

On partitioning the set  $[n] = \{1, \dots, n\}$   
into subsets of size at most  $m$   
such that all sums are powers of  $m$

---

Vladimir Gurvich

National Research University Higher School of Economics, Moscow, Russia  
Rutgers University, New Brunswick, NJ

**Gurvich, V. and Naumova, M., 2026.**

**On partitioning the set  $[n] = \{1, \dots, n\}$**

**into subsets of size at most  $m$**

**such that all sums are powers of  $m$**

Available at <https://arxiv.org/abs/2508.00946>

Given integers  $m > 1$  and  $n > 0$ , set  $[n] = \{1, \dots, n\}$ .

A subset of  $[n]$  is called *m-good*

if its size is at most  $m$  and the sum of its elements is a power of  $m$ .

In particular, the singleton  $\{1\}$  is *m-good* for any  $m$ .

A partition of  $[n]$  is called *m-good* if it consists of *m-good* subsets.

Every  $n$  admits a unique 2-good partition, and

for a fixed  $m > 3$ , there is no *m-good* partition for infinitely many  $n$ .

Less is known for  $m = 3$ .

A 3-good partition of  $[n]$  exists when  $n \equiv 1 \pmod{3}$ .

We conjecture that 3-good partitions of  $[n]$  exists for all  $n$  and verify this by computations for  $n \leq 844$ . Dmitry Rybin.

If  $n \not\equiv 1 \pmod{3}$ , a minimal counter-example (if any) must satisfy:

(i)  $n = 3^t + 3k + 2$  for  $t > 0$ , where

(ii)  $0 \leq k < \frac{3^{t-1}-1}{2}$ , and

(iii)  $k \neq \frac{3^\ell-1}{2}$  for all nonnegative integers  $\ell < t$ .

For example,  $\{\{1, 2, 6\}, \{3\}, \{4, 5\}\}$  is a 3-good partition of  $[6]$ .

To simplify notation, we will write  $21 = (1 + 2 + 6) + (3) + (4 + 5)$  instead.

The number  $T_n = 1 + \dots + n = n(n+1)/2$  is called *triangular*.

We will denote a partition of  $[n]$  by representing  $T_n$  as the sum of the corresponding clauses.

As a brief reminder,  
the value on the left side of each equation below  
is the triangular number  $T_n$ ,  
representing the total sum of the elements in the partitioned set  $[n]$ .

The first nine and seven  $m$ -good partitions for  $m = 2$  and 3 are as follows:

$$1 = (1), 3 = (1) + (2), 6 = (1 + 3) + (2), 10 = (1 + 3) + (2) + (4),$$

$$15 = (1) + (2) + (3 + 5) + (4), 21 = (1) + (2 + 6) + (3 + 5) + (4),$$

$$28 = (1 + 7) + (2 + 6) + (3 + 5) + (4), 36 = (1 + 7) + (2 + 6) + (3 + 5) + (4) + (8),$$

$$45 = (1) + (2 + 6) + (3 + 5) + (4) + (8) + (7 + 9), \dots$$

$$1 = (1), 3 = (1+2), 6 = (1+2)+(3), 10 = (1)+(2+3+4), 15 = (1+2)+(3)+(4+5),$$

$$21 = (1 + 2) + (3 + 6) + (4 + 5) = (1 + 2 + 6) + (3) + (4 + 5),$$

$$28 = (1) + (2 + 7) + (3 + 6) + (4 + 5), \dots$$

There exist two distinct 3-good partitions for  $n = 6$ .

Obviously, there are no  $m$ -good partitions of  $[n]$  for  $n = 2$  and  $m > 3$ .

The following stronger claim shows that

for any fixed  $m > 3$  there exists no  $m$ -good partition for infinitely many  $n$ .

**Proposition 1** *For integer  $m > 3$ ,  $t \geq 0$ , and  $n = 2(mt + 1)$ , there is no  $m$ -good partition of  $[n]$ .*

**Proof.** Indeed, if  $m > 3$  then

$$T_n = n(n+1)/2 = (mt+1)(2mt+3) \equiv 3 \pmod{m},$$

while it must be 0 or 1  $\pmod{m}$ , by the definition of  $m$ -good partitions.  $\square$

The above argument does not work for  $m = 2$  and 3,  
since  $T_n$  is congruent to 0 or 1, both modulo 2 and modulo 3.

In the first case this statement is trivial,

in the second one it is implied by the obvious identity

$$T_n \pmod{3} = (n \pmod{3}) \pmod{2}.$$

**Proposition 2** *A unique 2-good partition exists for any  $n$ .*

**Proof.** Without loss of generality (wlog) assume that  $n = 2^t + i$ , where  $t \geq 0$  and  $0 \leq i < 2^t$ .

It is easy to verify that, for any  $j$  such that  $1 \leq j \leq i$ , number  $2^t + j$  can form a pair only with  $2^t - j$ , in any 2-good partition of  $[n]$ .

In case  $j = 0$ , both numbers equal  $2^t$  and, hence,  $2^t$  must participate in a 2-good partition of  $[n]$  as a singleton.

In any case, in  $i + 1$  steps, the problem of 2-good partitioning of  $[n]$  is reduced to the same problem for  $[n - 2i - 1]$ , which is a strict subset of  $[n]$ .

Applying this reduction recursively, we obtain a unique 2-good partition of  $[n]$ .

□

From now on we will consider only the case  $m = 3$ , which looks difficult.

**Conjecture 1** *A 3-good partition of  $[n]$  exists for every positive integer  $n$ .*

We will prove the following partial results:

**Theorem 1** *A 3-good partition of  $[n]$  exists when  $n \equiv 1 \pmod{3}$ .*

*Otherwise, if  $n \not\equiv 1 \pmod{3}$ , a minimal counter-example (if any) must satisfy:*

*(i)  $n = 3^t + 3k + 2$ , where  $t > 0$  and*

*(ii)  $0 \leq k \leq \frac{3^{t-1}-1}{2}$ ;*

*moreover, (iii)  $k \neq \frac{(3^\ell-1)}{2}$  for all nonnegative integers  $\ell \leq t$ .*

We make several simple preliminary observations regarding this statement:

Obviously, (i) implies that  $n \equiv 2 \pmod{3}$ , and (ii, iii) can be rewritten as:

$$(ii') \quad 3k + 2 < \frac{3^{t+1}+1}{2} \text{ and}$$

$$(iii') \quad 3k + 2 \neq \frac{3^{\ell+1}+1}{2} \text{ for } 0 \leq \ell < t.$$

The cases  $k = 0$  and  $k = \frac{3^{t-1}-1}{2}$  of (ii) are equivalent with  $\ell = 0$  and  $\ell = t$  of (iii).

For  $\ell > 0$ , the values  $\frac{3^\ell+1}{2}$ ,  $\frac{3^\ell-1}{2}$ , and  $\frac{3^{\ell-1}-1}{2} = 3 \cdot \frac{3^{\ell-1}-1}{2}$

are three successive nonnegative decreasing integers congruent to 2, 1, and 0 modulo 3, respectively.

Statement (iii) excludes  $k = 0, 1, 4, 13, 40, \dots$  and, respectively,  $3k + 2 = 2, 5, 14, 41, 122, \dots$

Thus, the first numbers not covered by Theorem 1 are

$n = 35, 38, 89, 92, 98, 101, \dots$

We also prove the following related result.

A *3-good quasi-partition* of  $[n]$  is a family of 3-good subsets of  $[n]$  that cover powers of 3 once or twice and other numbers exactly once.

**Proposition 3** *For each  $n = 3^t + 3k + 2$  such that  $t > 0$  and  $0 \leq k < \frac{3^{t-1}-1}{2}$  there exists a 3-good quasi-partition of  $[n]$  in which  $3^t$  is covered twice.*

We also obtain the following results on the uniqueness of 3-good partitions.

**Theorem 2** *A 3-good partition of  $[n]$  is unique for*

$$n \in N_u = \{1, 2, 3, 4, 3^t - 4, 3^t - 2, 3^t - 1, 3^t, 3^t + 1, 3^t + 2, 3^t + 3, 3^t + 5; \ t \geq 2\}.$$

**Theorem 3** *Set  $[n]$  admits exactly two 3-good partitions for*

$$n = 3^t - 3, \ t \geq 2; \ n = 3^t - 6, \ t \geq 3; \ n = 3t - 15, \ t \geq 4.$$

As for  $n = 3^t + 4$ , computations show that there are two 3-good partitions for  $n = 13$  but already 14 for  $n = 31$ .

**Conjecture 2** *All other numbers  
(except listed in the above two theorems and 13)  
admit more than two 3-good partitions.*

This conjecture and Conjecture 1 were verified up to  $n = 844$   
by a computer code written by Dmitry Rybin.

Several subcases not covered by Theorem 1 were resolved  
by a more powerful reduction suggested by Oleg Polubasoff. LATER.

## Proof of Theorem 1 and Proposition 3

Denote by  $\text{part}(n)$  a 3-good partition of  $[n]$ .

Assume for contradiction that there exists an  $n$  that does not admit a 3-good partition.

Choose the minimal such  $n$  and call it *critical*.

**Reduction.** Obviously,  $n$  is not critical if

$$\text{part}(n) = \text{part}(n') + (C_1) + \dots + (C_\ell),$$

where all these  $\ell$  added parts are good and partition the interval  $[n' + 1, n]$ .

Such a representation of  $\text{part}(n)$  will be called a reduction of  $n$  to  $n'$ .

For example,  $n = 3^t$  and  $3^t + 1$  admit reductions

$$\text{part}(3^t) = \text{part}(3^t - 1) + (3^t) \text{ and } \text{part}(3^t + 1) = \text{part}(3^t - 2) + ((3^t - 1) + 3^t + (3^t + 1)).$$

Also,

$$\text{part}(3) = \text{part}(2) + (3),$$

$$\text{part}(4) = \text{part}(1) + (2 + 3 + 4),$$

$$\text{part}(5) = \text{part}(3) + (4 + 5),$$

$$\text{part}(6) = \text{part}(2) + (3 + 6) + (4 + 5),$$

$$\text{part}(7) = \text{part}(1) + (2 + 7) + (3 + 6) + (4 + 5),$$

$$\text{part}(9) = \text{part}(8) + (9),$$

$$\text{part}(10) = \text{part}(7) + (8 + 9 + 10).$$

Note that  $n = 3^t - 1$  is not reducible, yet,  $[n]$  admits a unique 3-good partition:

$$\text{part}(3^t - 1) = (1 + (3^t - 1)) + (2 + (3^t - 2)) + \cdots + \left( \frac{3^t - 1}{2} + \frac{3^t + 1}{2} \right).$$

For example:

$$\text{part}(2) = (1 + 2), \quad \text{part}(8) = (1 + 8) + (2 + 7) + (3 + 6) + (4 + 5), \quad \dots$$

The next non-reducible number is  $n = 11$ ,  
which has a more intricate 3-good partition:

$$\text{part}(11) = (1 + 8) + (2 + 7) + (3) + (4 + 5) + (6 + 10 + 11) + (9).$$

In general, for  $n = 3^t + 2$ , we have:

$$\begin{aligned} \text{part}(3^t + 2) = & (1 + (3^t - 1)) + (2 + (3^t - 2)) + (3) + (4 + (3^t - 4)) + \cdots + \left( \frac{3^t - 1}{2} + \frac{3^t + 1}{2} \right) \\ & + ((3^t - 3) + (3^t + 1) + (3^t + 2)) + (3^t). \end{aligned}$$

## Proof of Theorem 1 (ii)

Clearly, for each  $n > 0$  there is a unique  $t > 0$  such that  $3^{t-1} \leq n < 3^t$ .

Theorem 1 (ii) asserts

that inequalities  $3^{t-1} \leq n \leq \frac{3^t-1}{2}$  must hold whenever  $[n]$  is critical.

For example, consider the following reductions:

$$\text{part}(14) = \text{part}(12) + (13 + 14),$$

$$\text{part}(15) = \text{part}(11) + (12 + 15) + (13 + 14),$$

$$\text{part}(16) = \text{part}(10) + (11 + 16) + (12 + 15) + (13 + 14), \dots$$

$$\text{part}(25) = \text{part}(1) + (2 + 25) + (3 + 24) + (4 + 23) + \dots + (12 + 15) + (13 + 14).$$

In general, in accordance with Theorem 1 (ii), any integer  $n$  satisfying  $\frac{3^t+1}{2} \leq n < 3^t$ , is reducible.

Indeed, the reduction is given by the following formula:

$$\text{part} \left( \frac{3^t+1}{2} + \ell \right) = \text{part} \left( \frac{3^t-1}{2} - \ell - 1 \right) + \sum_{i=0}^{\ell} \left( \left( \frac{3^t-1}{2} - i \right) + \left( \frac{3^t+1}{2} + i \right) \right),$$

where  $t \geq 0$  and  $0 \leq \ell < \frac{3^t+1}{2}$ . This establishes the claim.  $\square$

Note that no new triples appear in the reduction.

In other words, the 3-good partitions

$$\text{part}\left(\frac{3^t + 1}{2} + \ell\right) \quad \text{and} \quad \text{part}\left(\frac{3^t - 1}{2} - \ell - 1\right)$$

contain the same set of triples.

There are the triple-free 3-good partitions  $\text{part}(3^t)$  and  $\text{part}(3^t - 1)$ .

The following statement is obvious but important:

$n \equiv 1 \pmod{3}$  if and only if  $n' \equiv 1 \pmod{3}$ , while

$n \equiv 0 \pmod{3}$  if and only if  $n' \equiv 2 \pmod{3}$ , and vice versa.

For example,  $15 \equiv 0 \pmod{3}$  is reduced to  $11 \equiv 2 \pmod{3}$ .

## 3-good partitions of symmetric integer intervals

Let  $[-\ell; \ell] = \{-\ell, \dots, -1, 0, 1, \dots, \ell\}$

be the symmetric integer interval of size  $2\ell + 1$ .

For  $\ell \equiv 1 \pmod{3}$ , we call a partition of  $[-\ell; \ell]$ , denoted by  $\text{part}[-\ell; \ell]$ , *3-good* if each of its parts is of size 3 and the sum of the numbers in it equals 0.

For  $\ell \equiv 0 \pmod{3}$ , we slightly modify this definition by adding one special part  $\{0\}$  of size 1, containing only 0.

Note that for  $\ell \equiv 2 \pmod{3}$  the size of  $[-\ell; \ell]$  is  $2\ell + 1 \equiv 2 \pmod{3}$ . In this case, we have no 3-good partitions.

**Lemma 1** *If  $\ell \equiv 0$  or  $1 \pmod{3}$  then  $[-\ell; \ell]$  admits a 3-good partition.*

**Proof.** Assume first that  $\ell = 3k$ .

Then, the partition  $\text{part}[-3k; 3k]$  is given by:

$$\begin{aligned} \text{part}[-3k; 3k] = & (0) + \sum_{i=1}^{k-1} ((-3k + i) + (1 + i) + (3k - 1 - 2i)) \\ & + \sum_{i=1}^{k-1} ((-2k + i) + (-k + i) + (3k - 2i)). \end{aligned}$$

For example, if  $k = 1$  and  $k = 2$ , we obtain:

$$\begin{aligned} \text{part}[-3; 3] &= (-3 + 1 + 2) + (-2 + (-1) + 3) + (0), \\ \text{part}[-6; 6] &= (-6 + 1 + 5) + (-5 + 2 + 3) + (-4 + (-2) + 6) \\ &\quad + (-3 + (-1) + 4) + (0). \end{aligned}$$

For  $\ell = 3k + 1$  replace  $(0)$  by  $(-\ell + 0 + \ell)$ . □

However,  $n \equiv 2 \pmod{3}$  are not reducible in this way.

Indeed, in this case, interval  $[-n; n]$  admits no 3-good partitions, since its size is  $2n + 1 \equiv 2 \pmod{3}$ .

In particular, this holds for  $n = 3^t + 3k + 2$  for any  $t > 0$  and  $k \geq 0$ .

## Proof of Theorem 1 (i)

Let us show that  $n \equiv 2 \pmod{3}$  must hold for each critical  $n$ .

If  $n \equiv 0$  or  $1 \pmod{3}$  then reductions work.

Consider, for example, the following reductions:

$$\text{part}(6) = \text{part}(2) + (3 + 6) + (4 + 5),$$

$$\text{part}(12) = \text{part}(5) + (6 + 10 + 11) + (7 + 8 + 12) + (9),$$

$$\text{part}(30) = \text{part}(23) + (24 + 28 + 29) + (25 + 26 + 30) + (27),$$

$$\begin{aligned} \text{part}(33) = & \text{part}(20) + (21 + 28 + 32) + (22 + 29 + 30) + (23 + 25 + 33) \\ & + (24 + 26 + 31) + (27). \end{aligned}$$

More generally, for  $n = 3^t + 3$ , we have:

$$\begin{aligned} \text{part}(3^t + 3) &= \text{part}(3^t - 4) \\ &+ ((3^t - 3) + (3^t + 1) + (3^t + 2)) + ((3^t - 2) + (3^t - 1) + (3^t + 3)). \end{aligned}$$

Eliminating terms involving  $3^t$  yields the 3-good partition:

$$\text{part}[-3; 3] = (-3 + 1 + 2) + (-2 - 1 + 3) + (0).$$

Consider also the following reductions:

$$\text{part}(7) = \text{part}(1) + (2 + 7) + (3 + 6) + (4 + 5),$$

$$\text{part}(13) = \text{part}(4) + (5 + 9 + 13) + (6 + 10 + 11) + (7 + 8 + 12) + (9),$$

$$\text{part}(31) = \text{part}(22) + (23 + 27 + 31) + (24 + 28 + 29) + (25 + 26 + 30),$$

$$\text{part}(34) = \text{part}(19) + (20 + 30 + 31) + (21 + 28 + 32) + (22 + 26 + 33) + (23 + 24 +$$

More generally, for  $n = 3^t + 4$ , we have:

$$\text{part}(3^t+4) = \text{part}(3^t-5) + ((3^t-4)+3^t+(3^t+4)) + ((3^t-3)+(3^t+1)+(3^t+2)) + ((3^t-2)$$

Eliminating all instances of  $3^t$  yields the 3-good partition:

$$\text{part}[-4; 4] = (-4 + 0 + 4) + (-3 + 1 + 2) + (-2 - 1 + 3).$$

In general, the reductions for  $n = 3^t + 3k$  and  $n = 3^t + 3k + 1$  are given by the following formulas.

$$\begin{aligned} \text{part}(3^t + 3k) &= \text{part}(3^t - 3k - 1) + (3^t) \\ &\quad + \sum_{i=0}^{k-1} ((3^t - 3k + i) + (3^t + i + 1) + (3^t + 3k - 2i - 1)) \\ &\quad + \sum_{i=0}^{k-1} ((3^t - 2k + i) + (3^t - k + i) + (3^t + 3k - 2i)) . \end{aligned}$$

$$\begin{aligned} \text{part}(3^t + 3k + 1) &= \text{part}(3^t - 3k - 1) \\ &\quad + ((3^t - 3k - 1) + 3^t + (3^t + 3k + 1)) \\ &\quad + \sum_{i=0}^{k-1} ((3^t - 3k + i) + (3^t + i + 1) + (3^t + 3k - 2i - 1)) \\ &\quad + \sum_{i=0}^{k-1} ((3^t - 2k + i) + (3^t - k + i) + (3^t + 3k - 2i)) . \end{aligned}$$

In both cases, the condition  $3^{t-1} \geq 2k + 1$  must hold,

which is equivalent to  $k \leq \frac{3^{t-1}-1}{2}$ .

Recall that if  $3^t + 1 \geq k > \frac{3^{t-1}-1}{2}$ , then the previous reduction applies.

Thus, both above reductions apply to all  $n$ , except those that, for some nonnegative  $t$  and  $k$ , satisfy the following conditions:

$$3^t + 1 < n < \frac{3^{t+1} + 1}{2} \quad \text{and} \quad n = 3^t + 3k + 2,$$

Furthermore, for both reductions we have:

$n \equiv 1 \pmod{3}$  if and only if  $n' \equiv 1 \pmod{3}$ , while

$n \equiv 0 \pmod{3}$  if and only if  $n' \equiv 2 \pmod{3}$ , and vice versa.

Thus,  $n$  admits a 3-good partition whenever  $n \equiv 1 \pmod{3}$ .

### Proof of Proposition 3

Earlier we constructed a 3-good partition of the symmetric interval  $[-\ell; \ell]$  for  $\ell = 3k$  and  $\ell = 3k + 1$ .

Shifting this symmetric interval by  $3^t$  (such that  $3^t > \ell$ ) we obtain a 3-good partition of the interval  $[3^t - \ell; 3^t + \ell]$ , that is, a reduction from  $n = 3^t + \ell$  to  $n' = 3^t - 2\ell - 1$ .

Although for  $\ell = 3k + 2$  no 3-good partition of the interval  $[-\ell; \ell]$  exists, still, it is easy to obtain a 3-good quasi-partition.

Generate a 3-good partition for  $\ell - 1$  and add the 3-good triple  $\{-\ell, 0, \ell\}$ . We obtain a quasi-partition since 0 is covered twice,

Shifting by  $3^t$  we obtain a reduction for the 3-good quasi-partitioning for  $n = 3^t + 3k + 2$ .

We may need to apply such reduction several times.

Thus, several distinct powers of 3 may be covered twice. □

Later we will construct another 3-good partition of the symmetric interval  $[-\ell; \ell]$  for  $\ell = 3k + 1$ , which explains the existence of numerous 3-good partitions.

Proof of Theorem 1 (iii)

For  $\ell > 0$ , the values  $\frac{3^\ell+1}{2}$ ,  $\frac{3^\ell-1}{2}$ , and  $\frac{3^\ell-1}{2} - 1 = 3 \cdot \frac{3^{\ell-1}-1}{2}$  are three decreasing integers congruent to 2, 1, and 0 modulo 3, respectively.

Hence,  $n = 3^t + \frac{3^\ell+1}{2} \equiv 2 \pmod{3}$ .

Assume that  $n$  is critical.

We obtain a contradiction reducing it to  $n - 2$  as follows.

Start partitioning  $[n]$  with a triple

$T = ((3^t + \frac{3^\ell+1}{2}) + (3^t + \frac{3^\ell-1}{2}) + (3^t - 3^\ell))$  whose sum is  $3^{t+1}$ .

Since  $n$  is critical, there exists a 3-good partition of  $[n - 2]$ .

One of its parts  $T'$  contains the number  $(3^t - 3^\ell)$ .

Then, it must also contain either  $3^\ell$  or two numbers whose sum is  $3^\ell$ .

Thus, we can move  $(3^t - 3^\ell)$  from  $T'$  to  $T$  to obtain a 3-good partition of  $[n]$ .

□

More 3-good partitions of the symmetric interval  $[\ell; \ell]$ , where  $\ell = 3k + 1$

First, for  $n = 3k + 1$ , we will partition  $[n]$  into  $k$  triples

$$T_i = (a_i, b_i, c_i), i = 1, \dots, k$$

such that  $a_i = b_i + c_i$  and one singleton  $L = \lceil (3k + 1)/2 \rceil$ .

Define the triples as follows:

$c_i$  is a sequence of length  $k$  in which the numbers that are not multiples of 3 go in the monotone increasing order and

then the numbers that are multiples of 3 go in the monotone decreasing order.

For example, for  $k = 0, 1, \dots, 11$  we obtain:

$\emptyset$ ; 1; 1, 2; 1, 2, 3; 1, 2, 4, 3; 1, 2, 4, 5, 3; 1, 2, 4, 5, 6, 3; 1, 2, 4, 5, 7, 6, 3; 1, 2, 4, 5, 7, 8, 6, 3;  
1, 2, 4, 5, 7, 8, 9, 6, 3; 1, 2, 4, 5, 7, 8, 10, 9, 6, 3; 1, 2, 4, 5, 7, 8, 10, 11, 9, 6, 3; ...

Thus, we have  $c_i$ ; now to define  $a_i$  and  $b_i$  for  $i = 1, \dots, k$   
we proceed recursively as follows.

Delete  $L$ , and  $[k]$  from  $[n]$ .

For  $i = 1, \dots, k$  define  $a_i$  as the maximum of the remaining numbers and  
set  $b_i = a_i - c_i$ . Then proceed with  $i + 1$ .

For example, for  $k = 0, 1, \dots, 7$  we obtain the following  $L = L_k$ , and  $(a_i, b_i, c_i)$  for  $i \in [k]$ :

$k = 0$  : 1;

$k = 1$  : 2; (4, 3, 1);

$k = 2$  : 4; (7, 6, 1), (5, 3, 2);

$k = 3$  : 5; (10, 9, 1), (8, 6, 2), (7, 4, 3);

$k = 4$  : 7; (13, 12, 1), (11, 9, 2), (10, 6, 4), (8, 5, 3);

$k = 5$  : 8; (16, 15, 1), (14, 12, 2), (13, 9, 4), (11, 6, 5), (10, 7, 3);

$k = 6$  : 10; (19, 18, 1), (17, 15, 2), (16, 12, 4), (14, 9, 5), (13, 7, 6), (11, 8, 3);

$k = 7$  :

11; (22, 21, 1), (20, 18, 2), (19, 15, 4), (17, 12, 5), (16, 9, 7), (14, 8, 6), (13, 10, 3).

Let us consider also a slightly different but equivalent representation:

Sequence  $a_i$  is of length  $k$ , monotone decreasing from  $3k + 1$  and contains no numbers multiple of 3. For  $k = 1, \dots, 7$  we have:

4; 7,5; 10,8,7; 13,11,10,8; 16,14,13,11,10; 19,17,16,14,13,11;  
22,20,19,17,16,14,13...

Sequence  $c_i$  is of length  $k$ , monotone increasing from 1 and contains no numbers multiple of 3. For  $k = 1, \dots, 7$  we have:

1; 1,2; 1,2,4; 1,2,4,5; 1,2,4,5,7; 1,2,4,5,7,8; 1,2,4,5,7,8,10;...

Sequence  $b_i$  is of length  $k$ , monotone decreasing from  $3k$  and contains only numbers multiple of 3. For  $k = 1, \dots, 7$  we have:

3; 6,3; 9,6,3; 12,9,6,3; 15,12,9,6,3; 18,15,12,9,6,3; 21,18,15,12,9,6,3;...

We obtain the same table as before modulo possible permutations in several last clauses of each row. For  $k = 0, 1, \dots, 7$  we have:

$k = 0$  : 1;

$k = 1$  : 2; (4, 3, 1);

$k = 2$  : 4; (7, 6, 1), (5, 3, 2);

$k = 3$  : 5; (10, 9, 1), (8, 6, 2), (7, 3, 4);

$k = 4$  : 7; (13, 12, 1), (11, 9, 2), (10, 6, 4), (8, 3, 5);

$k = 5$  : 8; (16, 15, 1), (14, 12, 2), (13, 9, 4), (11, 6, 5), (10, 3, 7);

$k = 6$  : 10; (19, 18, 1), (17, 15, 2), (16, 12, 4), (14, 9, 5), (13, 6, 7), (11, 3, 8);

$k = 7$  :

11; (22, 21, 1), (20, 18, 2), (19, 15, 4), (17, 12, 5), (16, 9, 7), (14, 6, 8), (13, 3, 10);  $\dots$

It is not difficult to verify that the desired partitions are well defined for all  $k$ .

**Remark 1** *This construction is of independent interest. It can be applied for  $\ell = 3k + 2$  and  $\ell = 3k$ , too. In the first case we simply add the second singleton  $3k + 2$ ; in the second case we partition  $3k$  numbers into  $k - 1$  triples and three singletons. No partition into  $k$  triplets  $(a_i, b_i, c_i)$  such that  $a_i = b_i + c_i$  for  $i = 1, \dots, k$  exists.*

Replace each triple  $(a_i, b_i, c_i)$  by two zero-sum triples  $(a_i, -b_i, -c_i)$  and  $(-a_i, b_i, c_i)$  and  $L$  - by the zero-sum triple  $\{L, 0, -L\}$ , we obtain  $2k + 1$  zero-sum triples that partition the symmetric interval  $[-(3k + 1); 3k + 1]$ .

## Uniqueness

**Case  $n = 3^t - 4$ .** The following 3-good partition is unique:

$$(1 + 2) + (3) + (4 + (3^t - 4)) + (5 + (3^t - 5)) + \cdots + \left( \frac{3^t - 1}{2} + \frac{3^t + 1}{2} \right).$$

**Case  $n = 3^t - 2$ .** The following 3-good partition is unique:

$$(1) + (2 + (3^t - 2)) + (3 + (3^t - 3)) + \cdots + \left( \frac{3^t - 1}{2} + \frac{3^t + 1}{2} \right).$$

**Case  $n = 3^t - 1$ .** A unique 3-good partition consists of pairs only:

$$(1 + (3^t - 1)) + (2 + (3^t - 2)) + \cdots + \left( \frac{3^t - 1}{2} + \frac{3^t + 1}{2} \right).$$

**Case  $n = 3^t$ .** A unique 3-good partition exists. Obviously,  $(3^t)$  can appear only as a singleton, after which the considered case is reduced to the previous one. The empty 3-good packing is forced in this case too.

**Case  $n = 3^t + 1$ .** A unique 3-good partition exists.

Obviously,  $(3^t + 1, 3^t, 3^t - 1)$  must appear,

since there is no other way to include  $3^t + 1$ .

Then, the considered case is reduced to  $n' = 3^t - 2$ .

In other words, the 3-good triple  $(3^t + 1, 3^t, 3^t - 1)$  is forced. □

**Case  $n = 3^t + 2$ .** A unique 3-good partition exists:

$$((3^t + 2) + (3^t + 1) + (3^t - 3)) + (3^t) + (1 + (3^t - 1)) + (2 + (3^t - 2)) + (3) + (4 + (3^t - 4)) + \cdots + \left(\frac{3^t - 1}{2} + \frac{3^t + 1}{2}\right).$$

**Case  $n = 3^t + 3$ .** A unique 3-good partition exists:

$$((3^t + 3) + (3^t - 1) + (3^t - 2)) + ((3^t + 2) + (3^t + 1) + (3^t - 3)) + (3^t) + (4 + (3^t - 4)) + \cdots + \left(\frac{3^t - 1}{2} + \frac{3^t + 1}{2}\right).$$

**Case  $n = 3^t + 5$ .** A unique 3-good partition exists.

It contains the triple  $((3^t + 5) + (3^t + 4) + (3^t - 9))$ .

Furthermore, a unique 3-good partition for  $n' = 3^t + 3$  contains the pair  $(9 + (3^t - 9))$ .

Although  $3^t - 9$  is already taken,  
the remaining number  $9 = 3^2$  is a legal singleton.

## Cases with exactly two 3-good partitions.

**Case**  $n = 3^t + 4$ .

It is easily seen that  $n = 13$  admits exactly two 3-good partitions:

$(13 + 8 + 6) + (12 + 10 + 5) + (11 + 9 + 7) + (4 + 3 + 2) + (1)$  and

$(13 + 9 + 5) + (12 + 8 + 7) + (11 + 10 + 6) + (4 + 3 + 2) + (1),$

Computations show that  $n = 31$  admits already fourteen 3-good partitions.

**Case**  $n = 3^t - 3$ ,  $t \geq 2$ . There are exactly two 3-good partitions:

$(1 + 2) + (3 + (3^t - 3)) + (4 + (3^t - 4)) + (5 + (3^t - 5)) + \cdots + \left(\frac{3^t-1}{2} + \frac{3^t+1}{2}\right)$   
and

$(1 + 2 + (3^t - 3)) + (3) + (4 + (3^t - 4)) + (5 + (3^t - 5)) + \cdots + \left(\frac{3^t-1}{2} + \frac{3^t+1}{2}\right).$

Case  $n = 3^t - 6$ .

There exist only the following two 3-good partitions of  $[n]$ :

$$(1 + 2) + (3) + (4 + 5) + (6 + (3^t - 6)) + (7 + (3^t - 7)) + \dots;$$

$$(1 + 2) + (3) + (9) +$$

$$(6 + (3^t - 6)) + (7 + (3^t - 7)) + (8 + (3^t - 8)) + (4 + 5 + (3^t - 9)) + (10 + (3^t - 10)) + \dots$$

In the previous case,  $n = 3^t - 3$ , we swapped 3 and  $2 + 1$ .

In the considered case 9 and  $5 + 4$ .

Case  $n = 3^t - 3\frac{3^{\ell+1}}{2} = 3(3^{t-1} - \frac{3^{\ell+1}}{2})$ ,  $\ell \geq 0$ ,  $t \geq \ell + 2$ .

The previous two cases,  $n = 3^t - 3$  and  $n = 3^t - 6$ , correspond to  $\ell = 0$  and 1.

There exist the following two reductions

of  $n$  to  $n' = 3\frac{3^{\ell+1}}{2} - 1$  and to  $n' - 2$ :

$$\begin{aligned} \text{part}(n) &= \text{part}(n') + \sum_i (i + (3^t - i)), \\ \text{part}(n) &= \text{part}(n' - 2) + \\ &\left( (3^t - 3^{\ell+1}) + \frac{3^{\ell+1} + 1}{2} + \frac{3^{\ell+1} - 1}{2} \right) + (3^{\ell+1}) + \sum_i (i + (3^t - i)); \end{aligned}$$

where summations include all  $i$  such that  $n' - 2 < i < 3^t/2$  in both cases and  $i \notin \{n' - 1, n', 3^{\ell+1}\}$  in the second one.

$\text{part}(n' - 2)$  and  $\text{part}(n')$  are both unique only when  $\ell = 0, 1, 2$ , that is, for  $n = 3^t - 3, 3^t - 6, 3^t - 15$ .

In these three cases we have  $n' = 2, 5, 14$  and  $n' - 2 = 0, 3, 12$ .

All six admit a unique 3-good partition each.

If  $\ell = 3$  then  $n' = 39$  and  $[n']$  admits 3091 3-good partitions, according to our computations.

Oleg Polubasoff suggested a more general method.

To reduce  $\text{part}(n)$  to  $\text{part}(n')$ , where  $n > n'$ , we are allowed to delete some sets from  $\text{part}(n')$  and then add some sets to obtain  $\text{part}(n)$ .

This method can be successfully applied to the smallest open cases

$n = 3^t + 8$  and  $3^t + 11$ ,

as well as to the known case  $n = 3^t + 14$ , for  $t \geq 3$  as follows:

$$\begin{aligned}
\text{part}(3^t + 8) = & \text{part}(3^t) - \\
& - [(3 + (3^t - 3)) + (9 + (3^t - 9)) + (11 + (3^t - 11)) + (13 + (3^t - 13))] + \\
& + (9) + (3 + 11 + 13) + \\
& + ((3^t - 3) + (3^t + 1) + (3^t + 2)) + ((3^t - 9) + (3^t + 3) + (3^t + 6)) \\
& + ((3^t - 11) + (3^t + 4) + (3^t + 7)) + ((3^t - 13) + (3^t + 5) + (3^t + 8));
\end{aligned}$$

$$\begin{aligned}
\text{part}(3^t + 11) = & \text{part}(3^t - 3) - \\
& - [(3 + (3^t - 3)) + (9 + (3^t - 9)) + (12 + (3^t - 12))] + \\
& + (13 + (3^t - 13)) + (14 + (3^t - 14)) + (15 + (3^t - 15)) + \\
& + (12 + 15) + (13 + 14) + (3) + (9) + \\
& + ((3^t - 3) + 3^t + (3^t + 3)) + ((3^t - 15) + (3^t + 6) + (3^t + 9)) \\
& + ((3^t - 14) + (3^t + 4) + (3^t + 10)) + ((3^t - 13) + (3^t + 2) + (3^t + 11)) \\
& + ((3^t - 12) + (3^t + 5) + (3^t + 7)) + ((3^t - 9) + (3^t + 1) + (3^t + 8));
\end{aligned}$$

$$\text{part}(3^t + 14) = \text{part}(3^t + 12) - [27 + (3^t - 27)] + (27) + ((3^t - 27) + (3^t + 13) + (3^t + 14)).$$

Recall that there exists a unique 3-good partition for  $[3^t]$  and exactly two for  $[3^t - 3]$ , and note that these partitions contain all sets that were subtracted in the first two reductions.

The third one was already explained earlier.

This reduction allows us to extend Theorem 1 (iii) as follows:

$$n = 3^t + \frac{1}{2}(3^{t_1} + 3^{t_2}) - 1, \text{ where } 0 < t_1 < t_2 < t.$$

Obviously, case  $t_1 = 1$  corresponds to Theorem 1 (iii) and that  $n \equiv 2 \pmod{3}$  for all feasible  $t_1, t_2$ , and  $t$ .

The first numbers not covered by the above statements are

$$n = 101, 104, 110, 113, 116, 119; 263, 266, 275, 278, 281, 290, 293, 299, \dots$$

Generalizing further, we obtain  $n = 3^t + \left(\sum_{i=1}^j 3^{\ell_i}\right) - 1$ , where  $0 < t_1 < \dots < t_j < t$ .

In other words,  $n = 3^t + q - 1$ , where  $q$  is a number whose ternary representation contains only digits 0 and 1 but not 2. Still  $n \equiv 2 \pmod{3}$ .

It remains open if the generalized reduction is sufficient to prove Conjecture 1.

Latest news: 9/11/2026, a Lean-verified AI proof of Conjecture 1 appeared online. It is based on the Skolem and Langford sequences discovered in 1990-s.

This work is anonymous, more precisely, by Astra, no human authors are mentioned.

### Acknowledgments:

This paper is an output of the research project (HSE-BR-2025-024) implemented as part of the Basic Research Program at HSE University.

We thank Oleg Polubasoff for important suggestions and valuable insights, some of which we include in the last two sections with his kind permission.

We are also thankful to Dmitry Rybin for his interest in our work and important computations, verifying our conjectures up to  $n = 844$ .

Alder, H.L., 1969. Partition identities—from Euler to the present.  
The American Mathematical Monthly, 76(7), pp.733–746.

Andrews, G.E., Eriksson, K., 2004. Integer partitions. Cambridge Univ. Press.

Baker C. A., Extended Skolem sequences, J. Combin. Design 3 (5) 1995  
363–379.

Gurvich V. and Naumova M., Partitioning set  $[n] = \{1, \dots, n\}$  into subsets of size at most  $m$  such that all sums are powers of  $m$ . Available at <https://arxiv.org/abs/2508.00946> (2025) the first version, 23 pp.

Lean-verified proof, On three-good partitions of every initial interval,  
<https://anonymous.4open.science/r/LeanProof-019E/output/pdf/three-good-partitions.pdf> Astra (2026) 4 pp.

Linek V. and Jiang Z., Extended Langford Sequences with Small Defects,  
J. Combin. Theory A 84 (1998) 38–54.

Polubasoff, O., Private communications (2025).

Rybin, D., Private communications (2025).