO}(q)$ is a maximal torus. For this and for a description of a Borel subgroup, describe $\mathrm{Lie}(\mathrm{SO}(q))$.

Similar arguments work for $\mathrm{Sp}(\omega)$, where

$$\omega((x_1, \dots, x_{2n}), (y_1, \dots, y_{2n})) = x_1 y_{2n} - x_{2n} y_1 + \dots + x_n y_{n+1} - x_{n+1} y_n.$$

1.8.32. One may assume that G is connected and H is connected (right action of a finite group preserves affineness). Then $G = T \leftthreetimes U$, $H = T_1 \leftthreetimes U_1$, $U_1 \subseteq U$ and, up to conjugation, $T_1 \subseteq T$. In particular, any character of H may be extended to G. There is a pair (V, v), where V is a G-module and H is the stabilizer of $\langle v \rangle$, where H acts by a character χ. Let χ' be an extension of χ to G. Consider the tensor product of V with a one-dimensional G-module corresponding to the character

$-\chi'$. Then H is the stabilizer of $V \otimes 1$, and G/H is quasiaffine. Further, there is an open G-equivariant embedding $G/H \hookrightarrow X$, where X is an affine G-variety. Take $Y := X \setminus (G/H)$, and consider the ideal $\mathbb{I}(Y) \lhd \mathcal{O}(X)$. There is a non-zero G-eigenvector f in $\mathbb{I}(Y)$. Thus $G/H = X_f$.

1.9.26. $T = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{K}^\times \right\}$; $N = T \cup \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} T$; $U_n = \left\{ \begin{pmatrix} \epsilon & a \\ 0 & \epsilon^{-1} \end{pmatrix} : \epsilon^n = 1, a \in \mathbb{K} \right\}$;

$B = \left\{ \begin{pmatrix} t & a \\ 0 & t^{-1} \end{pmatrix} : t \in \mathbb{K}^\times, a \in \mathbb{K} \right\}$; SL(2).

First we classify algebraic Lie subalgebras $\mathfrak{g}$ in $\mathfrak{sl}(2)$ (and thus connected subgroups). If $\mathfrak{g} = \langle x \rangle$, then either x is nilpotent (then it is unique up to conjugation, and we get $U = U_1$) or semisimple (it is unique up to conjugation and proportionality, and we get T). Assume that $\mathfrak{g} = \langle x, y \rangle$. If x is semisimple, then we may set $x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $y = aE_{12} + bE_{21}$. Here $[x, y] = 2aE_{12} - 2bE_{21} \in \mathfrak{g}$, hence $ab = 0$ and we obtain B. Finally, if x and y are nilpotent, then G^0 is unipotent, but a maximal unipotent subgroup in SL(2) is one-dimensional.

In order to describe non-connected subgroups, we need to calculate the normalizers: $N_{SL(2)}U = B$, $N_{\mathrm{SL}(2)}T = N$, $N_{\mathrm{SL}(2)}B = B$.

1.9.27. Denote $U_k = \langle e_1, \dots, e_k \rangle$ and $A(i)$ the operator induced by $A \in P_{k_1,\dots,k_s}$ on $U_{k_i}/U_{k_{i-1}}$. Consider $H = \{A \in P_{k_1,\dots,k_s} : A(i) = \mathrm{id},\ i = 1, \dots, s\}$. Clearly, H is a closed normal unipotent subgroup in $P_{k_1,\dots,k_s}$, and

$$P_{k_1,\dots,k_s}/H \cong \mathrm{GL}(k_1) \times \mathrm{GL}(k_2 - k_1) \times \cdots \times \mathrm{GL}(k_s - k_{s-1})$$

is reductive. Hence $H = R^u(P_{k_1,\dots,k_s})$. Similar arguments show that

$$R(P_{k_1,\dots,k_s}) = (Z(\mathrm{GL}(k_1)) \times \cdots \times Z(\mathrm{GL}(k_s - k_{s-1}))) \rightthreetimes H.$$

1.9.28. See Exercise 1.3.30 (b).

1.9.29. If $p : G \to G/R^u(G)$ and $H \lhd G/R^u(G)$ is a normal unipotent subgroup, then, by Lemma 1.9.1, $p^{-1}(H)$ is a normal unipotent subgroup of G.

1.9.30. Take $G_1 = G_a$ and $G_2 = G_m$.

1.9.31. If G_1 and G_2 are reductive, then $G_1 \times G_2$ satisfies condition (5) of Theorem 1.9.16. For $H \lhd G$, one has $R^u(H) \lhd G$. Finally, if $H \lhd G$, then any representation of G/H may be considered as a representation of G, thus G/H satisfies Theorem 1.9.16 (4).

1.9.32. $\phi(R^u(F))$ is a normal unipotent subgroup of G. Thus $R^u(F) \subseteq \mathrm{Ker}\phi$, a contradiction with Lemma 1.6.11.

1.9.33. Consider $p : G \to G/R(G)$. Since $G/R(G)$ is semisimple, so is its tangent algebra, and $\mathfrak{r}(\mathfrak{g}) \subseteq \mathrm{Ker}(dp) = \mathrm{Lie}(R(G))$. Now we shall show that if $R(G)$ is a solvable algebraic group, then $\mathrm{Lie}(R(G))$ is a solvable Lie algebra. Indeed, $R(G)$ contains a non-trivial closed normal commutative subgroup $A \lhd R(G)$; $\mathfrak{a} = \mathrm{Lie}(A)$ is a commutative ideal of $\mathrm{Lie}(R(G))$. Consider $R(G)/A$ with $\mathrm{Lie}(R(G)/A) = \mathrm{Lie}(R(G))/\mathfrak{a}$ and apply induction.

1.9.34. Study the linear span of an orbit, Exercise 1.3.27.

1.9.35. Semisimple classical groups are : SL(n), O(n) ($n \neq 2$), SO(n) ($n \neq 2$) and Sp($2n$) (calculate the center using Exercise 1.9.34 and the Schur Lemma). Simple classical groups are SL(n), O(n) ($n \neq 2, 4$), SO(n) ($n \neq 2, 4$) and Sp($2n$). In order to prove it, one should study ideals in the corresponding Lie algebra.

1.9.36. One may assume that H and N are connected. Applying Proposition 1.2.6 to HN we get that HN is closed. Note that the group $H \times N$ acts transitively on $H \times N$, $(h, n) \circ (h_1, n_1) = (hh_1, n_1 n^{-1})$, on HN, $(h, n) \circ h_1 n_1 = hh_1 n_1 n^{-1}$, and the morphism $\phi : H \times N \to HN$, $\phi(h, n) = hn$ is $(H \times N)$-equivariant. Theorem 3.0.33 implies that differential of ϕ at (e, e) is surjective. But $\mathrm{Lie}(H \times N) = \mathrm{Lie}(H) \oplus \mathrm{Lie}(N)$ (a direct sum of vector spaces) and the differential of the restriction of ϕ on $H \times \{e\}$ (resp. on $\{e\} \times N$) defines the embedding $\mathrm{Lie}(H) \subset \mathrm{Lie}(G)$ (resp. $\mathrm{Lie}(N) \subset \mathrm{Lie}(G)$).

1.9.37. If $x \in \mathfrak{g}$ is nilpotent, then $G(x) \subset G$, a contradiction. Thus all elements of $\mathfrak{g}$ are semisimple. It is sufficient to prove that G^0 is commutative. If it is not the case, then $\mathfrak{g}$ is not commutative (otherwise all elements of $\mathfrak{g}$ may be diagonalized simultaneously and G^0 is contained in $D(n)$). Take any $x \in \mathfrak{g} \setminus \mathfrak{z}(\mathfrak{g})$ and let $\mathfrak{t}$ be a maximal commutative algebraic Lie subalgebra in $\mathfrak{g}$ containing x. Then $\mathfrak{t}$ is the tangent algebra of a subtorus $T \subset G^0$. The $\mathrm{Ad}(T)$-module $\mathfrak{g}$ is a direct sum of one-dimensional submodules. This shows that

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha, \quad \text{where} \quad \mathfrak{g}_\alpha := \{x \in \mathfrak{g} : [y, x] = \alpha(y)x\} \quad \text{for} \quad \text{all} \quad y \in \mathfrak{t},$$

where Δ is a finite set of non-zero linear functions on $\mathfrak{t}$. It follows from the Jacobi Identity that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. Since $\beta + s\alpha \notin \Delta$ for all $\alpha, \beta \in \Delta$ and sufficiently big s, the operator $\mathrm{ad}(x)$ is nilpotent for any $x \in \mathfrak{g}_\alpha$. But this operator acts non-trivially on $\mathfrak{t}$, thus it is non-zero. Then it can not be an image of a semisimple element of $\mathfrak{g}$, a contradiction.

2.1.9. $\mathbb{K}[x_1, x_1x_2, x_1x_2^2, x_1x_2^3, \dots]$ or $\langle x_1^i x_2^j \,:\, i \le \sqrt{2}j\rangle$.

2.1.10. Let $p : \mathcal{O}(X) \to \mathcal{O}(X)^G$ be G-invariant projection. Prove that any simple G-submodule of non-zero type in $\mathcal{O}(X)$ lies in $\operatorname{Ker} p$.

2.1.11. By Exercise 2.1.10, it is sufficient to check that the formula

$$f \to \frac{1}{|G|}\sum_{g\in G} g\cdot f$$

defines a G-invariant projection $\mathcal{O}(X) \to \mathcal{O}(X)^G$.

2.1.12. Clearly, $\mathcal{O}(\mathbb{K}^n)^{\mathrm{GL}(n)} = \mathbb{K}$ and $f \to f((0,\dots,0))$ is a G-invariant map.

2.1.13. For $n = 2$ it is sufficient to prove that

$$\{x_1^k, (x_1+x_2)^k, \dots, (x_1+kx_2)^k\}$$

is a basis in the space of homogeneous polynomials of degree k. Here one may use the Vandermonde determinant. For $n > 2$ use induction.

2.1.14. Let f be a homogeneous invariant of degree k. There are linear forms $l_1, \dots, l_N$ with $f = l_1^k + \dots + l_N^k$. Then

$$f = \sum_{i=1}^{N}\Big(\frac{1}{|G|}\sum_{g\in G}(g\cdot l_i)^k\Big).$$

Any $\sum_{g\in G}(g\cdot l_i)^k$ is a symmetric polynomial in $g\cdot l_i$, and it may be expressed in elementary symmetric polynomials in $g\cdot l_i$. But these elementary symmetric polynomials are homogeneous invariants of degree $\le |G|$.

2.1.15. (a) The invariant bilinear symmetric form q defines a quadratic invariant on $\mathbb{K}^n$: $F(x) := q(x,x)$. Consider the standard orthogonal basis $e_1, \dots, e_n$, set $S = \langle e_1\rangle$, and prove that $\mathcal{O}(\mathbb{K}^n)^{\mathrm{SO}(n)} = \mathbb{K}[F]$.

(b) The action $\mathrm{Sp}(2n) : \mathbb{K}^{2n}$ is transitive on the set of non-zero vectors, so $\mathcal{O}(\mathbb{K}^{2n})^{\mathrm{Sp}(2n)} = \mathbb{K}$.

(c) For $s < n$ the action has open orbit and $\mathcal{O}(X)^{\mathrm{SL}(n)} = \mathbb{K}$. If $s = n$, then define $F(v_1, \dots, v_n) := \det(v_1, \dots, v_n)$, set $S = \langle (e_1, \dots, e_n)\rangle$, and prove that $\mathcal{O}(X)^{\mathrm{SL}(n)} = \mathbb{K}[F]$.

2.1.16. (a) $x_1^n, x_1^{n-1}x_2, \dots, x_1x_2^{n-1}, x_2^n$;

(b) x_1^n, x_2^n, x_1x_2.

2.1.17. $x_1x_3^3, x_1^2x_4^3, x_2x_3, x_2^2x_4, x_1x_2x_4^2, x_1x_3x_4$

Prove that there exists a generating set consisting of monomials, and generating monomials correspond to non-decomposable non-negative integer solutions of the equation $3a_1 + a_2 - a_3 - 2a_4 = 0$.

2.2.11. Put $X = \mathrm{SL}(2)$ and $G = U(2)$ with the action by right translations. Here $\mathrm{Spec}(\mathcal{O}(X)^G) \cong \mathbb{K}^2$, but $\pi(X) = \mathbb{K}^2 \setminus \{0\}$ (cf. Proposition 1.4.6 and Example 1.4.8).

2.2.12. All semisimple elements in $\pi^{-1}(y)$ are conjugate to the same diagonal matrix, thus form one $\mathrm{GL}(n)$-orbit O'. By Corollary 1.7.12, the centralizer of a semisimple element in $\pi^{-1}(y)$ has the largest dimension among centralizers of elements in $\pi^{-1}(y)$, thus the orbit O' has the smallest dimension.

2.2.13. Since for any $f \in \mathcal{O}(X)$ all coefficients of the polynomial

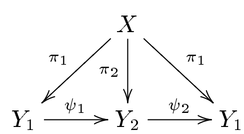
$$F_f(T) = \prod_{g \in G} (T - g \cdot f)$$

are G-invariants, the extension $\mathcal{O}(X)^G \subseteq \mathcal{O}(X)$ is integral. Moreover, any G-orbit is closed in X.

2.2.14. Consider $\mathbb{K}^\times : \mathbb{K}^3$, $t \cdot (x_1, x_2, x_3) = (tx_1, tx_2, t^{-1}x_3)$.

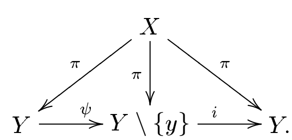
2.2.15. Follow the proof of Theorem 2.2.7.

2.2.16. Let $\pi_1 : X \to Y_1$ and $\pi_2 : X \to Y_2$ be categorical quotients. Applying Definition 2.2.5 to the G-invariant morphisms $\pi_2 : X \to Y_2$ and $\pi_1 : X \to Y_1$, we get morphisms ψ_1 and ψ_2:



The uniqueness implies that $\psi_2 \circ \psi_1 = \mathrm{id}_{Y_1}$. Similarly, $\psi_1 \circ \psi_2 = \mathrm{id}_{Y_2}$.

2.2.17. Assume that $y \in Y := X/\!/G$, $y \notin \pi(X)$. Take $Z := Y \setminus \{y\}$. Then $\pi : X \to Z$ is a G-invariant morphism, and there is $\psi : Y \to Y \setminus \{y\}$ making the diagram commutative:



Let $i : Y \setminus \{y\} \to Y$ be the embedding. Then uniqueness implies that $i \circ \psi = \mathrm{id}_Y$. But $(i \circ \psi)(y) \neq y$, a contradiction.

2.2.18. The condition $\mathcal{O}(\mathbb{K}^n)^{T^m} \neq \mathbb{K}$ is equivalent to the existence of a non-constant monomial $x_1^{a_1} \dots x_n^{a_n}$ with $a_1\chi_1 + \dots + a_n\chi_n = 0$.

2.2.19. It is sufficient to prove that $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)^G$ is an isomorphism. Since ϕ is dominant, the homomorphism ϕ^* is injective. Take any $f \in \mathcal{O}(X)^G$. We know, that f is constant on a generic fiber of ϕ. Corollary 3.0.29 guarantees that there is $\tau \in \mathcal{O}(Y)$ with $\phi^*(\tau) = f$.

2.2.20. For $s = n$, consider $\phi : X \to \mathrm{Sym}(n)$, $\phi(v_1, \dots, v_n) = (q(v_i, v_j))$. This morphism is surjective, because any symmetric matrix S has a form $A^T A$ for some $A \in \mathrm{Mat}(n \times n)$. Moreover, for a non-degenerate S the preimage $\phi^{-1}(S)$ consists of a unique G-orbit: $A^T A = B^T B \Rightarrow A = (A^T)^{-1} B^T B$, $(A^T)^{-1} B^T \in \mathrm{O}(n)$. Now apply Igusa's criterion. For $s < n$, use Lemma 2.1.5 and surjectivity of ϕ.

2.2.21. Consider $X = G$ as an affine H-variety with the right H-action. All orbits of the action have the same dimension, thus are closed, and any fiber of the quotient morphism $\pi : G \to G/\!/H$ is an H-coset. Since π is the categorical quotient, there is a (bijective) morphism $\psi : G/\!/H \to G/H$, which is an isomorphism.

2.2.22. If $z_i \in \overline{Gx_i} \cap \overline{Gx_{i+1}}$, then $\phi(x_i) = \phi(z_i) = \phi(x_{i+1})$.

2.2.23. It is sufficient to prove that any two points on $\mathbb{P}^n$ are equivalent. Renumbering, one may assume that there is a one-parameter subgroup $\lambda : \mathbb{K}^\times \to T^m$ such that $\langle \lambda, \chi_1 \rangle \le \langle \lambda, \chi_i \rangle \le \langle \lambda, \chi_2 \rangle$ and $\langle \lambda, \chi_1 \rangle < \langle \lambda, \chi_2 \rangle$. For any $x \in \mathbb{P}^n$ denote by x_{χ_j} the point $[y_1 : \dots : y_{n+1}]$ with $y_i = x_i$ for $\langle \lambda, \chi_i \rangle = \langle \lambda, \chi_j \rangle$ and $y_i = 0$ otherwise. Check that for any $x, y \in \mathbb{P}^n$

$$x \sim x_{\chi_1} \sim (x_{\chi_1} + y_{\chi_2}) \sim y_{\chi_2} \sim y.$$

2.3.17. Consider the diagonal action $\mathrm{SL}(2) : \mathbb{K}^2 \oplus \mathbb{K}^2$, and set $Z = \langle e_1 \rangle \oplus \langle e_1 \rangle$, $X = \overline{\mathrm{SL}(2)Z}$. Here $\mathcal{O}(X)^{\mathrm{SL}(2)} = \mathbb{K}$, because any orbit in Z contains zero in its closure. But X does not have open SL(2)-orbit, so $\mathbb{K}(X)^{\mathrm{SL}(2)} \neq \mathbb{K}$.

2.3.18. Use Proposition 2.3.6.

2.3.19. Assume that G-orbits in X are of the same dimension. Then (G1), (G2) and (G4) follow from Theorem 2.2.2, and (G3) follows from Theorem 3.0.30. In fact, one may remark that $\pi(U) = \pi(GU)$ and deduces (G3) from Theorem 2.2.2 (2).

2.3.20. Follows from Exercise 2.3.19.

2.3.21. If $Q\mathcal{O}(X)^G = \mathbb{K}(X)^G$, then elements of $\mathcal{O}(X)^G$ separate G-orbits on the open subset U of Corollary 2.3.14. By Lemma 2.3.13, the intersection of a generic fiber of π with U is dense in this fiber.

Conversely, if $W \subseteq X$ is an open subset consisting of G-orbits of maximal dimension, then generators of $\mathcal{O}(X)^G$ separate orbits on $W \cap \pi^{-1}(V)$, and thus generate $\mathbb{K}(X)^G$ (Proposition 2.3.16).

2.3.22. Let $f_1, \ldots, f_m$ be a generating set of $\mathbb{K}(X)^G$. There is an open subset $W \subseteq X/G$ such that all f_i are regular on $\pi^{-1}(W)$. Then

$$f_1, \ldots, f_m \in \mathcal{O}(\pi^{-1}(W))^G = \pi^*(\mathcal{O}(W)) \subseteq \pi^*(\mathbb{K}(X/G)).$$

For a categorical quotient, consider $\mathbb{K}^\times : \mathbb{K}^n$, $t \cdot (x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$.

2.3.23. If $\pi : \mathbb{K}^2 \setminus \{0\} \to Y$ is the geometric quotient, then Y is a curve. Since any curve is either affine or projective, the points $y_1 = \pi(Ge_1)$ and $y_2 = \pi(Ge_2)$ lies in a common affine chart $U \subseteq Y$. There is a function $f \in \mathcal{O}(U)$ with $f(y_1) \neq f(y_2)$. Then $\pi^*(f)$ is a rational invariant on $\mathbb{K}^2 \setminus \{0\}$ that separates Ge_1 and Ge_2. But $\mathbb{K}(\mathbb{K}^2 \setminus \{0\})^{\mathbb{K}^\times} = \mathbb{K}(\mathbb{K}^2)^{\mathbb{K}^\times} = \mathbb{K}(x_1x_2)$, a contradiction.

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