Constructive approaches to optimal first-order methods for (strongly) convex minimization

Adrien Taylor

Optimization without borders, Soshi – July 2021

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P. Giselsson
(Lund)

Yoel Drori

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Presentation based on joint works:

- ◇ "An optimal gradient method for smooth strongly convex minimization." (2021)
- ◇ "On the oracle complexity of smooth strongly convex minimization." (2021)

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Introduction based on joint works:

- ◇ "Smooth strongly convex interpolation and exact worst-case performance of first-order methods." (2017)
- ◇ "Exact worst-case performance of first-order methods for composite convex optimization." (2017)

Many (very) related works; much more careful bibliographical treatment in papers.

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- B. Polyak. "Introduction to optimization" (1964)
- Y. Nesterov. "A method of solving a convex programming problem with convergence rate $O(1/k^2)$." (1983)
- A. Nemirovsky, and B. Polyak. "Iterative methods for solving linear ill-posed problems under precise information." (1984)
- A. Nemirovsky. "Information-based complexity of linear operator equations." (1992)
- A. Nemirovsky. "Information-based complexity of convex programming." (lecture notes, 1995)
- Y. Nesterov. "Introductory Lectures on Convex Optimization." (2003/2018)
- Y. Drori, and M. Teboulle. "Performance of first-order methods for smooth convex minimization: a novel approach." (2014)
- D. Kim, and F. Fessler. "Optimized first-order methods for smooth convex minimization." (2016)
- L. Lessard, B. Recht, and A. Packard. "Analysis and design of optimization algorithms via integral quadratic constraints." (2016)
- B. Van Scoy, R. Freeman, K. Lynch. "The fastest known globally convergent first-order method for minimizing strongly convex functions." (2017)
- D. Kim, and F. Fessler. "Optimizing the efficiency of first-order methods for decreasing the gradient of smooth convex functions." (2021)

Take-home messages

Worst-cases are solutions to optimization problems

Acceleration/“optimal” methods by optimizing worst-cases

On worst-case analyses

Step-sizes optimization

Constructing lower bounds

Software

Concluding remarks

Analysis of a first-order method

Say we aim to solve

$$\min_{x \in \mathbb{R}^{\boldsymbol{d}}} f(x)$$

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x_1 &= x_0 - h_{1,0} f'(x_0) \\
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x_N &= x_{N-1} - \sum_{i=0}^{N-1} h_{N,i} f'(x_i),
\end{aligned} \tag{FOM}$$

for some coefficients $\{h_{i,j}\}$.

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Question 2: how to choose the step-sizes $\{h_{i,j}\}$?

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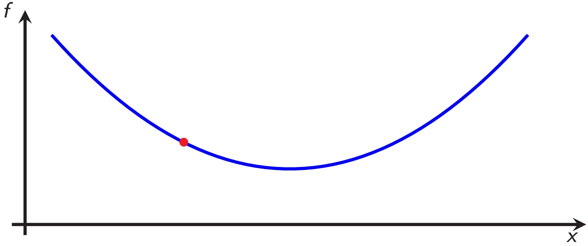

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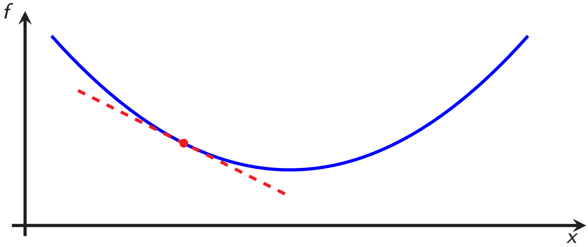

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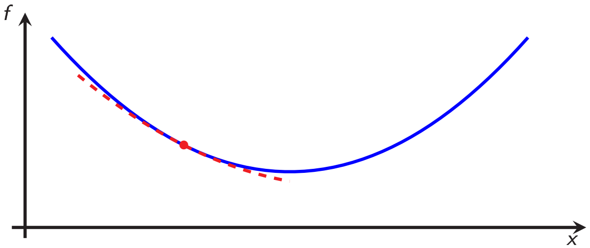

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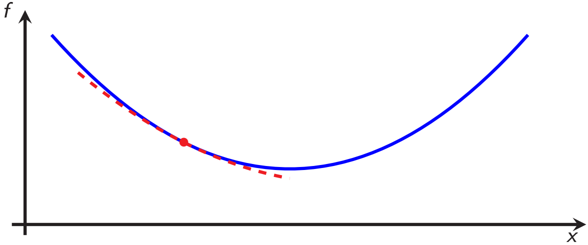

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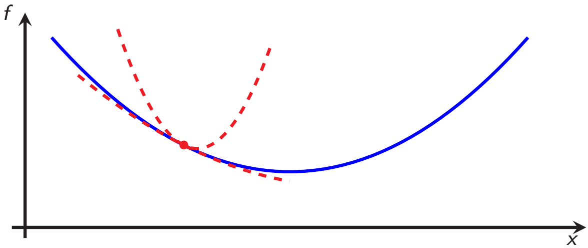

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Toy example: What is the smallest τ such that:

$$\|x_1 - x_\star\|^2 \leqslant \tau \|x_0 - x_\star\|^2,$$

for all

- L-smooth and μ-strongly convex function f (notation $f \in \mathcal{F}_{\mu,\mathrm{L}}$),
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Variables: f, x_0, x_1, $x_\star$; parameters: μ, L, $h_{1,0}$.

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Smooth strongly convex interpolation (or extension)

Consider an index set S, and its associated values $\{(x_i, g_i, f_i)\}_{i\in S}$ with coordinates x_i, (sub)gradients g_i and function values f_i.

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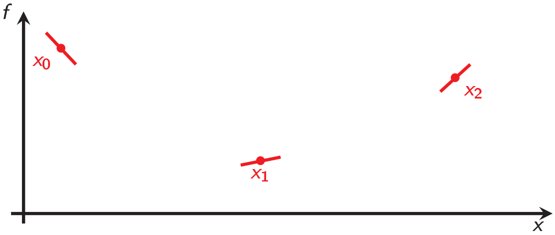

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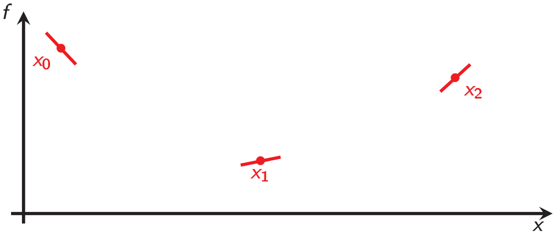

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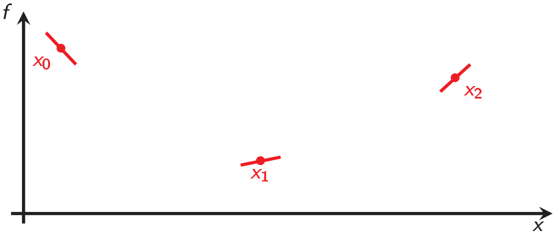

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- Simpler example: pick $\mu = 0$ and $L = \infty$ (just convexity):

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- Interpolation conditions allow removing red constraints

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- Same optimal value (no relaxation); but still non-convex quadratic problem.

Semidefinite lifting

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◇ Using the new variables $G \succcurlyeq 0$ and F

$$G = \begin{bmatrix} \|x_0 - x_\star\|^2 & \langle g_0, x_0 - x_\star\rangle \\ \langle g_0, x_0 - x_\star\rangle & \|g_0\|^2 \end{bmatrix}, \quad F = f_0 - f_\star,$$

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- previous problem can be reformulated as a 2×2 SDP

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- For $d = 1$ same optimal value by adding $\operatorname{rank}(G) \leqslant 1$.

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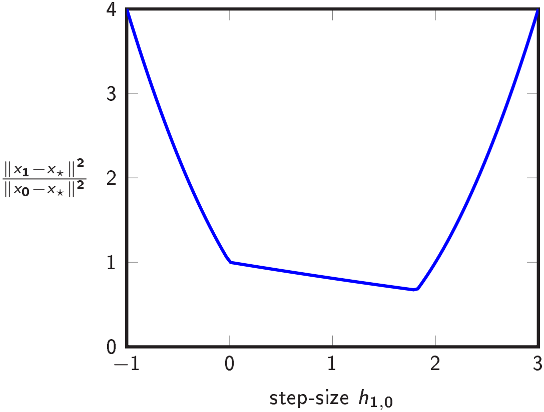

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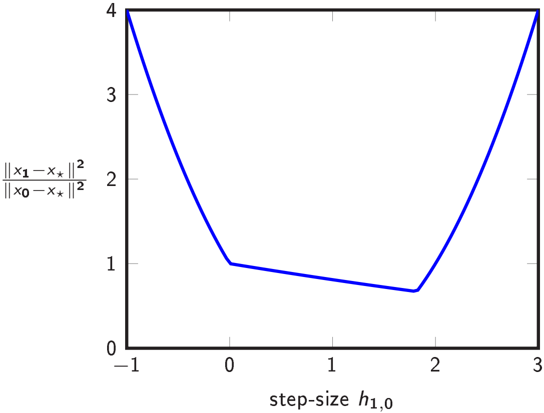

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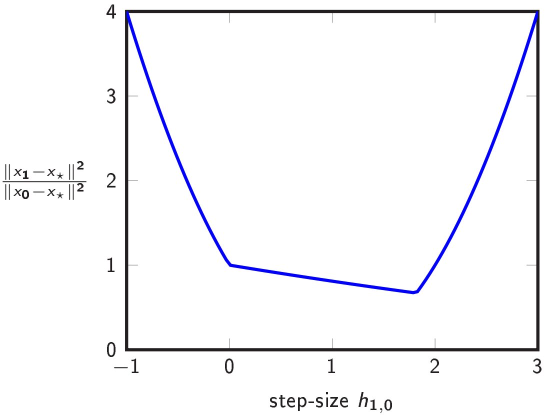

◇ Observation: numerics match $\max\{(1 - h_{1,0}L)^2, (1 - h_{1,0}\mu)^2\}$.

◇ We recover the celebrated $\frac{2}{L+\mu}$ as the optimal step-size.

Translation to worst-case guarantees

◇ Summary: we can compute for the smallest $\tau(h_{1,0})$ such that

$$\|x_1 - x_\star\|^2 \leq \tau(h_{1,0})\|x_0 - x_\star\|^2$$

is satisfied for all $x_0 \in \mathbb{R}^d$, $d \in \mathbb{N}$, $f \in \mathcal{F}_{\mu,L}$, and $x_1 = x_0 - h_{1,0} f'(x_0)$.

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- Note: many things can be said on such problems (obtaining rigorous proof, etc.)
  - Many details & references in `https://francisbach.com/computer-aided-analyses/`
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  - Performance-Estimation-Toolbox (PESTO) on Github.
- For now: what about minimizing $\tau(h_{1,0})$? And for more complicated methods?

Primal problem

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- Recall primal problem, with step-size optimization

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\min_{h_{1,0}} \ \max_{G,\,F} \quad & G_{1,1} + h_{1,0}^2 G_{2,2} - 2h_{1,0} G_{1,2} \\
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- "Simple" minimization problem by dualizing inner maximization.
- Introduce dual variables λ_1, λ_2 and τ for the linear constraints, and dualize.

Dual problem

- Dual problem is

$$\begin{aligned} \min_{\tau,\lambda_1,\lambda_2\geqslant 0} \ & \tau \\ \text{subject to } S = & \begin{bmatrix} \frac{\mu+L(\lambda_1\mu-1)}{L-\mu}+\tau & h_{1,0}-\frac{\lambda_1(\mu+L)}{2(L-\mu)} \\ h_{1,0}-\frac{\lambda_1(\mu+L)}{2(L-\mu)} & \frac{\lambda_1}{L-\mu}-h_{1,0}^2 \end{bmatrix} \preccurlyeq 0 \\ 0 = & \lambda_1-\lambda_2. \end{aligned}$$

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- Optimization jointly over $h_{1,0}$ "for free" (still linear SDP via Schur complement).

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- Recall first-order method of interest

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x_1 &= x_0 - h_{1,0} f'(x_0) \\
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- This time, more complicated, and nonconvex (example with $N = 2$ next slide).
- Idea: use numerical inspiration to find tractable relaxations/reformulations.

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for some $S_{1,1}, S_{1,2}, S_{1,3}, S_{2,2}, S_{2,3}, S_{3,3}$ (functions of $\tau, \lambda_1, \ldots, \lambda_6$ and $\{h_{i,j}\}$).

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◇ In particular

$$S_{1,2} = -\frac{L\lambda_3 - 2(L-\mu)h_{2,0} + \mu\lambda_1 + L\mu(\lambda_2+\lambda_5)h_{1,0}}{L-\mu}$$

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- LMI remains convex in some step-sizes ($h_{2,0}$ and $h_{2,1}$) but not in the others.

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- Recall again the first-order method of interest

$$
\begin{aligned}
x_1 &= x_0 - h_{1,0} f'(x_0) \\
x_2 &= x_1 - h_{2,0} f'(x_0) - h_{2,1} f'(x_1) \\
x_3 &= x_2 - h_{3,0} f'(x_0) - h_{3,1} f'(x_1) - h_{3,2} f'(x_2) \\
&\vdots \\
x_N &= x_{N-1} - \sum_{i=0}^{N-1} h_{N,i} f'(x_i).
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$$[h_{i,j}] = \begin{bmatrix}1.9187 & & & \\ 0.4098 & 2.1746 & & \\ 0.1796 & 0.5147 & 2.1746 & \\ 0.0627 & 0.1796 & 0.4098 & 1.9187\end{bmatrix}.$$

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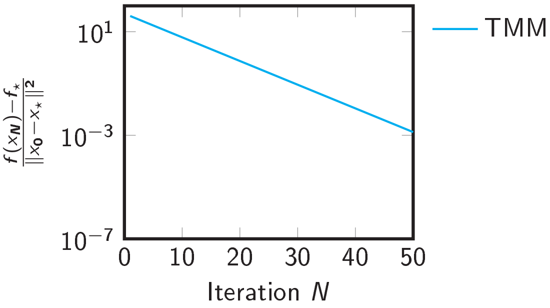

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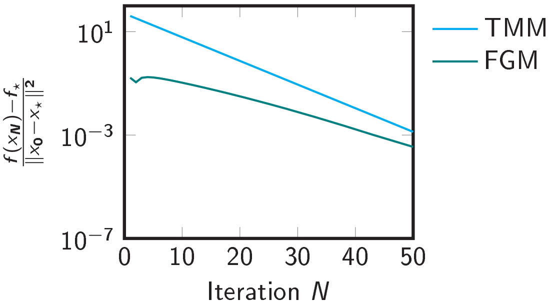

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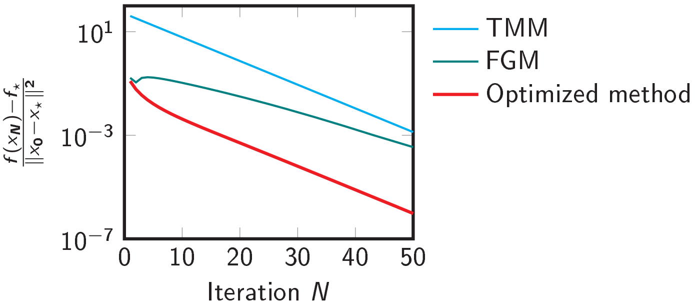

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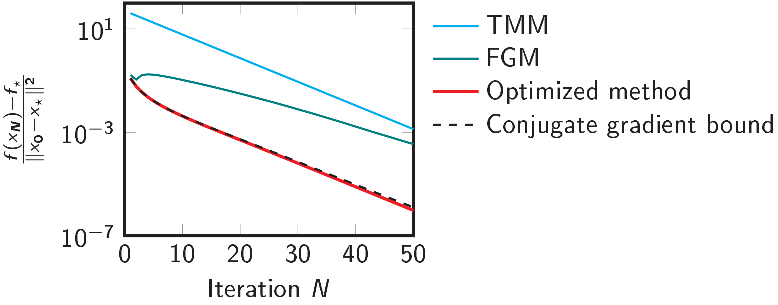

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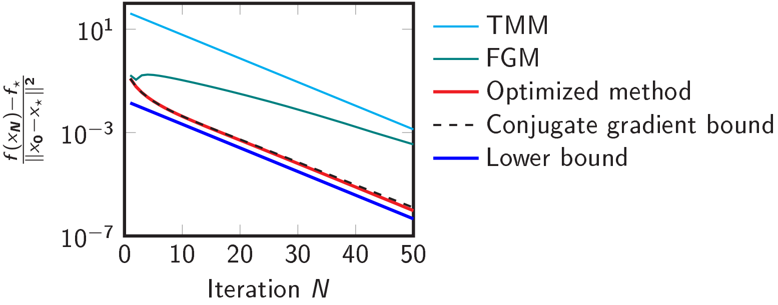

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- All details can be found in (T. & Drori, 2021), and (Drori & T., 2021).

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A few observations/limitations

Were we lucky? Some pieces might be missing!

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- see e.g.: A. Nemirovsky's "Information-based complexity of convex programming." (lecture notes, 1995)

On worst-case analyses

Step-sizes optimization

Constructing lower bounds

Software

Concluding remarks

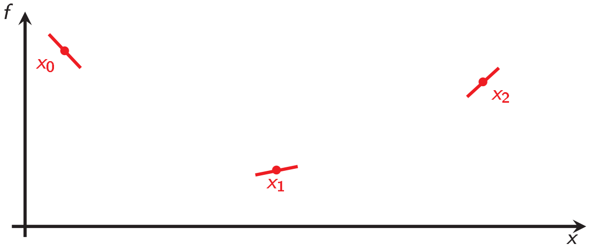
Shape of lower complexity bounds

Role of extension/interpolation results, so far?

- For obtaining *tight* SDP representation of the worst-case computation problem.
- We can infer shapes for the worst-case functions!
  - Why? Let's flashback into the interpolation/extension problem!

Reminder: smooth strongly convex interpolation/extension

Consider a set S, and its associated values $\{(x_i, g_i, f_i)\}_{i\in S}$ with coordinates x_i, subgradients g_i and function values f_i.



? Possible to find a $f \in \mathcal{F}_{\mu,L}$ s.t.

$$f(x_i) = f_i, \quad \text{and} \quad g_i \in \partial f(x_i), \qquad \forall i \in S.$$

Special case: convex interpolation problem

Conditions for $\{(x_i, g_i, f_i)\}_{i \in S}$ to be interpolable by a function $f \in \mathcal{F}_{0,\infty}$ (proper, closed and convex function) ?

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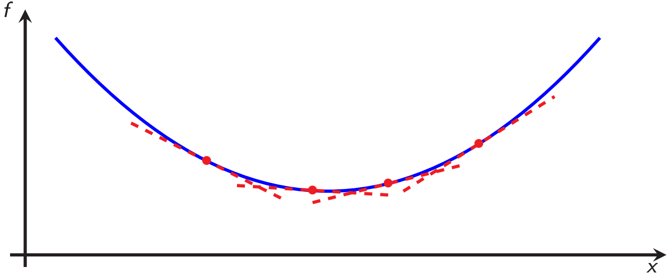

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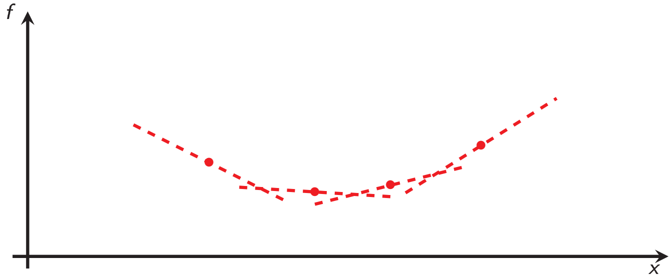

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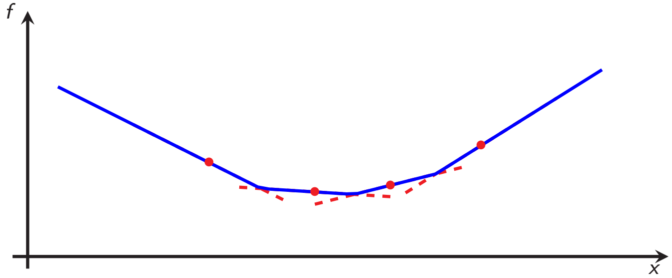

Conditions $f_i \geq f_j + \langle g_j, x_i - x_j \rangle$ is nec. and suff.

Explicit construction:

$$f(x) = \max_j \left\{ f_j + \langle g_j, x - x_j \rangle \right\},$$

Not unique.

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  - those constraints fit into a SDP;
  - such functions are sometimes referred to as being "zero-chain".

Avoiding semidefinite programming modeling steps?

- ◇ Performance Estimation Toolbox (PESTO) available on

  AdrienTaylor/Performance-Estimation-Toolbox

  Contains about 50 examples.
- ◇ Python version should be available during summer.

Example: convergence of Douglas-Rachford

```
% (0) Initialize an empty PEP
P=pep();

N = 1;
% (1) Set up the class of monotone inclusions
paramA.L  =  1; paramA.mu = 0; % A is 1-Lipschitz and 0-strongly monotone
paramB.mu = .1;                % B is .1-strongly monotone

A = P.DeclareFunction('LipschitzStronglyMonotone',paramA);
B = P.DeclareFunction('StronglyMonotone',paramB);

w  = cell(N+1,1);   wp = cell(N+1,1);
x  = cell(N,1);     xp = cell(N,1);
y  = cell(N,1);     yp = cell(N,1);

% (2) Set up the starting points
w{1}     = P.StartingPoint(); wp{1}    = P.StartingPoint();
P.InitialCondition((w{1}-wp{1})^2<=1);

% (3) Algorithm
lambda = 1.3;       % step size (in the resolvents)
theta  = .9;         % overrelaxation

for k = 1 : N
    x{k}    = proximal_step(w{k},B,lambda);
    y{k}    = proximal_step(2*x{k}-w{k},A,lambda);
    w{k+1}  = w{k}-theta*(x{k}-y{k});

    xp{k}   = proximal_step(wp{k},B,lambda);
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end

% (4) Set up the performance measure: ||z0-z1||^2
P.PerformanceMetric((w{k+1}-wp{k+1})^2);

% (5) Solve the PEP
P.solve()

% (6) Evaluate the output
double((w{k+1}-wp{k+1})^2)    % worst-case contraction factor
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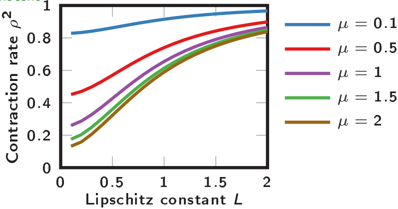

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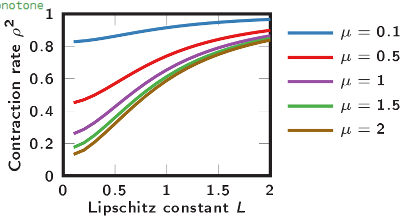

- ✔ fast prototyping ($\sim$ 20 effective lines)
- ✔ quick analyses ($\sim$ 10 minutes)
- ✔ computer-aided proofs (multipliers)

Current library of examples within PESTO

Includes... but not limited to

- subgradient, gradient, heavy-ball, fast gradient, optimized gradient methods,
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- Douglas-Rachford/three operator splitting,
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Among others, see works by Drori, Teboulle, Kim, Fessler, Ryu, Lieder, Lessard, Recht, Packard, Van Scoy, Cyrus, Gu, Yang, etc.

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Performance estimation's philosophy

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- ⋄ closed-form solutions might be involved.

Note: many links with theory on quadratics (Chebyshev methods).

Take-home messages

Worst-cases are solutions to optimization problems

Acceleration/“optimal” methods by optimizing worst-cases

Design of theoretical methods via numerical experiments

Short bibliography

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Presentation mainly based on

- T., Y. Drori. "An optimal gradient method for smooth strongly convex minimization." (2021)
- Y. Drori, T. "On the oracle complexity of smooth strongly convex minimization." (2021)
- T., J. Hendrickx, F. Glineur. "Smooth strongly convex interpolation and exact worst-case performance of first-order methods." (2017)

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Many (very) related works; much more careful bibliographical treatment in papers.

- ◇ Y. Nesterov. "A method of solving a convex programming problem with convergence rate $O(1/k^2)$." (1983)
- ◇ A. Nemirovsky, and B. Polyak. "Iterative methods for solving linear ill-posed problems under precise information." (1984)
- ◇ A. Nemirovsky. "Information-based complexity of linear operator equations." (1992)
- ◇ A. Nemirovsky. "Information-based complexity of convex programming." (lecture notes, 1995)
- ◇ Y. Nesterov. "Introductory Lectures on Convex Optimization." (2003/2018)
- ◇ Y. Drori, and M. Teboulle. "Performance of first-order methods for smooth convex minimization: a novel approach." (2014)
- ◇ D. Kim, and F. Fessler. "Optimized first-order methods for smooth convex minimization." (2017)
- ◇ B. Van Scoy, R. Freeman, K. Lynch. "The fastest known globally convergent first-order method for minimizing strongly convex functions" (2017)
- ◇ D. Kim, and F. Fessler. "Optimizing the efficiency of first-order methods for decreasing the gradient of smooth convex functions." (2021)

Thanks! Questions?

www.di.ens.fr/∼ataylor/

AdrienTaylor/Performance-Estimation-Toolbox on Github