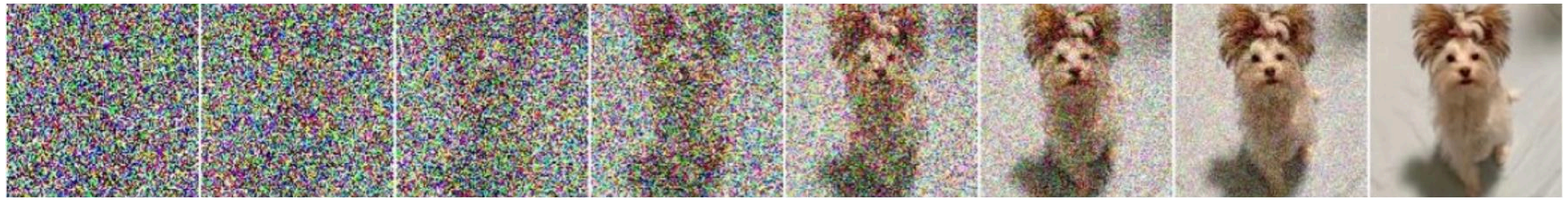
Schrödinger Bridges and Stochastic Control

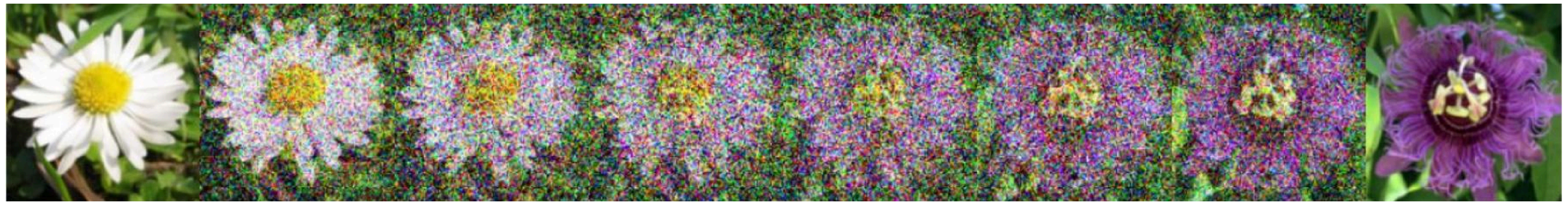
Denis Rakitin

Translation between distributions

Between what?

- Generative modeling (noise → data)
- Image2Image / editing (data → data)


[1]


[2]

Additional settings

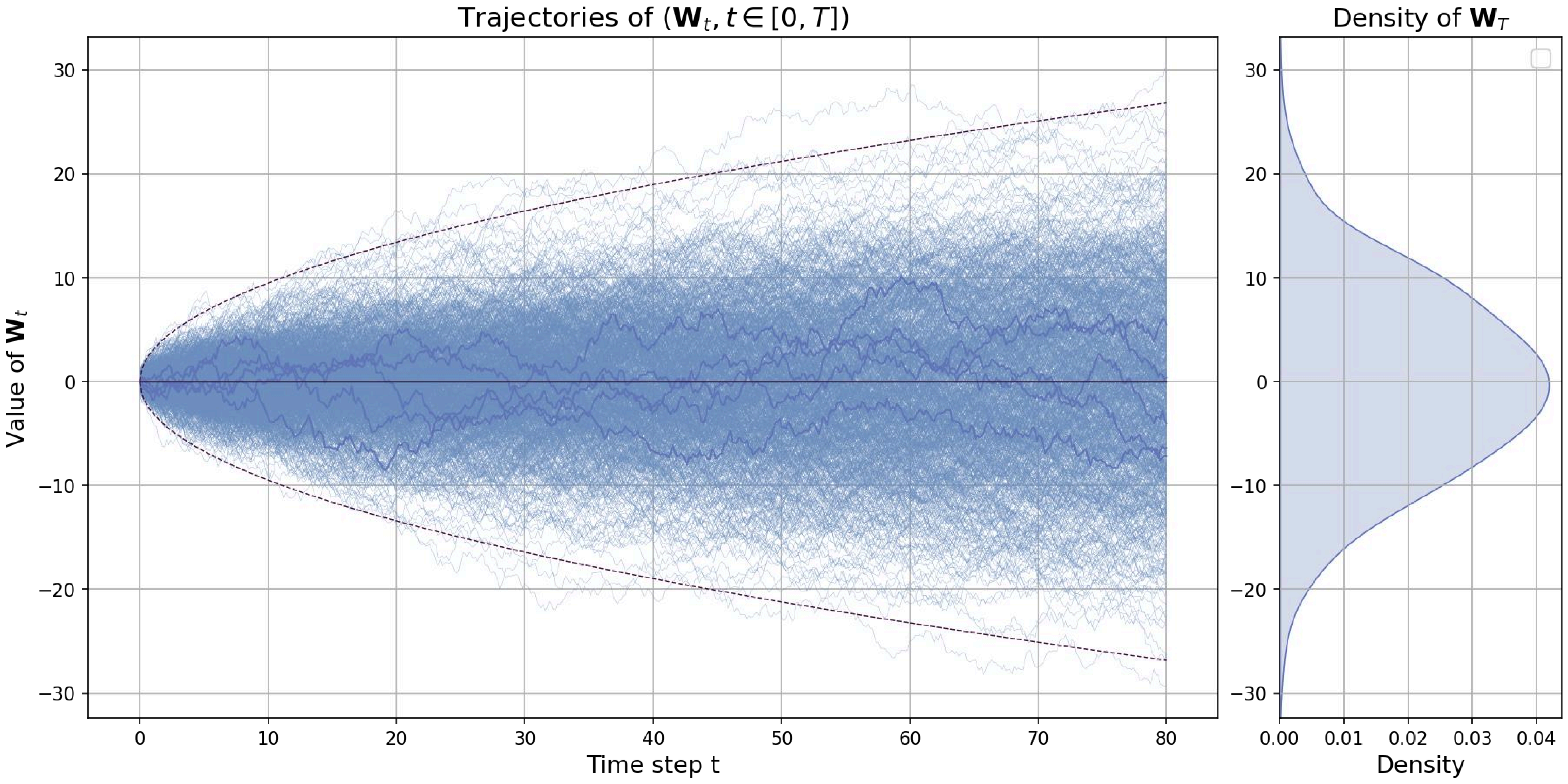
- RL fine-tuning
- Data-free sampling

[3]

Continuous time: Wiener process

Process $\mathbf{W}_t$ with the properties of "random walk":

- $\mathbf{W}_t - \mathbf{W}_s \perp (\mathbf{W}_\tau, \tau \in [0, s])$; $\mathbf{W}_t - \mathbf{W}_s \sim \mathcal{N}(0, (t-s)I)$



Stochastic Differential Equations

SDEs: stochastic dynamics that are

- Markovian
- Deterministic velocity + random walk

Stochastic Differential Equations

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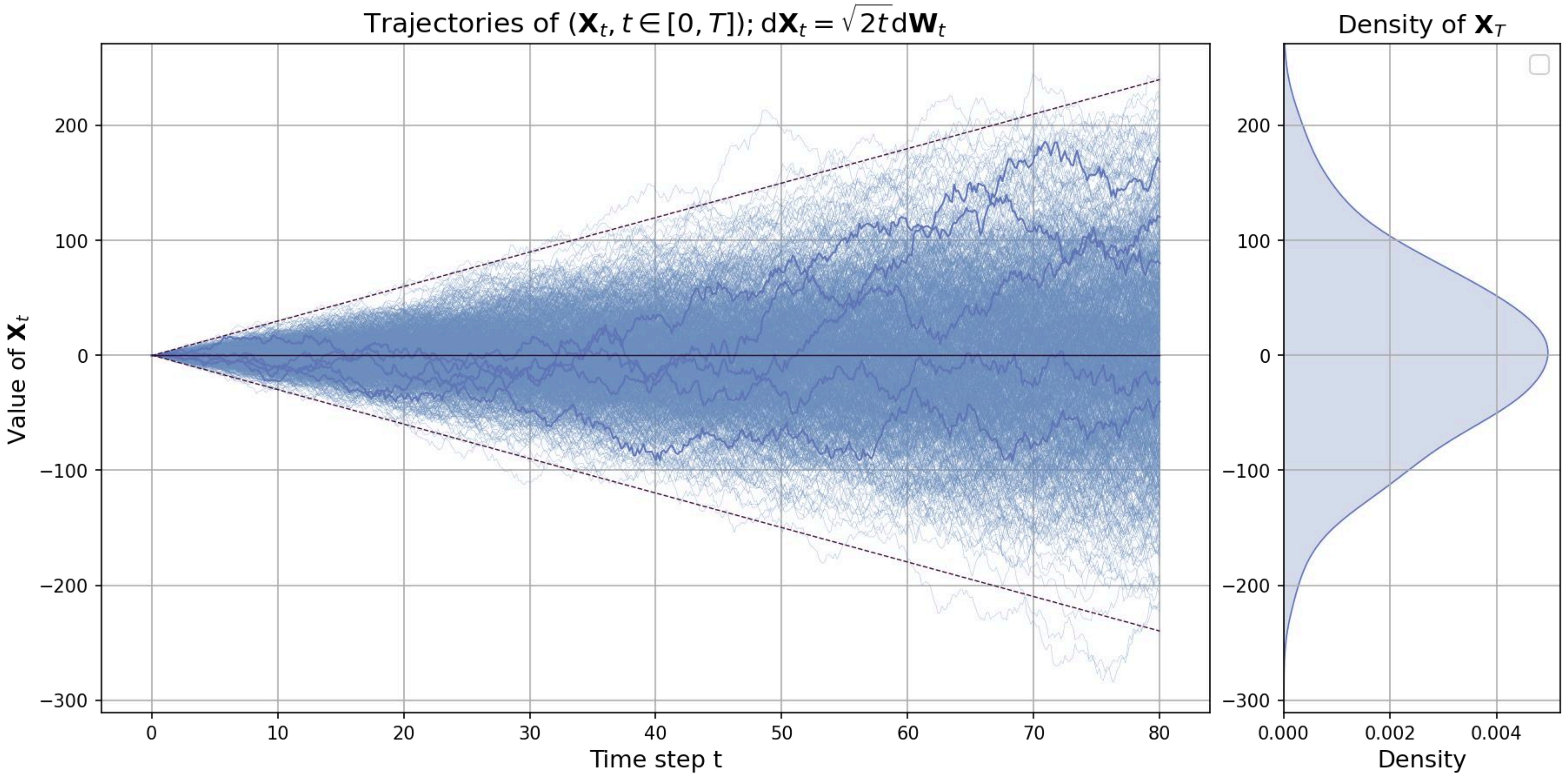
(Pseudo)-definition + Euler scheme

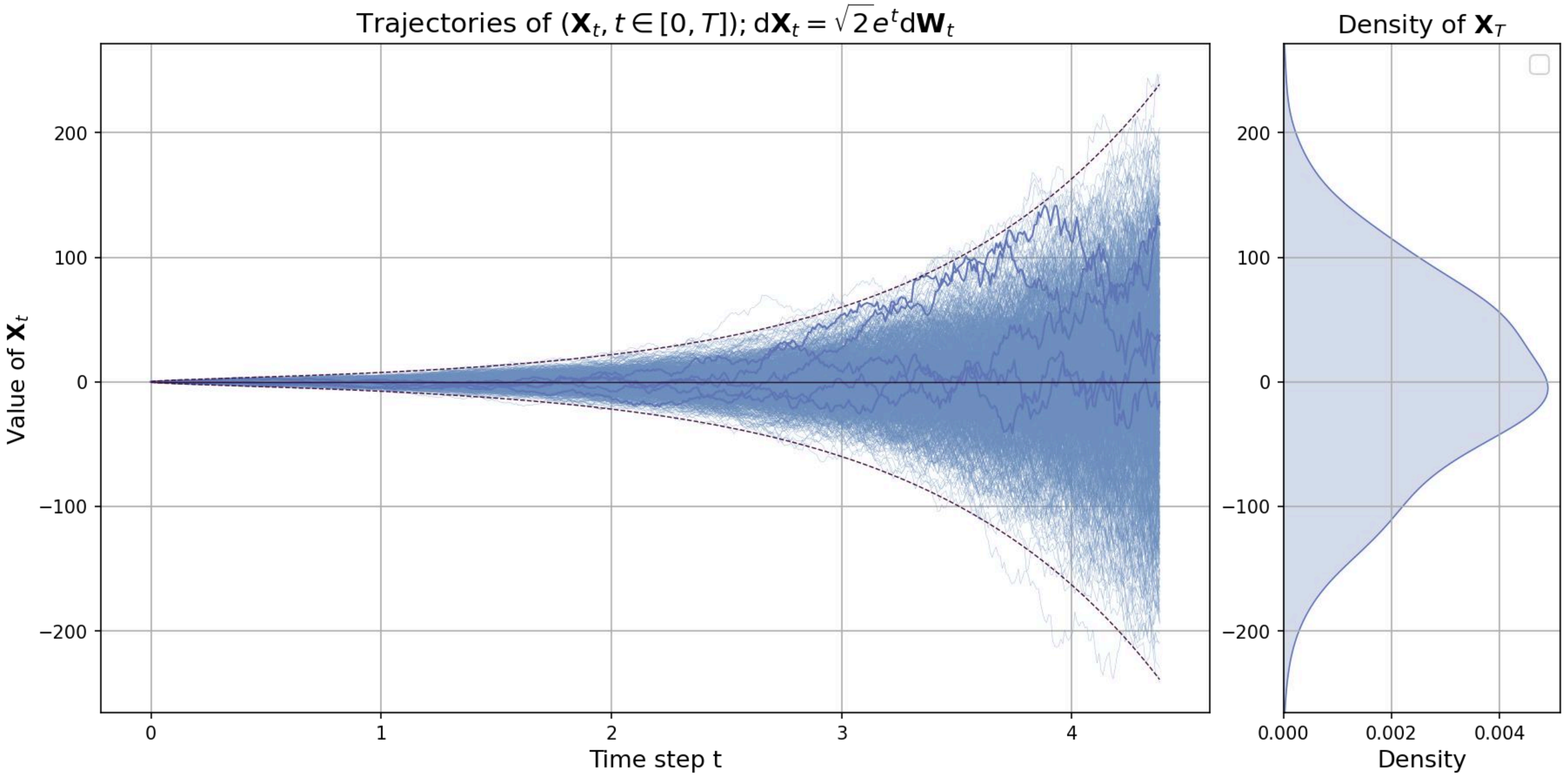
- $\mathrm{d}\mathbf{X}_t = f_t(\mathbf{X}_t)\mathrm{d}t + g(t)\mathrm{d}\mathbf{W}_t$
- $$\begin{aligned}\mathbf{X}_{t+h} &\approx \mathbf{X}_t + hf_t(\mathbf{X}_t) + g(t)(\mathbf{W}_{t+h} - \mathbf{W}_t) \\ &\approx \mathbf{X}_t + hf_t(\mathbf{X}_t) + \sqrt{h}g(t)\boldsymbol{\epsilon}_t; \quad \boldsymbol{\epsilon}_t \sim \mathcal{N}(0, I)\end{aligned}$$

Variance Exploding SDE

$$\mathrm{d}\mathbf{X}_t = g(t)\mathrm{d}\mathbf{W}_t \quad \mathbf{X}_{t+h} \approx \mathbf{X}_t + \sqrt{h}g(t)\boldsymbol{\epsilon}_t$$

$$p_{\mathbf{X}_t|\mathbf{X}_0}(\mathbf{x}_t|\mathbf{x}_0) = \mathcal{N}\left(\mathbf{x}_t|\mathbf{x}_0, \int_0^t g^2(s)\mathrm{d}s\right)$$

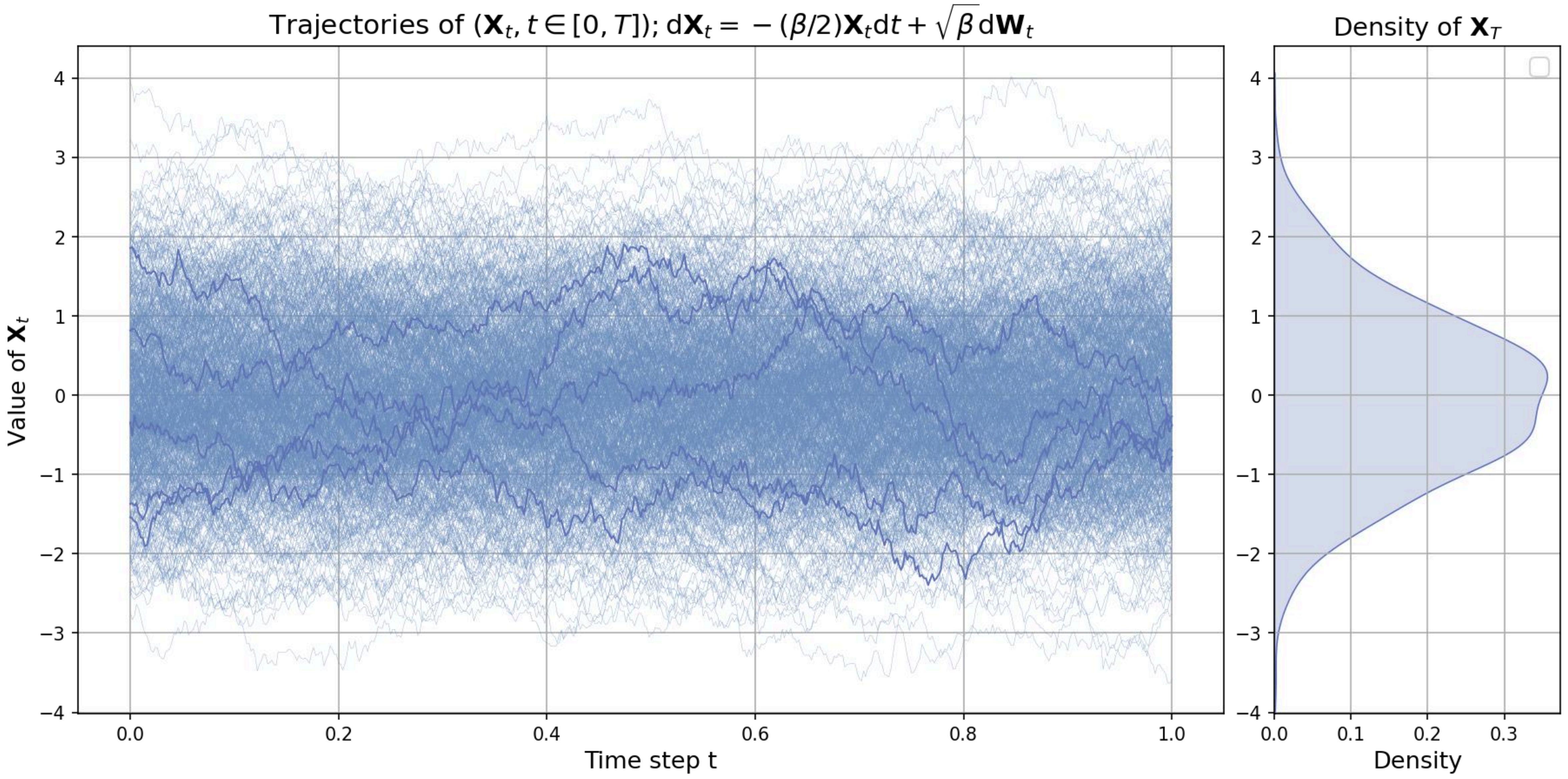


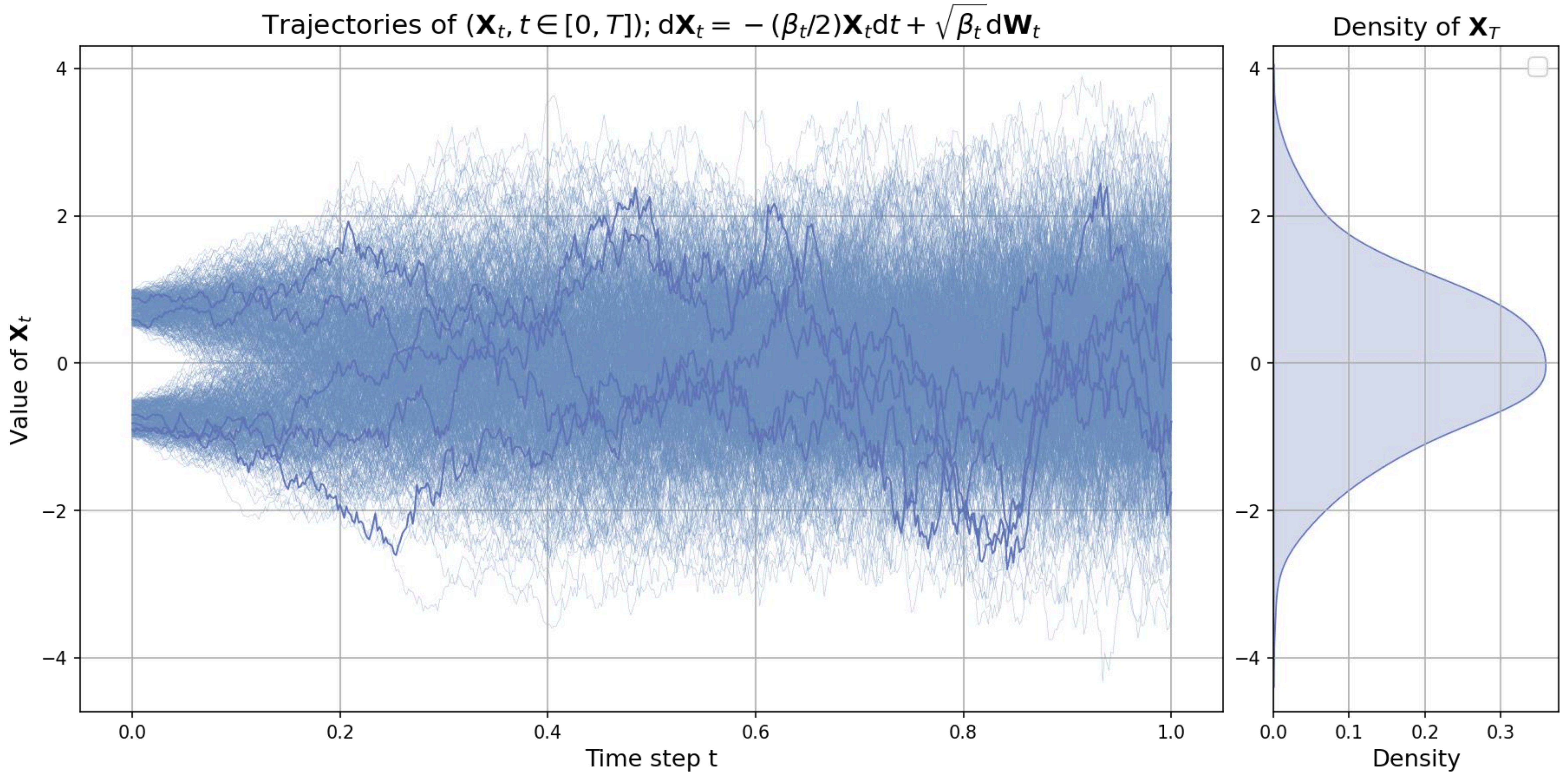


Variance Preserving SDE

$$\mathrm{d}\mathbf{X}_t = -\frac{\beta_t}{2}\mathbf{X}_t \mathrm{d}t + \sqrt{\beta_t}\mathrm{d}\mathbf{W}_t \quad \mathbf{X}_{t+h} \approx \mathbf{X}_t\left(1 - \frac{h\beta_t}{2}\right) + \sqrt{h\beta_t}\boldsymbol{\epsilon}_t$$

$$p_{\mathbf{X}_t|\mathbf{X}_0}(\mathbf{x}_t|\mathbf{x}_0) = \mathcal{N}\left(\mathbf{x}_t|\sqrt{\alpha_t}\mathbf{x}_0, (1-\alpha_t)I\right); \quad \alpha_t = \exp\left(-\int_0^t \beta_s \mathrm{d}s\right)$$





Schrödinger Bridges (SB)

Let $\mathrm{d}\mathbf{Z}_t = f_t^{\mathrm{base}}(\mathbf{Z}_t)\mathrm{d}t + \sqrt{\gamma}\mathrm{d}\mathbf{W}_t$ be some "base" SDE (e.g. Wiener process)

Goal: manipulate $\mathbf{Z}_t$ to perform transition between the desired distributions

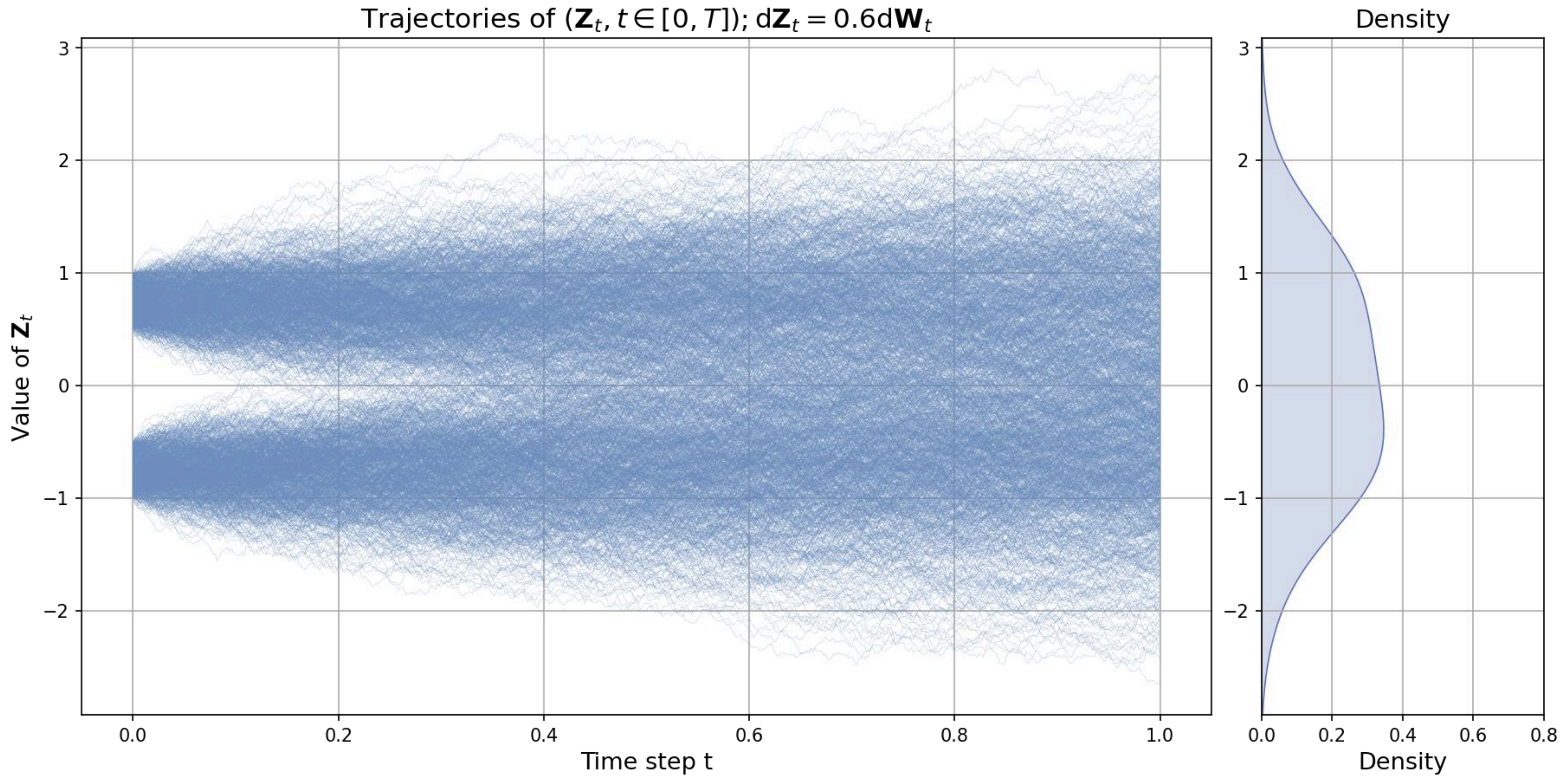
Formalization:

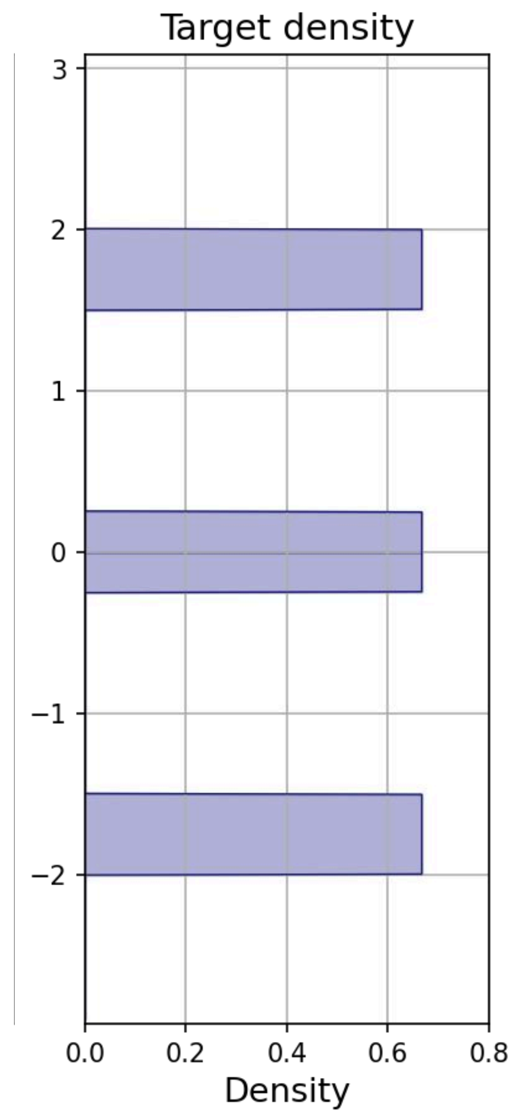
$$\begin{cases} \mathrm{KL}(\mathsf{P}_{\mathbf{X}} \| \mathsf{P}_{\mathbf{Z}}) \to \min\limits_{\mathsf{P}_{\mathbf{X}}}; \\ p_{\mathbf{X}_0} = p^{\mathcal{S}}; p_{\mathbf{X}_1} = p^{\mathcal{T}} \end{cases}$$

Schrödinger Bridges (SB)

$$\mathrm{d}\mathbf{Z}_t = f_t^{\mathrm{base}}(\mathbf{Z}_t)\mathrm{d}t + \sqrt{\gamma}\mathrm{d}\mathbf{W}_t$$

$$\mathbf{X}_t \text{ solves } \begin{cases} \mathrm{KL}(\mathsf{P}_{\mathbf{X}} \| \mathsf{P}_{\mathbf{Z}}) \to \min\limits_{\mathsf{P}_{\mathbf{X}}}; \\ p_{\mathbf{X}_0} = p^{\mathcal{S}}; p_{\mathbf{X}_1} = p^{\mathcal{T}} \end{cases}$$





Schrödinger Bridges (SB)

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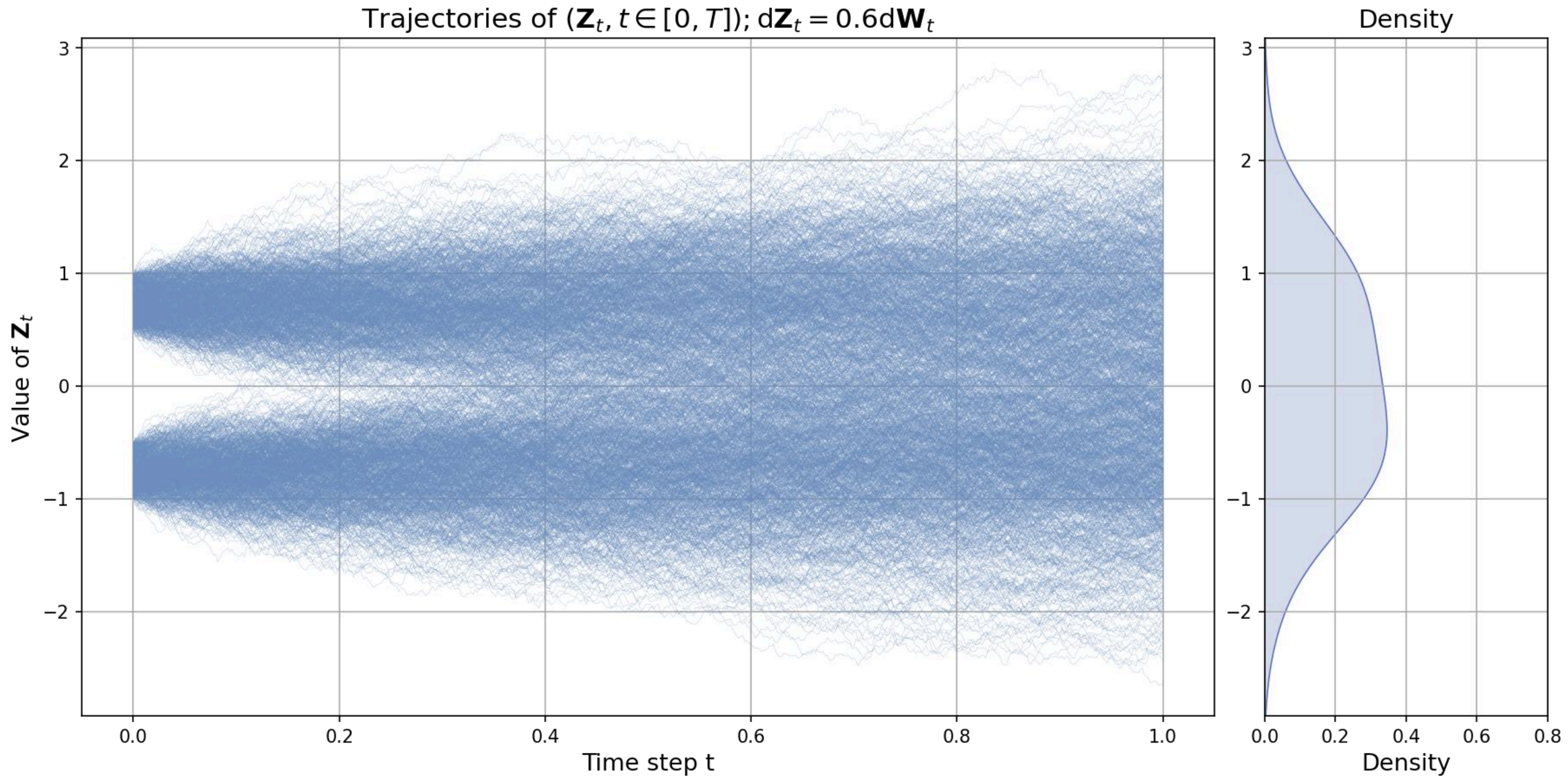
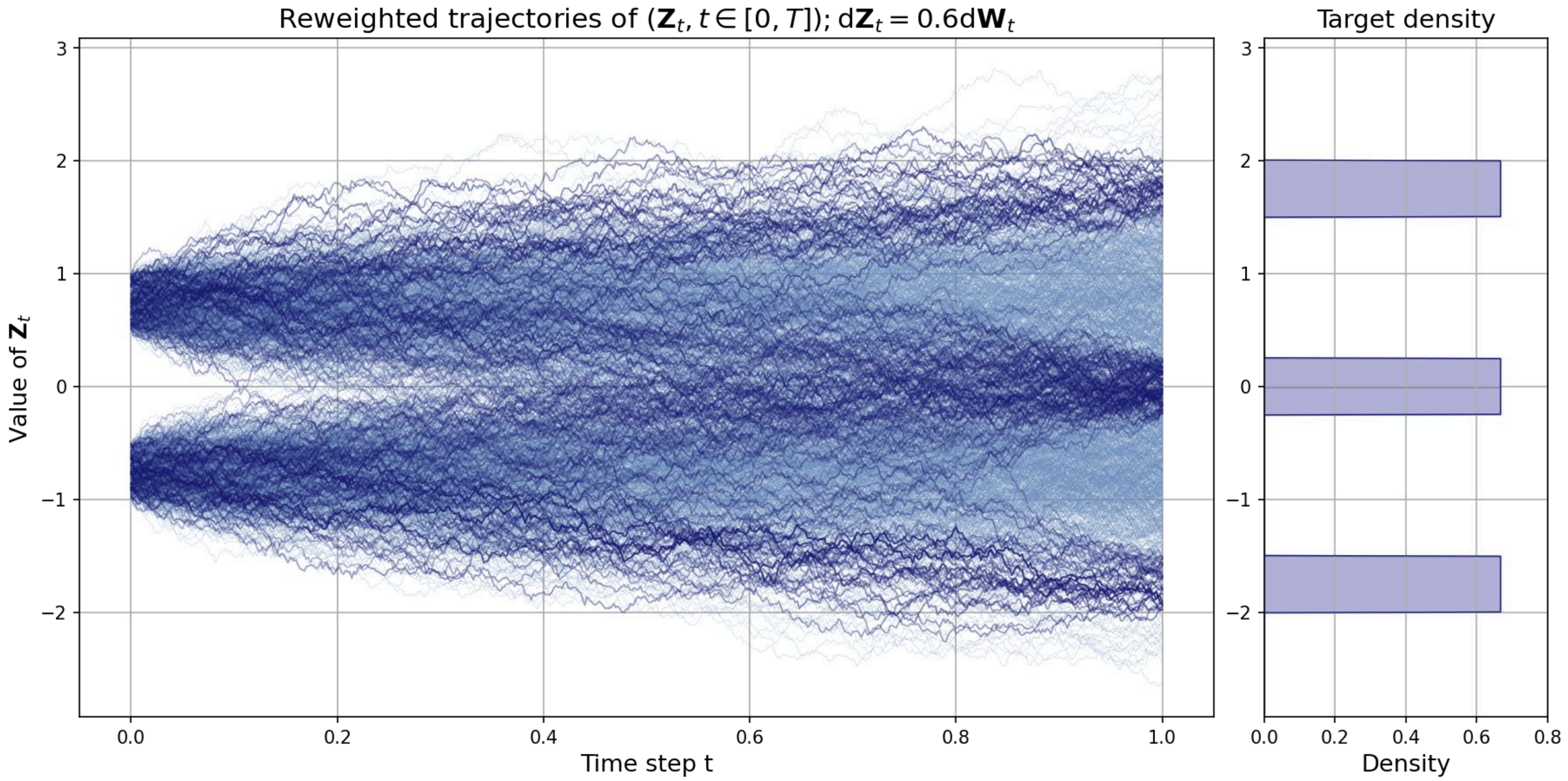
Reweighting as generalization of conditioning

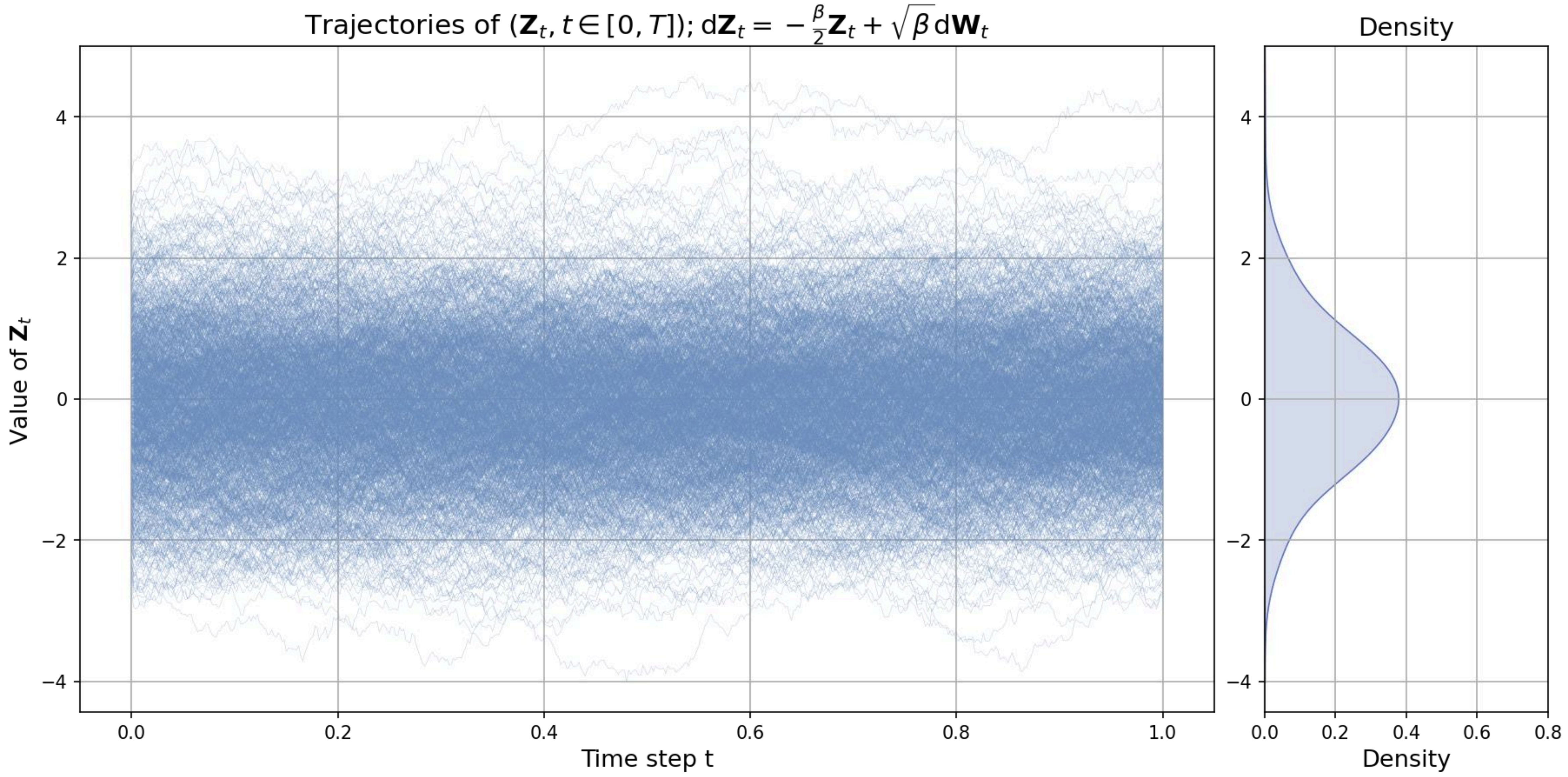




Applications: generative modeling

$$\mathrm{d}\mathbf{Z}_t = -\frac{\beta_t}{2}\mathbf{Z}_t \mathrm{d}t + \sqrt{\beta_t}\mathrm{d}\mathbf{W}_t$$

$$\begin{cases} \mathrm{KL}(\mathsf{P}_{\mathbf{X}} \| \mathsf{P}_{\mathbf{Z}}) \to \min\limits_{\mathsf{P}_{\mathbf{X}}}; \\ p_{\mathbf{X}_0} = \mathcal{N}(0, I);\ p_{\mathbf{X}_1} = p^{\text{data}} \end{cases} \Longrightarrow$$



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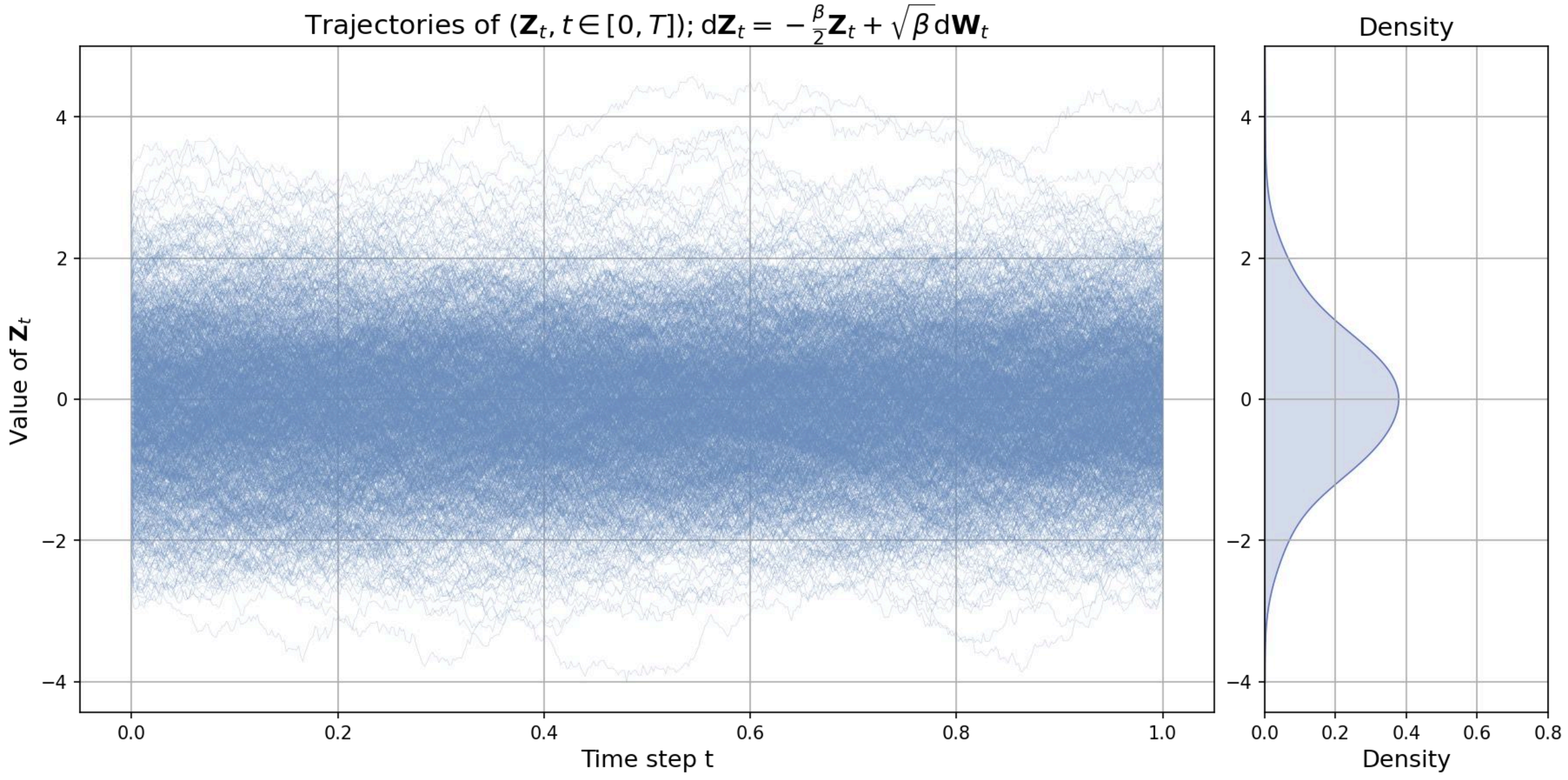
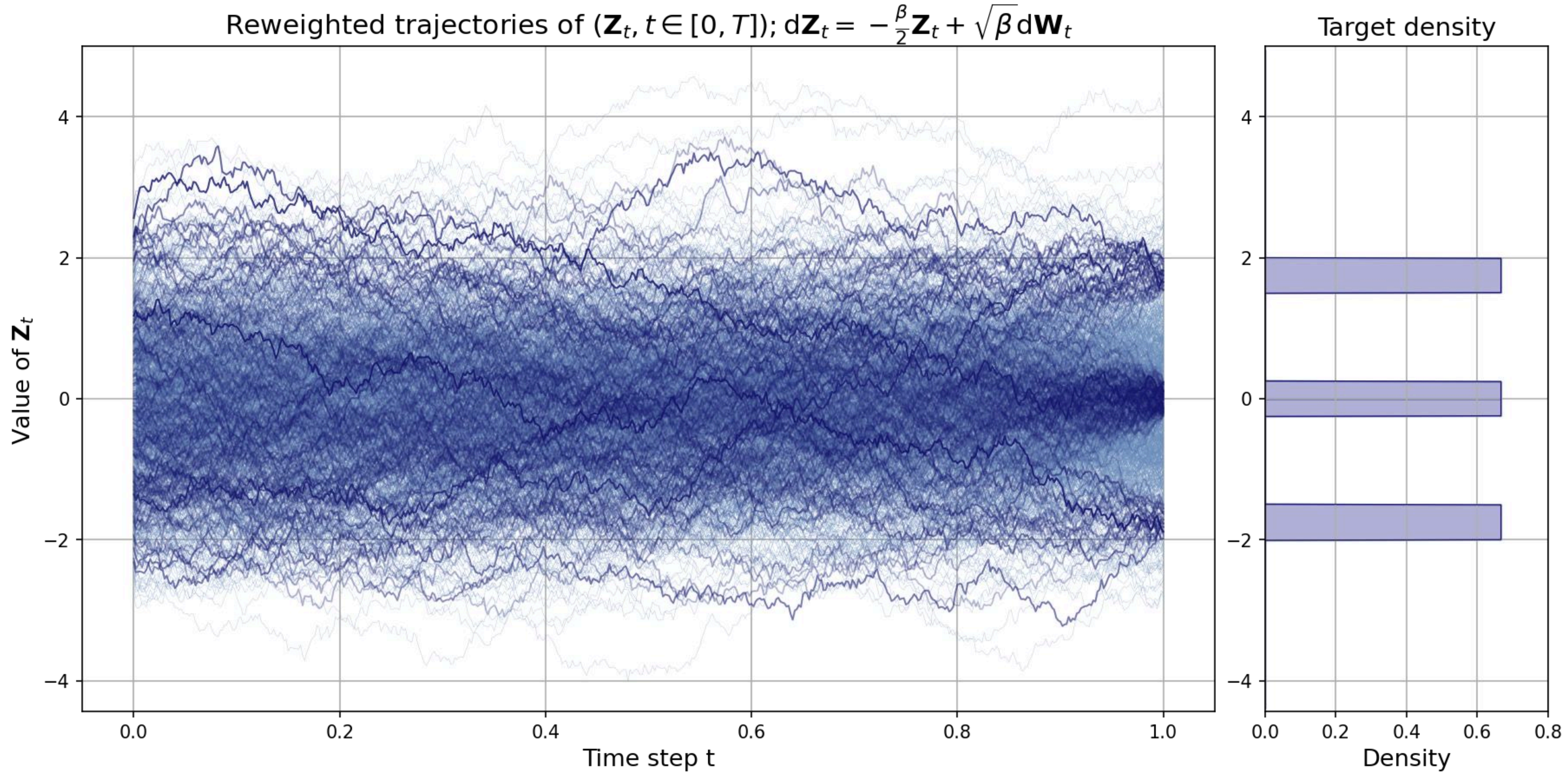
$\Longrightarrow$

Generative modeling
(Diffusion models)





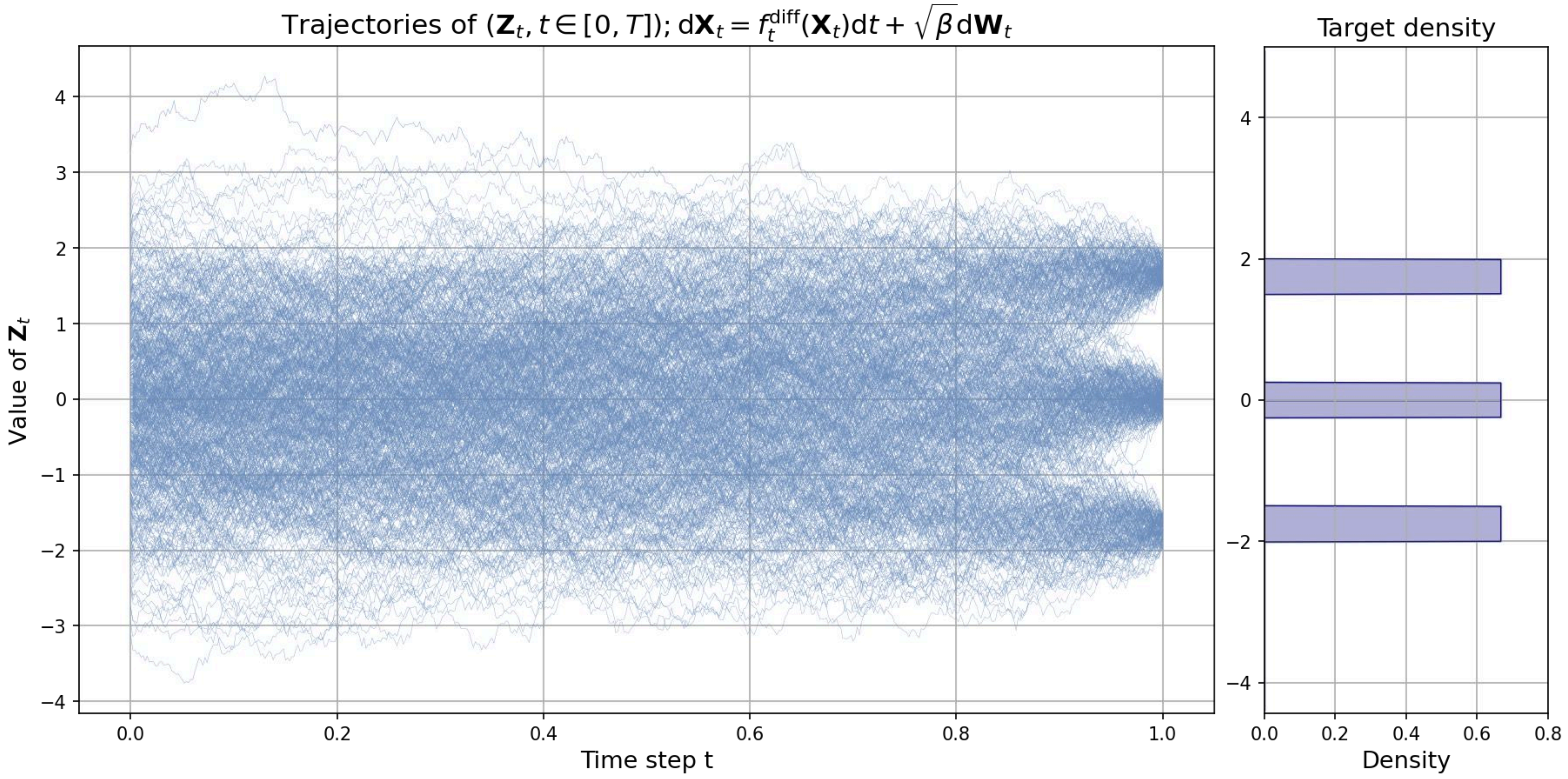
Applications: finetuning

$$\mathrm{d}\mathbf{Z}_t = f_t^{\mathrm{diff}}(\mathbf{Z}_t)\mathrm{d}t + g(t)\mathrm{d}\mathbf{W}_t$$

$$\begin{cases} \mathrm{KL}(\mathsf{P}_{\mathbf{X}} \| \mathsf{P}_{\mathbf{Z}}) \to \min\limits_{\mathsf{P}_{\mathbf{X}}}; \\ p_{\mathbf{X}_0} = \mathcal{N}(0, I);\ p_{\mathbf{X}_1}(\mathbf{x}_1) \propto p^{\mathrm{data}}(\mathbf{x}_1) \cdot \exp(R(\mathbf{x}_1)) \end{cases}$$

$$\Rightarrow$$

$$R(\mathbf{x}_1) = |\mathbf{x}_1|$$



Applications: finetuning

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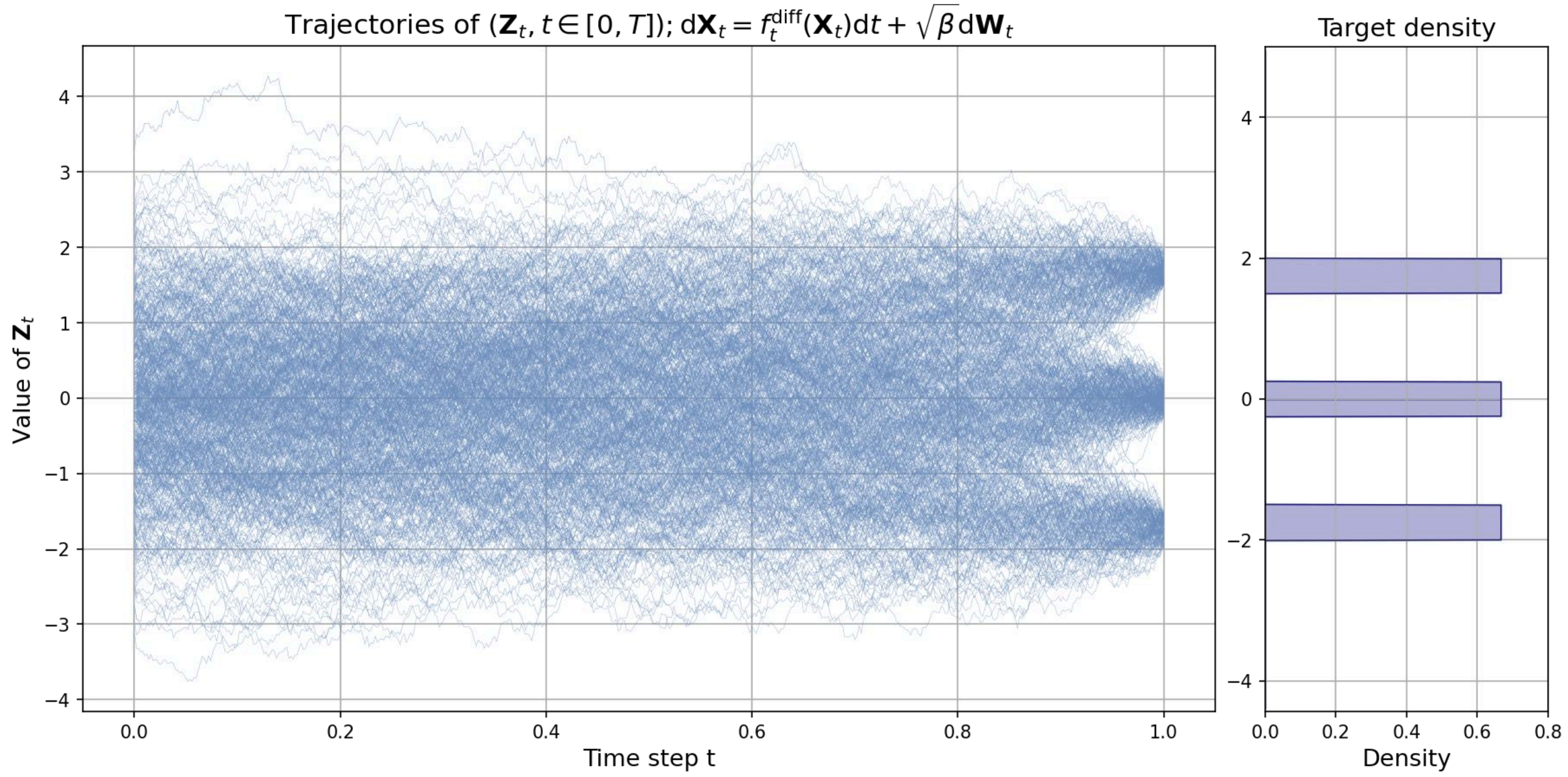
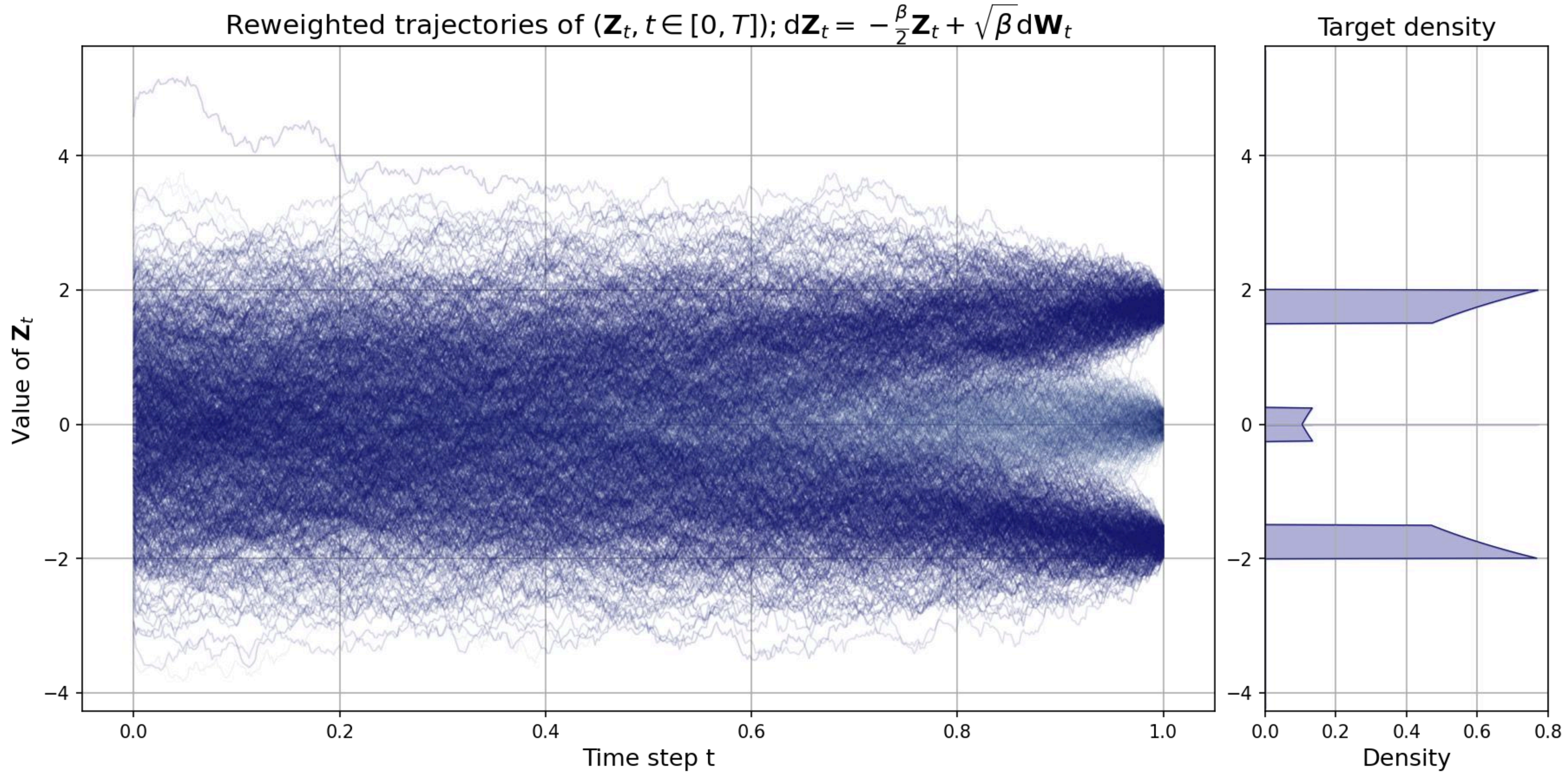
$\Rightarrow$

Diffusion models +
fine-tuning on reward

$$R(\mathbf{x}_1) = |\mathbf{x}_1|$$





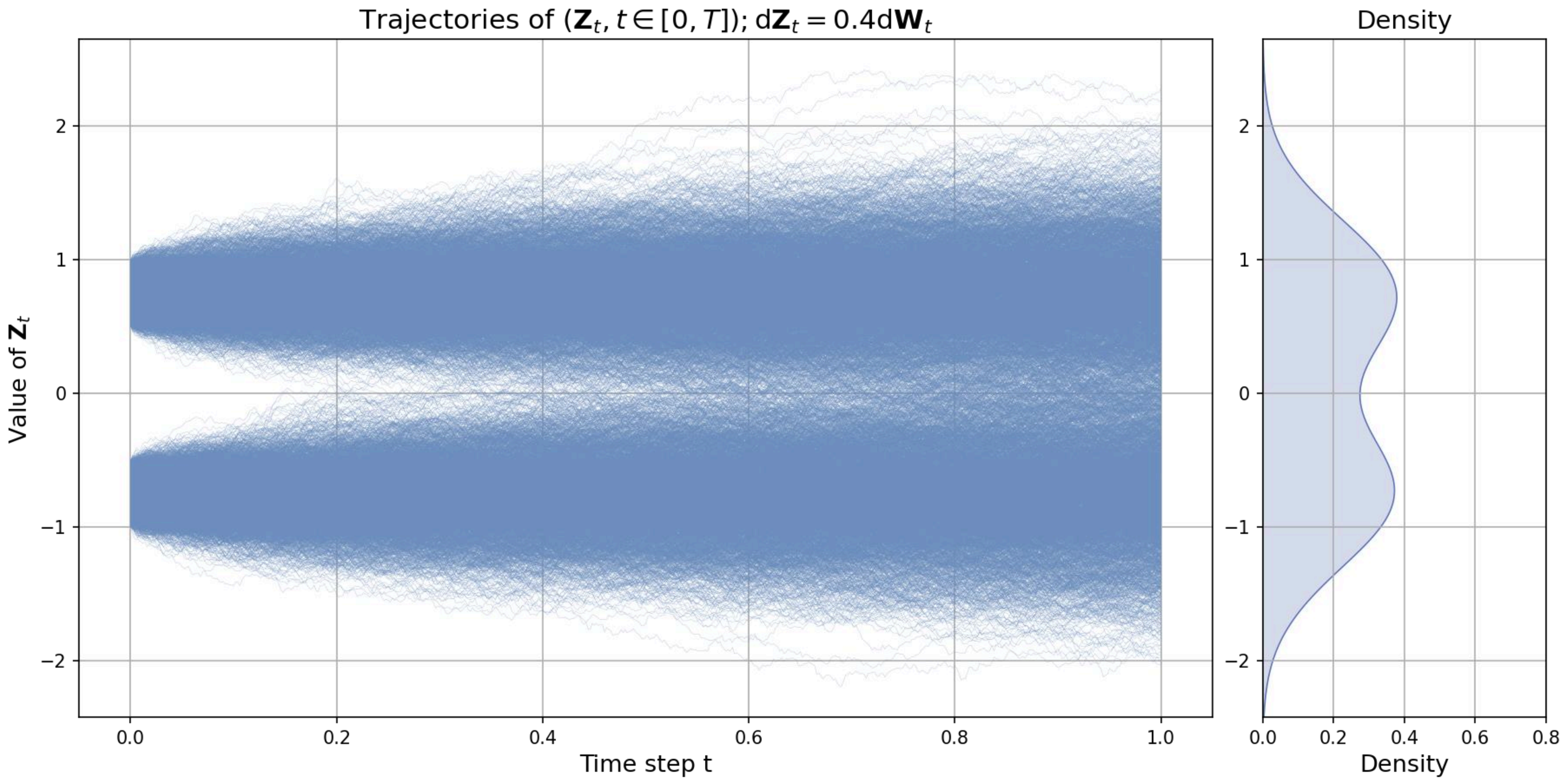
Applications: optimal transport

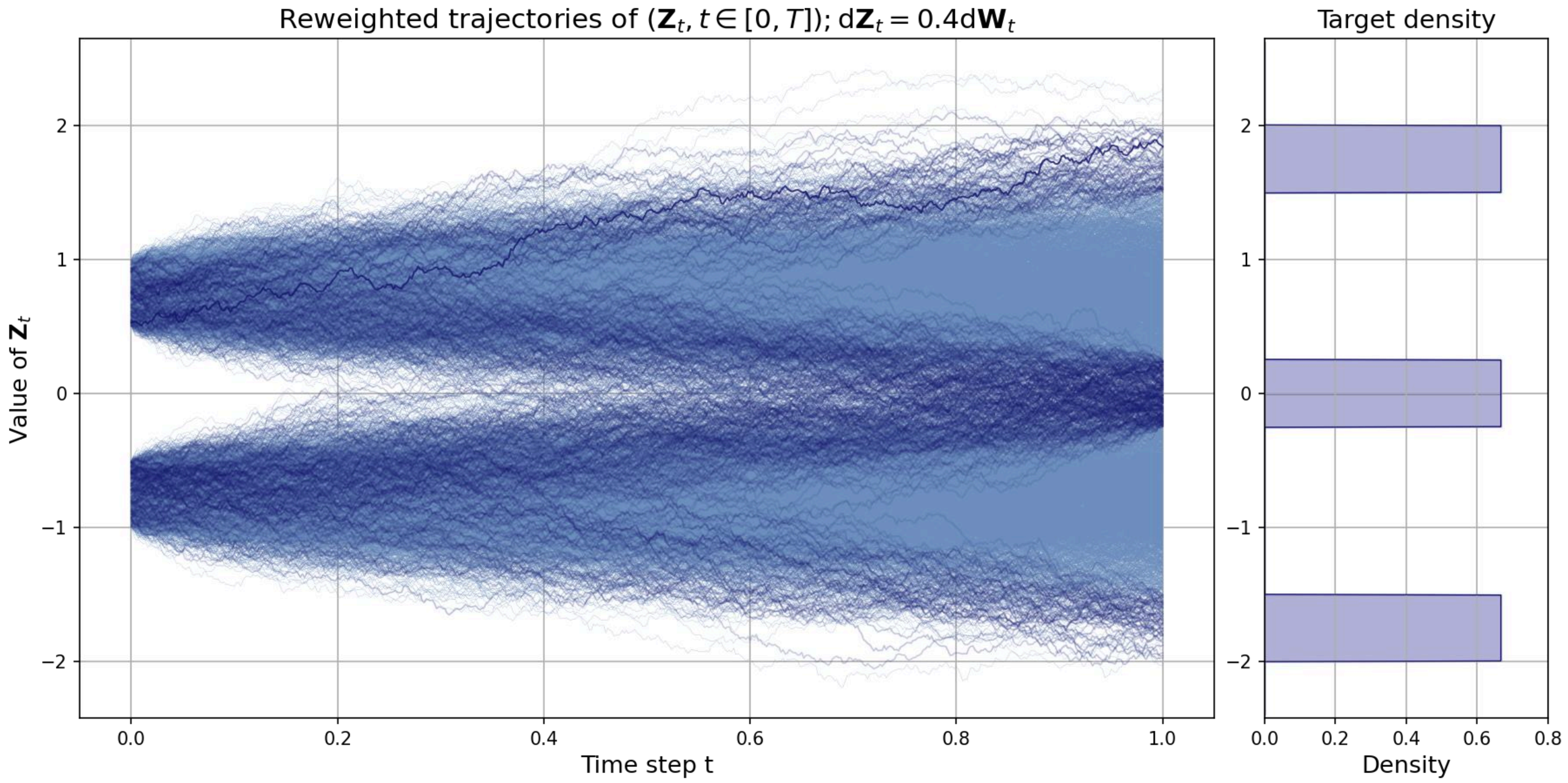
$$\mathrm{d}\mathbf{Z}_t = \sqrt{\gamma}\mathrm{d}\mathbf{W}_t$$

$$\begin{cases} \mathrm{KL}(\mathsf{P}_{\mathbf{X}} \| \mathsf{P}_{\mathbf{Z}}) \to \min\limits_{\mathsf{P}_{\mathbf{X}}}; \\ p_{\mathbf{X}_0} = p^{\mathcal{S}}; p_{\mathbf{X}_1} = p^{\mathcal{T}} \end{cases}$$

$\Rightarrow$

Optimal transport:
target density + in/out similarity





Stochastic optimal control (SOC)

$$\begin{cases} \mathbb{E}_{\mathsf{P}^f}\left[\int_0^1 V_t(\mathbf{X}_t^f)\mathrm{d}t + \beta(\mathbf{X}_1^f)\right] \to \min_f; \\ \mathrm{d}\mathbf{X}_t^f = \left(f_t^{\mathrm{base}}(\mathbf{X}_t^f) + g(t) f_t(\mathbf{X}_t^f)\right)\mathrm{d}t + g(t)\mathrm{d}\mathbf{W}_t \end{cases}$$

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Environment transition

Action

Stochastic optimal control (SOC)

Current reward

Final reward

$$
\begin{cases}
\mathbb{E}_{\mathsf{P}^f}\left[\int_0^1 V_t(\mathbf{X}_t^f)\mathrm{d}t + \beta(\mathbf{X}_1^f)\right] \to \min_f; \\
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\end{cases}
$$

Environment transition

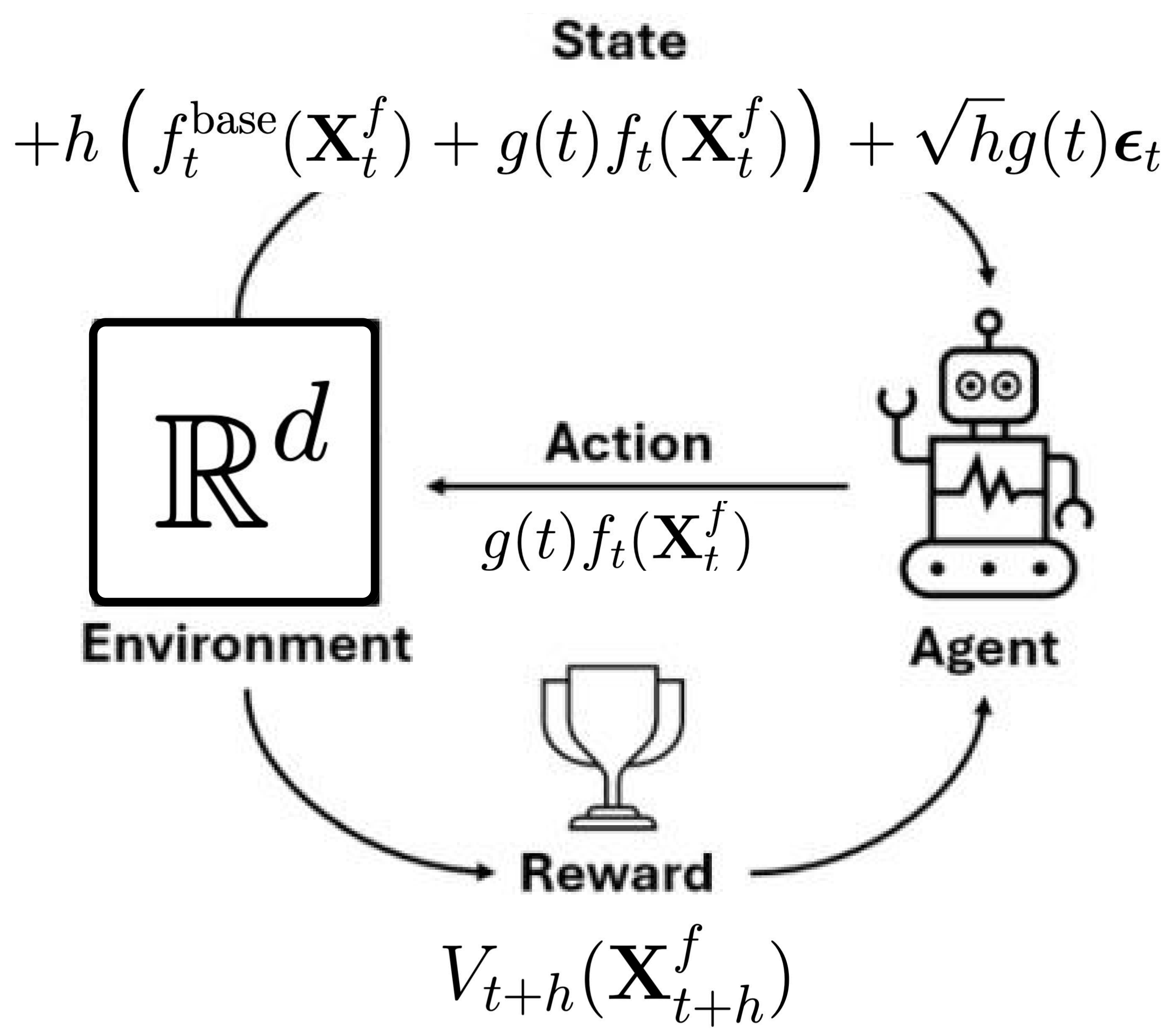
Action

Stochastic optimal control (SOC)

Current reward Final reward

$$\begin{cases} \mathbb{E}_{\mathbb{P}^f}\left[\int_0^1 V_t(\mathbf{X}_t^f)\mathrm{d}t + \beta(\mathbf{X}_1^f)\right] \to \min_f; \\ \mathrm{d}\mathbf{X}_t^f = \left(f_t^{\text{base}}(\mathbf{X}_t^f) + g(t) f_t(\mathbf{X}_t^f)\right)\mathrm{d}t + g(t)\mathrm{d}\mathbf{W}_t \end{cases}$$

Environment transition Action



SB through SOC

$$\mathrm{d}\mathbf{Z}_t = f_t^{\mathrm{base}}(\mathbf{Z}_t)\mathrm{d}t + \sqrt{\gamma}\mathrm{d}\mathbf{W}_t$$

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SB through SOC

$$d\mathbf{Z}_t = f_t^{\text{base}}(\mathbf{Z}_t)dt + \sqrt{\gamma}d\mathbf{W}_t$$

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[4]

$$\Leftrightarrow \begin{cases} \mathbb{E}_{\mathsf{P}^f}\left[\int\limits_0^1 \frac{1}{2}\|f_t(\mathbf{X}_t^f)\|^2 dt + \beta(\mathbf{X}_1^f)\right] \to \min\limits_f \\ d\mathbf{X}_t^f = \left(f_t^{\text{base}}(\mathbf{X}_t^f) + g(t)f_t(\mathbf{X}_t^f)\right)dt + g(t)d\mathbf{W}_t; \end{cases}$$

- For some β that defines $p^{\mathcal{T}}$

SB through SOC

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- For some β that defines $p^{\mathcal{T}}$
- β acts as "discriminator"

SB through SOC

$$d\mathbf{Z}_t = f_t^{\text{base}}(\mathbf{Z}_t)dt + \sqrt{\gamma}d\mathbf{W}_t$$

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- For some β that defines $p^{\mathcal{T}}$
- β acts as "discriminator"
- Known for generative modeling and fine-tuning

SOC: typical solutions

$$\begin{cases} \mathbb{E}_{\mathsf{P}^\theta}\left[\int\limits_0^1 \frac{1}{2}\|f_t^\theta(\mathbf{X}_t^\theta)\|^2 \mathrm{d}t + \beta(\mathbf{X}_1^\theta)\right] \to \min\limits_\theta \\ \mathrm{d}\mathbf{X}_t^\theta = \left(f_t^{\text{base}}(\mathbf{X}_t^\theta) + g(t) f_t^\theta(\mathbf{X}_t^\theta)\right) \mathrm{d}t + g(t)\mathrm{d}\mathbf{W}_t; \end{cases}$$

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$$\nabla_\theta \mathcal{L}(\theta) \approx \nabla_\theta \left(\sum_{i=0}^{N-1} \frac{1}{2}\|f_{t_i}^\theta(\mathbf{X}_{t_i}^\theta)\|^2 \Delta t_i + \beta(\mathbf{X}_1^\theta)\right)$$ [4]

$$\mathbf{X}_{t_{i+1}}^\theta = \mathbf{X}_{t_i}^\theta + \left(f_{t_i}^{\mathrm{base}}(\mathbf{X}_{t_i}^\theta) + g(t_i) f_{t_i}^\theta(\mathbf{X}_{t_i}^\theta)\right)\Delta t_i + g(t)\sqrt{\Delta t_i}\boldsymbol{\epsilon}_i$$

SOC: typical solutions

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Here: N forward, N backward

SOC: typical solutions

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Here: N forward, N backward

Possible: N forward, K backward [5]

SOC: typical solutions

$$\begin{cases} \mathbb{E}_{\mathsf{P}^\theta}\left[\int\limits_0^1 \frac{1}{2}\|f_t^\theta(\mathbf{X}_t^\theta)\|^2 \mathrm{d}t + \beta(\mathbf{X}_1^\theta)\right] \to \min\limits_\theta \\ \mathrm{d}\mathbf{X}_t^\theta = \left(f_t^{\mathrm{base}}(\mathbf{X}_t^\theta) + g(t) f_t^\theta(\mathbf{X}_t^\theta)\right)\mathrm{d}t + g(t)\mathrm{d}\mathbf{W}_t; \end{cases}$$

$$\nabla_\theta \mathcal{L}(\theta) \approx \nabla_\theta \left(\sum_{i=0}^{N-1} \frac{1}{2}\|f_{t_i}^\theta(\mathbf{X}_{t_i}^\theta)\|^2 \Delta t_i + \beta(\mathbf{X}_1^\theta)\right)$$ [4]

$$\mathbf{X}_{t_{i+1}}^\theta = \mathbf{X}_{t_i}^\theta + \left(f_{t_i}^{\mathrm{base}}(\mathbf{X}_{t_i}^\theta) + g(t_i) f_{t_i}^\theta(\mathbf{X}_{t_i}^\theta)\right)\Delta t_i + g(t)\sqrt{\Delta t_i}\boldsymbol{\epsilon}_i$$

Here: N forward, N backward

Possible: N forward, K backward [5]

Goal: K forward, K backward

References

[1] Song Y. et al. Score-based generative modeling through stochastic differential equations;

[2] Stochastic Interpolants: A Unifying Framework for Flows and Diffusions;

[3] DPOK: Reinforcement Learning for Fine-tuning Text-to-Image Diffusion Models;

[4] Entropic Neural Optimal Transport via Diffusion Processes;

[5] Adjoint Sampling: Highly Scalable Diffusion Samplers via Adjoint Matching;

[1+] Rakitin D., Oganov A. Diffusion-based Generative Models Course.