Консенсус в многоагентных системах

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Текст на последующих слайдах в основном английский (связано с изменением в программе школы)

Многоагентная система или мультиагентная?
Я считаю, что правильно первое. Аналоги:

- **Многомерное пространство (multidimensional)**
- **Многоканальная связь (multi-channel)**
- **Многовекторная внешняя политика (multi-vector)**
- **Многоуровневая защита (multi-level)**
- **Многокритериальная оптимизация (multi-criterial)**

«Мульти» в основном сохраняется там, где два корня слова по отдельности не переводят, например мультистабильность.

Агенты – кто (что) они?
В AI/CS – ряд свойств (ключевое – автономность)

В теории управления – просто подсистема сложной системы. Например, нервная клетка.

Минута ностальгии: ТМШ 2013г., оз. Сенеж

- Мое первое участие в летней школе и один из моих первых докладов по консенсусу и многоагентным системам

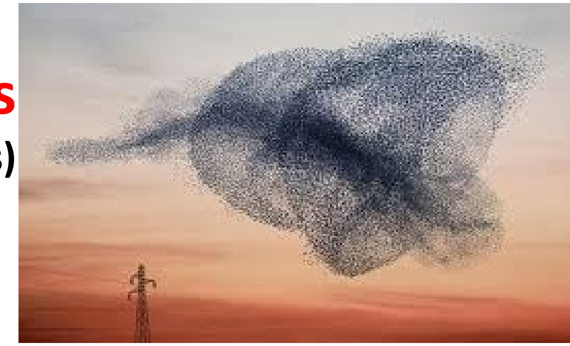
  http://www.youtube.com/watch?v=fNJ7MC0bqiM

- С тех многое изменилось – от названия школы до понимания того, для чего на самом деле нужны консенсусные алгоритмы.

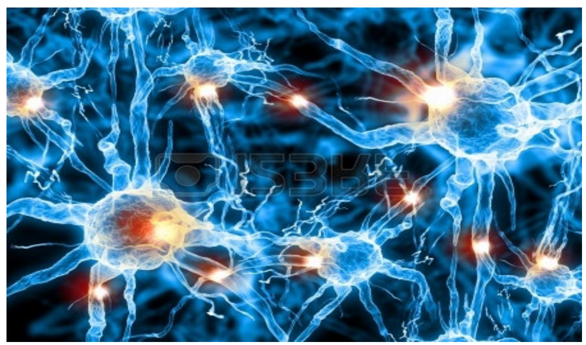
Large-Scale Complex Systems Dynamics

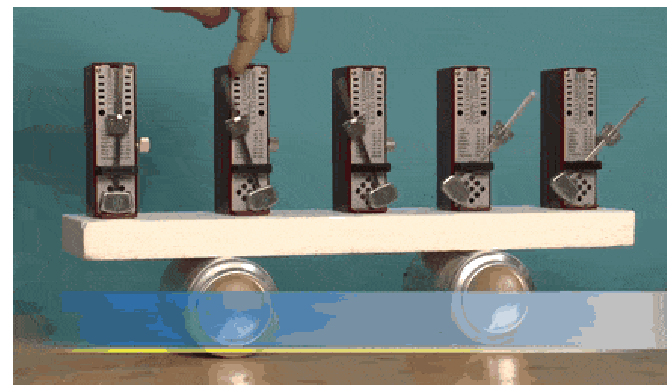
Swarms, herds, flocks

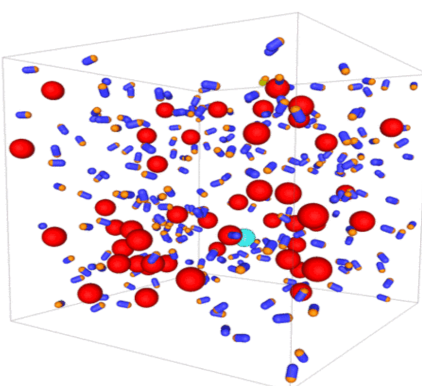
(biological or man-made agents)



Physical, Neuronal, Chemical Oscillators

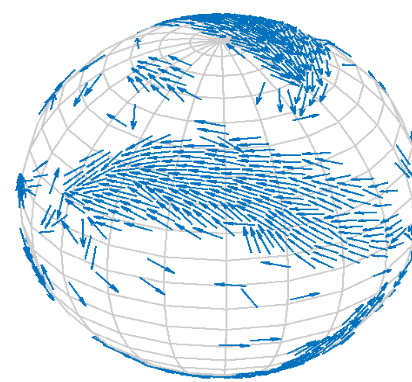






Active Matter

(smart materials and biopolymers, converting the energy from the environment into mechanical work)



About consensus:

- **Direct meaning:** harmony, concord, unaimity, solidarity;
- **In Multi-Agent Systems (MAS) and Complexity Science:** coordinated behavior of subsystems, synchrony in some variables of interest.

Applications in Natural and Social Sciences:
effects of spontaneous order (from synchronous fireflies to sociodynamics);

Приложения к социологии мы обсудим в пятницу!

Applications in Robotic and Computer Systems:
time synchronization, frequency synchronization in power grids, load balance, flocking and swarming of mobile robots, blockchain etc…

✓ distributed optimization and computing.

План лекции:

I. Пример: распределенное усреднение в сенсорных сетях. Алгоритмы итеративного усреднения.

II. Удивительная робастность алгоритмов усреднения: консенсус в неравенствах

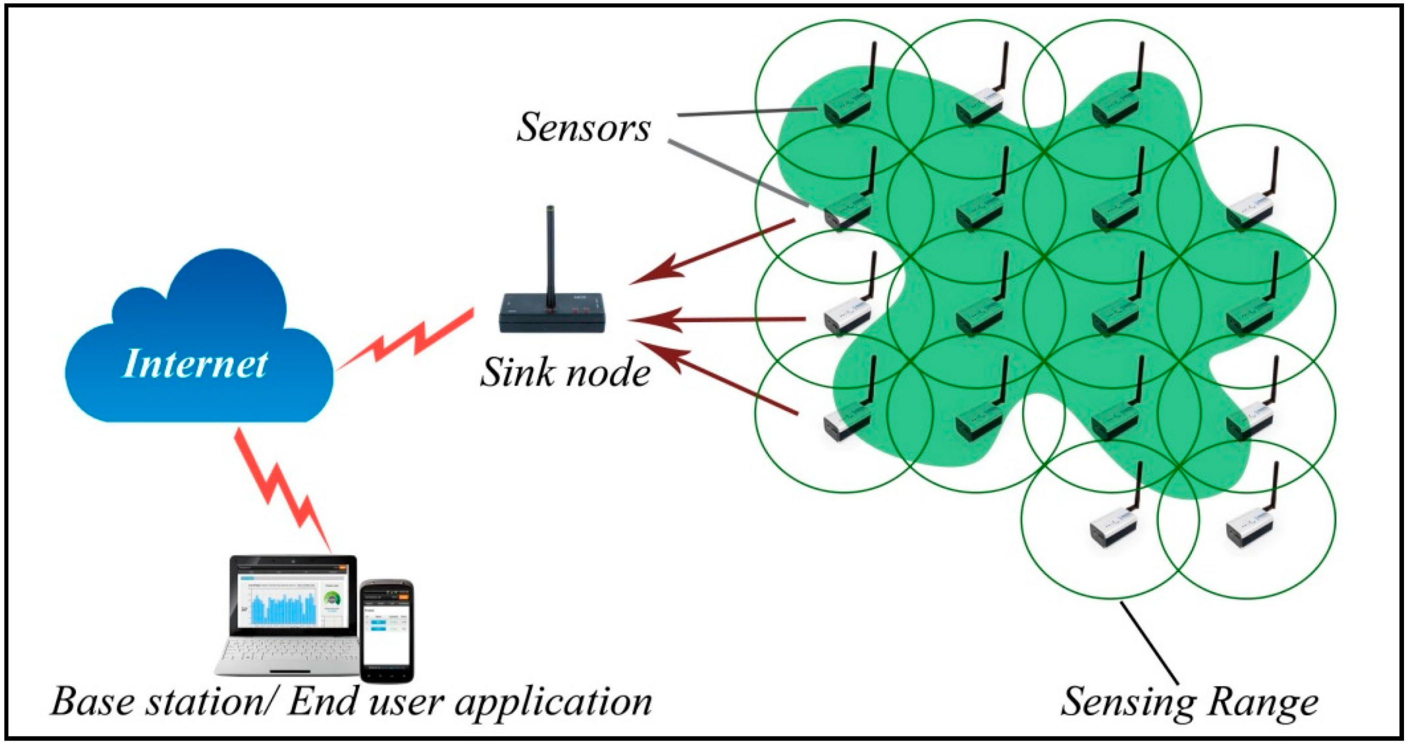
III. Консенсус с ограничениями. Распределенное решение линейных уравнений.

IV. Неподвижные точки парасжимающих отображений. В сторону распределенной оптимизации, или почему я об этом всем рассказываю на данной школе.

V. Заключение.

I. Distributed Averaging in Sensor Networks

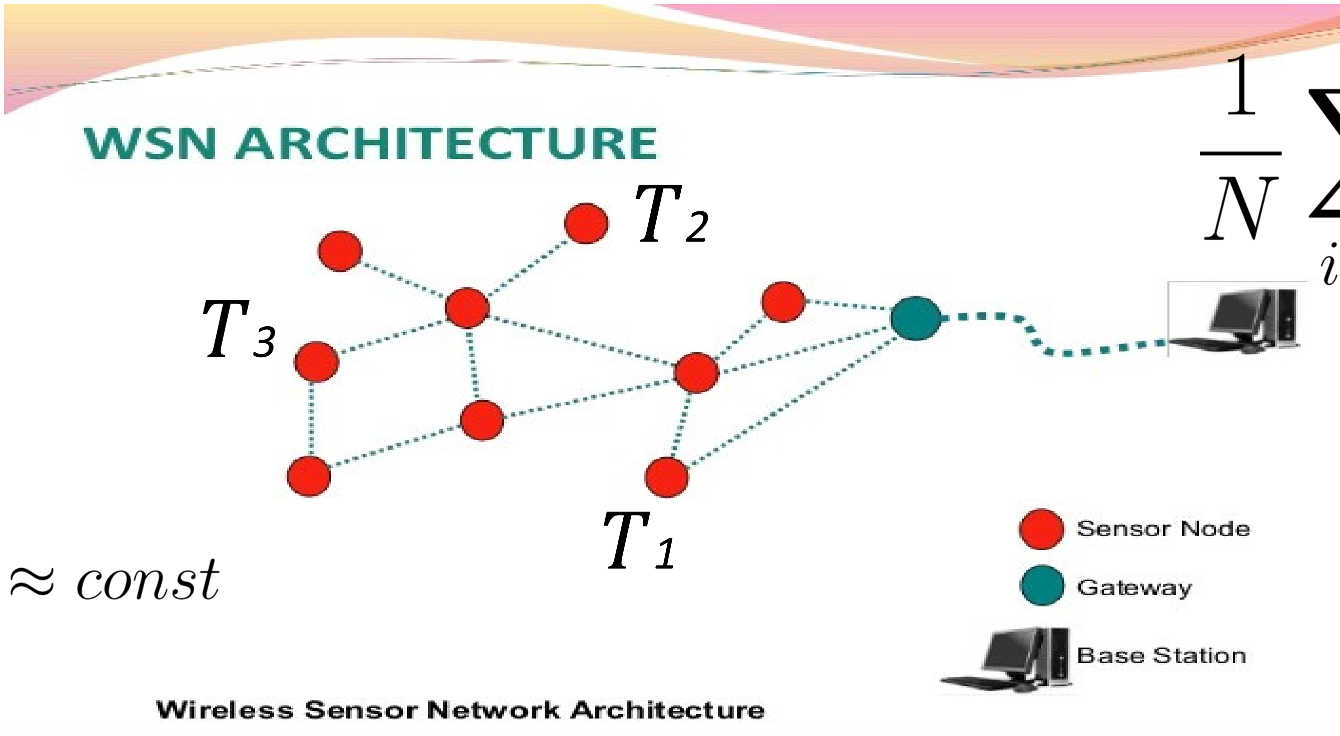
Wireless Sensor Network as a Self-Organizing System



- Huge number of simple agents **(sensors)** with limited ranges – motion sensors, thermometers, manometers, concentration meters, smoke detectors etc.;
- Communicate **wireless** (limited data-rate, nearest-neighbor connection, delays);
- Can be accessed remotely via a few gateways (sinks);
- Can **coordinate** in order to reach cooperative goals or enhance resilience/robustness.

Wireless Sensor Network. Averaging (I).

$$\frac{1}{N}\sum_{i=1}^{N} T_i = ?$$



$$T_i \approx const$$

Is the straightforward centralized solution that good?

- Global information: how much memory do we need? how long to wait?
- Complex protocols (guaranteed delivery, multi-path resolution etc.);
- Security issues (malicious attacks on the central unit allow to access to all information and change the result of computing).

Wireless Sensor Network. Averaging (II).

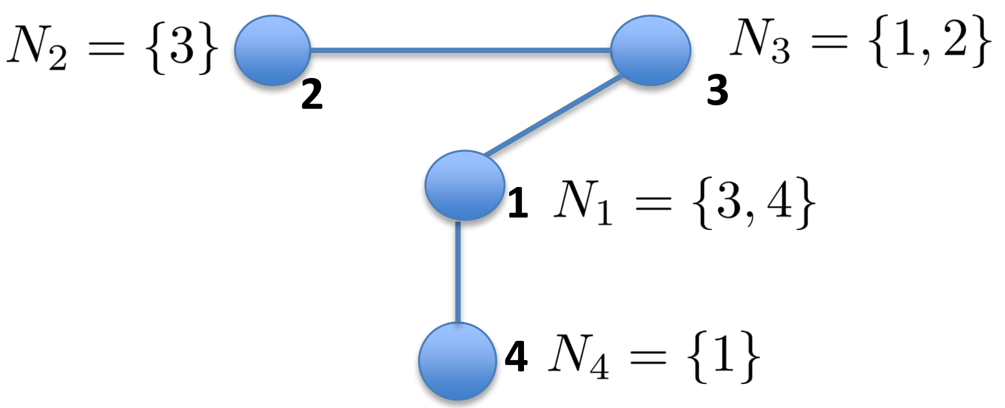
A "rule of thumb" decentralization: each sensor *i* computes the average

- **Initialize the process by the actual measurement:**

$$x_i(0) = T_i$$

- **At each stage, average the own value with the neighbors' values:**

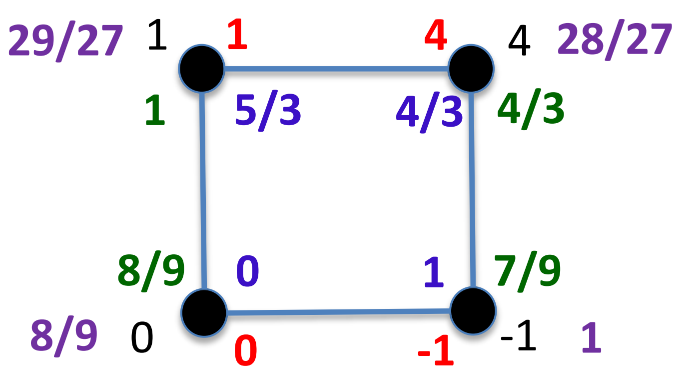
$$x_i(k+1) = \frac{1}{1+|N_i|}\left(x_i(k) + \sum_{j\in N_i} x_j(k)\right)$$



- **All sensors perform the operations simultaneously, k=0,1,2,…**

Wireless Sensor Network. Averaging (III).

A numerical example: let's see how it works on a small graph



$$N = 4, \bar{T} = \frac{1}{N}\sum_i T_i = 1,$$

$$D(k) = \max_i |x_i(k) - \bar{T}|$$

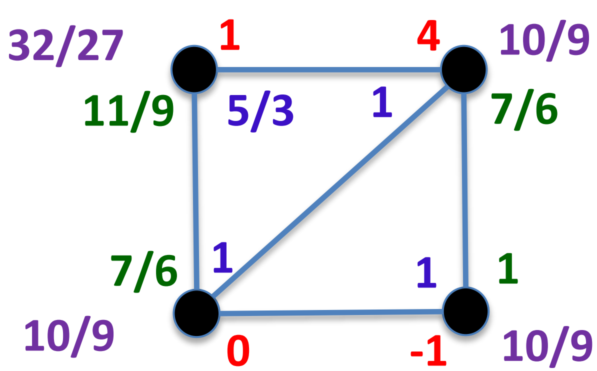
Step 0 (D=3)

Step 1 (D=1)

Step 2 (D=1/3)

Step 3 (D=1/9)

Algorithm seems to (and does) compute the mean value, but…



Step 0 (D=3)

Step 1 (D=2/3)

Step 2 (D=2/9)

Step 3 (D=5/27)

$$x_i(k) \geq 10/9 \quad \forall i = \overline{1,4} \;\; \forall k \geq 3.$$

Actually, all values converge to 1.14286…

Is the algorithm completely useless then?

Wireless Sensor Network. Averaging (IV).

To where and why does the vector *x(k)* converge? Look at its dynamics

- **At each stage, average the own value with the neighbors' values:**

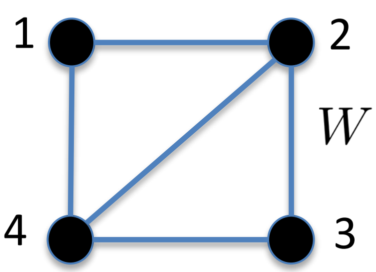
$$x_i(k+1) = \frac{1}{1+|N_i|}\left(x_i(k) + \sum_{j\in N_i} x_j(k)\right)$$

- **In the vector form, these dynamics shape into**

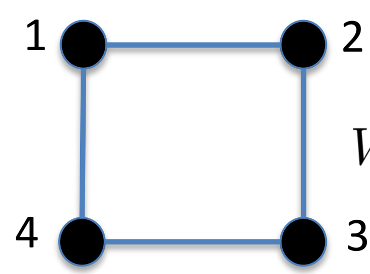
$$x(k+1) = Wx(k),$$

$$W = (w_{ij}),\ w_{ij} \geq 0,\ \sum_{j=1}^{N} w_{ij} = 1. \qquad (*)$$

- **Matrices obeying the condition (*) are said to be (row-)stochastic**



$$W = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$



$$W = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

a **doubly stochastic** matrix

Wireless Sensor Network. Averaging (V).

Consensus, Markov Chains, SIA (Fully Regular) Matrices

The DeGroot Model of Iterative Averaging (Opinion Pooling):

x(k) – vector of the agents' opinions about the value of something (e.g. mean value)

$$x(k+1) = Wx(k), \quad x(0) \in \mathbb{R}^N \qquad (+)$$

Dynamics of the Markov chain with transition matrix *W:*

$$\pi(k+1)^\top = \pi(k)^\top W, \quad \pi_i(0) \geq 0, \sum_i \pi_i(0) = 1 \qquad (!)$$

$$\pi_i(k) = \mathbb{P}(\text{chain in state } i \text{ at time } k)$$

The following three statements are equivalent (easy exercise from linear algebra):

- **algorithm (+) establishes consensus:** $\lim_{k\to\infty} x_i(k) = \xi_* = \xi_*[x(0)] \quad \forall x(0)$
- **the Markov chain (!) is ergodic:** $\lim_{k\to\infty} \pi(k) = \pi_* = const \quad \forall \pi(0)$
- **the matrix W is fully regular, or stochastic indecomposable aperiodic (SIA)**

$$\lim_{k\to\infty} W^k = \mathbb{1}_N \pi_*^\top = \begin{bmatrix} \pi_*^\top \\ \vdots \\ \pi_*^\top \end{bmatrix}$$

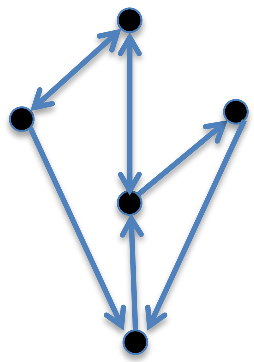
$$x(k) \xrightarrow[k\to\infty]{} \mathbb{1}_N \xi_*, \ \xi_* = \pi_*^\top x(0)$$

$$\pi(k)^\top \xrightarrow[k\to\infty]{} \pi_*^\top, \ \pi_*^\top W = \pi_*^\top.$$

Wireless Sensor Network. Averaging (VI).

Consensus in the DeGroot model: A simple sufficient condition

$$x(k+1) = Wx(k), \quad x(0) \in \mathbb{R}^N \qquad (+)$$



Each *NxN* matrix encodes some graph with *N* nodes

$$i \mapsto j \implies w_{ij} > 0$$

A stochastic matrix is **irreducible** if its graph is strongly connected.

Algorithms(+) establishes consensus, if *W* is irreducible and has a positive diagonal

$$w_{ii} > 0 \quad \forall i > 0$$

$$\lim_{k\to\infty} W^k = \mathbb{1}_N \pi_*^\top = \begin{bmatrix} \pi_*^\top \\ \vdots \\ \pi_*^\top \end{bmatrix} \implies \begin{cases} x(k) \xrightarrow[k\to\infty]{} \mathbb{1}_N \xi_*, \; \xi_* = \pi_*^\top x(0) \\ \pi_* \geq 0, \; \pi_*^\top \mathbb{1}_N = 1, \; \pi_*^\top W = \pi_*^\top. \end{cases}$$

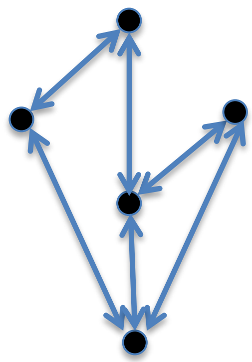
Average consensus (as desired) means double stochasticity of W:

$$\pi_* = N^{-1}\mathbb{1}_N \iff \mathbb{1}_N^\top W = \mathbb{1}_N^\top \iff \sum_i w_{ij} = 1 \, \forall j.$$

Wireless Sensor Network. Averaging (VII).

Solution for undirected connected interaction graphs

$$x(k+1) = Wx(k), \quad x(0) \in \mathbb{R}^N \qquad (+)$$



$$w_{ij} = \begin{cases} 1-\alpha|N_i|, & i=j \\ \alpha, & j \in N_i (\Leftrightarrow i \in N_j) \\ 0, & j \notin N_i \end{cases}, \quad \text{where } \alpha|N_i| < 1.$$

W is **symmetric** (thus doubly stochastic), **irreducible** (connectivity)

$$x_i(k+1) = (1-\alpha|N_i|)x_i(k) + \alpha \sum_{j \in N_i} x_j(k)$$

A solution to the averaging problem:

- **decentralized (no global information, except for the gain α);**
- **secure (transmissions only between neighbor agents), redundant;**
- **computes the exact value not in finite time, but converges exponentially;**
- **can be extended to general strongly connected graph (finding W is less trivial);**
- **other extensions: distributed filtering and inference (the measured value is generated by a non-trivial and partially uncertain systems, observations are noisy)**

II. Amazing Robustness. From Equations to Inequalities

Robustness of consensus algorithms

$$x(k+1) = Wx(k).$$

(always assumed: *W* is irreducible with positive diagonal entries)

1. **Bounded communication delay doesn't destroy consensus**

$$x_i(k+1) = w_{ii}x_i(k) + \sum_{j\neq i} w_{ij}x_j(t - \tau_{ij}(k)) \implies \lim_{k\to\infty} x_i(k) = \tilde{\xi}_* \, \forall i.$$

2. **Synchronization (not convergence!) is robust against small perturbations**

$$x(k+1) = Wx(k) + \eta(k) \implies$$

$$\overline{\lim_{k\to\infty}} |x_i(k) - x_j(k)| \leq c \overline{\lim_{k\to\infty}} |\eta(k)| \quad \forall i \forall j$$

3. **Amazing robustness against sign-preserving disturbances**

$$x(k+1) = Wx(k) + \eta(k), \quad \eta_i(k) \geq 0 \forall i \forall k \implies$$

$$\lim_{k\to\infty} x_i(k) = \tilde{\xi}_* = \tilde{\xi}_* \left[x(0), \eta(\cdot)\right] \leq \infty \quad \forall i.$$

Important notice: $\xi_* < \infty \implies \eta(k) \xrightarrow[k\to\infty]{} 0 \quad (\not\Longleftarrow)$

Theorem (A. Proskurnikov, M. Cao, 2017)

Let *W* be a square row-stochastic irreducible matrix with positive diagonal entries. Then, each solution to the system of recurrent inequalities

$$x(k+1) \geq Wx(k) \iff x_i(k+1) \geq \sum_j w_{ij} x_j(k)\, \forall i$$

converges to a vector of identical components

$$x(k) \xrightarrow[k\to\infty]{} \xi_* \mathbb{1}_N, \quad \xi_* \in \mathbb{R} \cup \{+\infty\}$$

If, additionally, $\xi_* < \infty$ **then the ``residual''** $x(k+1) - Wx(k) \xrightarrow[k\to\infty]{} 0$

The requirement of irreducibility (strong connectivity) cannot be relaxed

The result applies to the reversed inequalities as follows

$$x(k+1) \leq Wx(k) \quad (\xi_* \in \mathbb{R} \cup \{-\infty\})$$

Proceedings of IEEE Conference on Decision and Control CDC 2017, Melbourne

Sketch of the proof

$$x(k+1) \geq Wx(k) \iff x_i(k+1) \geq \sum_j w_{ij} x_j(k)\, \forall i$$

Important constructions: ordering permutation of the vector and the minimal positive entry

$$\min_i x_i(k) = y_1(k) \leq y_2(k) \leq \ldots \leq y_N(k) = \max_i x_i(k)$$

$$q := \min\{w_{ij} : w_{ij} > 0\}.$$

Observation 1: the minimum is not decreasing $y_1(k+1) \geq y_1(k) \implies \exists \xi_* = \lim_{k\to\infty} y_1(k)$

Observation 2: $w_{ij} > 0 \implies x_i(k+1) \geq q x_j(k) + (1-q) y_1(k) \qquad (*)$

$$\forall i \;\; w_{ii} > 0 \implies x_i(k+1) \geq q x_i(k) + (1-q) y_1(k) \qquad (!)$$

Observation 3: $y_1(k+1) \geq q y_2(k) + (1-q) y_1(k) \implies y_2(k) \xrightarrow[k\to\infty]{} \xi_*.$

$$\exists i_1 = i_1(k), j \neq i_1 : y_1(k) = x_{i_1}(k), w_{i_1 j} > 0 \overset{(*)}{\implies} x_{i_1}(k+1) \geq q y_2(k) + (1-q) y_1(k)$$

$$\forall m \neq i_1 : x_m(k) \geq y_2(k) \overset{(!)}{\implies} x_m(k+1) \geq q y_2(k) + (1-q) y_1(k).$$

Observation 4 (similar): $y_s(k+1) \geq q y_{s+1}(k) + (1-q) y_1(k)$

$$y_1(k), \ldots, y_s(k) \to \xi_* \implies y_{s+1}(k) \to \xi_*$$

III. Constrained Consensus and Distributed Equation Solving

Немного истории: классическая задача о нахождении общей точки выпуклых множеств, но в многоагентной постановке.

Предшествующие работы (основные).

- J. von Neumann, Functional Operators, Vol. II. The Geometry of Orthogonal Spaces, Princeton Univ. Press, 1933
- Kaczmarz S. Angen ̈aherte aufl ̇osung von systemen linearer gleichungen// Bull. Int. l'Acad. Polon. Sci. Lett. A. — 1937. — 35. — C. 355–357.
- Cimmino G. Calcolo approssiomatto per le soluzioni dei sistemi di equazioni lineari// La Ricerca Sci. XVI. Ser. II. — 1938. — 1. — C. 326–333.

Нахождение точки, принадлежащей пересечению нескольких гиперпространств

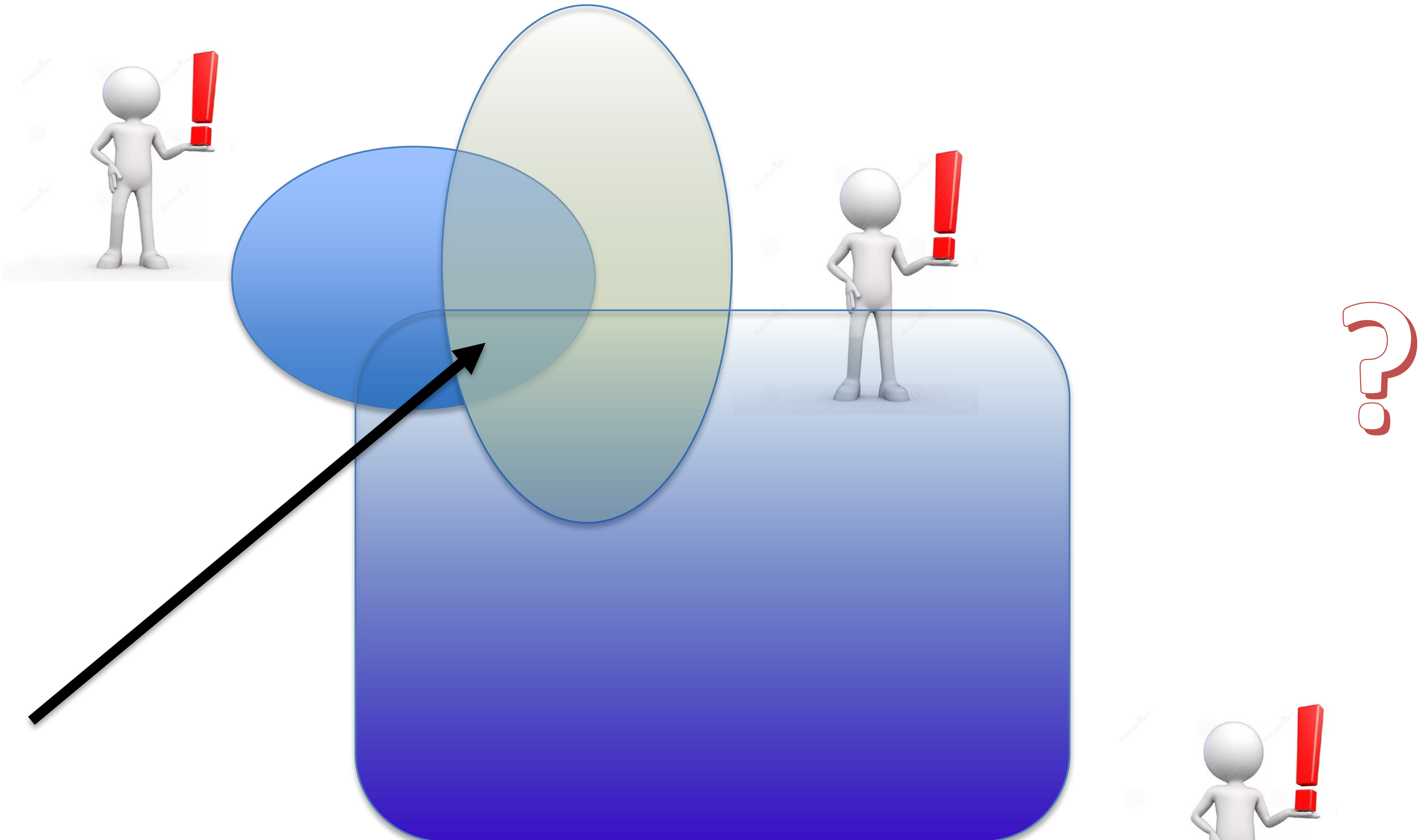
- Якубович В.А. Рекуррентные конечно-сходящиеся алгоритмы решения систем неравенств // ДАН СССР. 1966. Т.166. No 6. С. 1308–131

- Губин Л. Г., Поляк Б. Т., Райк Е. В. Метод проекции для нахождения общей точки выпуклых множеств// Журн. выч. мат. и мат. физ. — 1967. —7. — С. 1–24.

- Брэгман Л.М., Релаксационный метод нахождения общей точки выпуклых множеств и его применение для решения задач выпуклого программирования//Журн. выч. мат. и мат. физ. — 1967. —7. — С.620-631

Естественное обобщение: задача о консенсусе с выпуклыми ограничениями (группа A. Nedic, группа W. Ren, A.S. Morse, B.D.O. Andersson и др.) Консенсусный алгоритм + метод проекции.

Constrained Consensus (I):
A problem of rational decision making



Each of N agents has a convex set (constraint) in a decision space, which the agent prefers to keep private.

Find a point in the decision space, satisfying all of these constraints

Constrained Consensus (II):
Potential applications

Distributed solving of linear equations (e.g. PageRank, Katz or Eigenvector Centrality computation in a network)

$$a_i^\top x = b_i, \quad i = 1, \ldots, N.$$

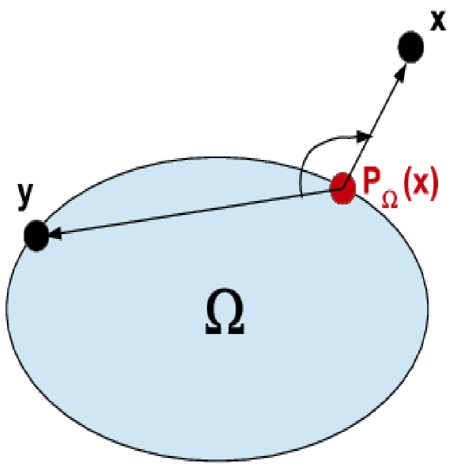
Agent i keeps confidential data $a_i \in \mathbb{R}^M, b_i \in \mathbb{R}.$ **Decision space is** $\{x\} = \mathbb{R}^M.$

The algorithm delivers some solution of the resulting system (assuming that it exists)

A board of directors are discussing a portfolio of investments. In total, 1 bln $ is to be distributed between M assets (actions, currencies etc.). Each director has a subjective expectation of the financial gain from each asset (can be negative), and aims to keep the expective profit non-negative. This leads to constraints:

$$a_i^\top x \geq 0, \quad x \geq 0, x^\top \mathbb{1}_M = 1.$$

Constrained Consensus (III): Distributed algorithms



Projection operator: returns the nearest point

$$\forall y \in \Omega \quad |x-y|^2 \geq |x - P_\Omega(x)|^2 + |y - P_\Omega(x)|^2$$

$$\Omega = \{y : a^\top y = b\} \implies P_\Omega(x) = x + \frac{b - a^\top x}{|a|^2} a$$

The agents' constraint sets:

$$\Xi_i = \overline{\mathrm{conv}}\, \Xi_i \subseteq \mathbb{R}^n, \; P_i = P_{\Xi_i},$$

$$\Xi_* = \Xi_1 \cap \ldots \cap \Xi_N \neq \emptyset$$

Consider the following two algorithms

$$\xi_i(k+1) = w_{ii} P_i(\xi_i(k)) + \sum_{j \neq i} w_{ij} \xi_j(k); \qquad (1)$$

$$\xi_i(k+1) = P_i\left(\sum\nolimits_j w_{ij}\xi_j(k)\right). \qquad (2)$$

Constrained Consensus (IV):

Why do the algorithms work? Consider (1) as a simpler one:

$$\xi_i(k+1) = w_{ii}P_i(\xi_i(k)) + \sum_{j\neq i} w_{ij}\xi_j(k). \qquad (1)$$

Observation 1. Algorithm (1) hides some inequality inside!

$$\xi_* \in \Xi_*, \quad x_i(k) = |\xi_i(k) - \xi_*|^2 \implies$$

$$0 \leq x_i(k+1) \leq w_{ii}|P_i(\xi_i(k)) - \xi_*|^2 + \sum_{j\neq i} w_{ij}|\xi_j(k) - \xi_*|^2 \leq$$

$$\leq \sum_j w_{ij}x_j(k) - w_{ii}|\xi_i(k) - P_i(\xi_i(k))|^2 \leq \sum_j w_{ij}x_j(k).$$

Observation 2. Consensus in this inequality implies that (1) is "almost linear"

$$\xi_i(k+1) = \sum_j w_{ij}\xi_j(k) + \varepsilon_i(k), \quad \varepsilon_i(k) = \xi_i(k) - P_i(\xi_i(k)) \to 0$$

Observation 3. We know the robustness property.
Hence, all agents' vectors converge to the desired set.

$$|\xi_i(k) - \xi_j(k)| \xrightarrow[k\to\infty]{} 0$$

$$\max_j d(\xi_i(k), \Xi_j) \to 0 \implies d(\xi_i(k), \Xi_*) = 0 \forall i$$

Constrained Consensus (V).

Theorem (almost proved on the previous slide)

$$\Xi_i = \overline{\mathrm{conv}}\,\Xi_i \subseteq \mathbb{R}^n,\; P_i = P_{\Xi_i},$$

$$\Xi_* = \Xi_1 \cap \ldots \cap \Xi_N \neq \emptyset$$

$$\xi_i(k+1) = w_{ii} P_i(\xi_i(k)) + \sum_{j \neq i} w_{ij} \xi_j(k); \qquad (1)$$

$$\xi_i(k+1) = P_i\left(\sum_j w_{ij} \xi_j(k)\right). \qquad (2)$$

Let *W* be a square row-stochastic irreducible matrix with positive diagonal entries. Then, both algorithms (1) and (2) provide the convergence

$$\xi_i(k) \xrightarrow[k \to \infty]{} \xi_* = \xi_* \left[(\xi_i(0))_{i=1}^N, W \right] \in \Xi_*.$$

- **Algorithm (1): Nedic, Parrilo, Ozdaglar, 2010, IEEE TAC (doubly stochastic W)**
- **Algorithm (2): You, Song, Tempo, 2016, IEEE CDC**
- **Modifications: Mou, Liu, Morse, 2015, IEEE TAC (only linear constraints)**

IV. Fixed Points of Paracontractive Maps. Towards Distributed Optimization

An Extension of the Constrained Consensus Protocols

$$\xi_i(k+1) = w_{ii} P_i(\xi_i(k)) + \sum_{j \neq i} w_{ij} \xi_j(k); \qquad (1)$$

$$\xi_i(k+1) = P_i\left(\sum\nolimits_j w_{ij} \xi_j(k)\right). \qquad (2)$$

Which properties of the projection operators have we actually used?

$$\Xi_i = \{x : P_i(x) = x\} \neq \emptyset, \quad \Xi_* = \bigcap_{i=1}^{N} \Xi_i \neq \emptyset.$$

$$\forall x \notin \Xi_i \, \forall y \in \Xi_i \quad |P_i(x) - P_i(y)| = |P_i(x) - y| < |x - y|.$$

(paracontraction property)

The previous theorem retains its validity if all "projectors" are continuous and paracontractive: if there exists a common fixed point, the algorithms (1), (2) deliver one of such points.

Constraint (fixed point) sets are unknown by the agents!

Fullmer, Liu, Morse, Proc. of IEEE CDC 2016, Proc. Of ACC 2017

Example:
Simultaneous Optimization of Smooth Strongly Convex Functions

$$P_i(x) = x - \alpha_i \nabla f_i(x)$$

$$m_i |x - y|^2 \leq (\nabla f_i(x) - \nabla f_i(y))^\top (x - y) \leq L_i |x - y|^2$$

$$0 < \alpha_i < 2m_i / L_i.$$

If the set where all functions achieve the global minimum is non-empty, then the algorithms (1a) and (2a) will compute one of the optimal points.

$$\xi_i(k+1) = \sum_j w_{ij} \xi_j(k) - \alpha_i \nabla f_i(\xi_i(k)); \qquad (1a)$$

$$\xi_i(k+1) = v_i(k) - \alpha_i \nabla f_i(v_i(k)), \quad v_i(k) = \sum_j w_{ij} \xi_j(k). \qquad (2a)$$

Optimization, but very special: each agent has own cost function, many technical assumptions, no constraints. What about more general cases?

Can we combine the general constrained consensus algorithm with the gradient descent?

Constrained Optimization via Constrained Consensus:

Nedic's theory

$$\sum_{i=1}^{N} f_i(\xi_i) \to min$$

$$\text{s.t.} \quad \xi_i \in \Xi_i \;\; \forall i = 1, \ldots, N.$$

Under certain assumptions, the following algorithm with time-varying step-size parameter finds a global optimum (generally, non-unique)

$$\xi_i(k+1) = P_{\Xi_i}\left[v_i(k) - \alpha_i(k)\nabla f_i(v_i(k))\right], \quad v_i(k) = \sum\nolimits_j w_{ij}\xi_j(k).$$

$$\sum_{k=1}^{\infty} \alpha_i(k) = \infty, \sum_{k=1}^{\infty} \alpha_i(k)^2 < \infty$$

Conditions are: double stochasticity of W, bounded (sub)gradients, compact sets

Alternative approaches exist, e.g. primal-dual (Lagrangian multipliers).

Nedic, Ozdaglar, Parrilo, 2010, IEEE TAC, vol.55, no.4

Conclusions and Extensions.

- Consensus algorithms open up the perspective to solve complex problems in a distributed way (e. g., solve systems of linear equations and/or inequalities)
- Consensus algorithms (under strong connectivity assumptions) are amazingly robust: consensus is not destroyed, replacing the equation by inequality
- Constrained consensus algorithms are closely related to distributed convex optimization, which has recently been thoroughly studied (deterministic/stochastic gradient descent, primal-dual methods)
- Consensus algorithms are closely related to models of social dynamics;
- Extensions to time-varying (e.g. repeatedly strongly connected) graphs and continuous time exist;
- Most challenging problem is synchronization of agents that have non-trivial heterogeneous dynamics (e.g. power generators, neurons, robots etc.)

Some Literature References

Consensus algorithms:

- Ren, W. , & Beard, R. Distributed consensus in multi-vehicle cooperative control: Theory and applications Springer-Verlag, 2008.
- Mesbahi, M. and Egerstedt, M. Graph Theoretic Methods in Multiagent Networks. Princeton Univ. Press, 2010.
- Proskurnikov, A. and Tempo, R. A tutorial on modeling and analysis of dynamic social networks. Part II, Annual Reviews in Control, 2018, vol. 45.

Consensus in recurrent and differential inequalities:

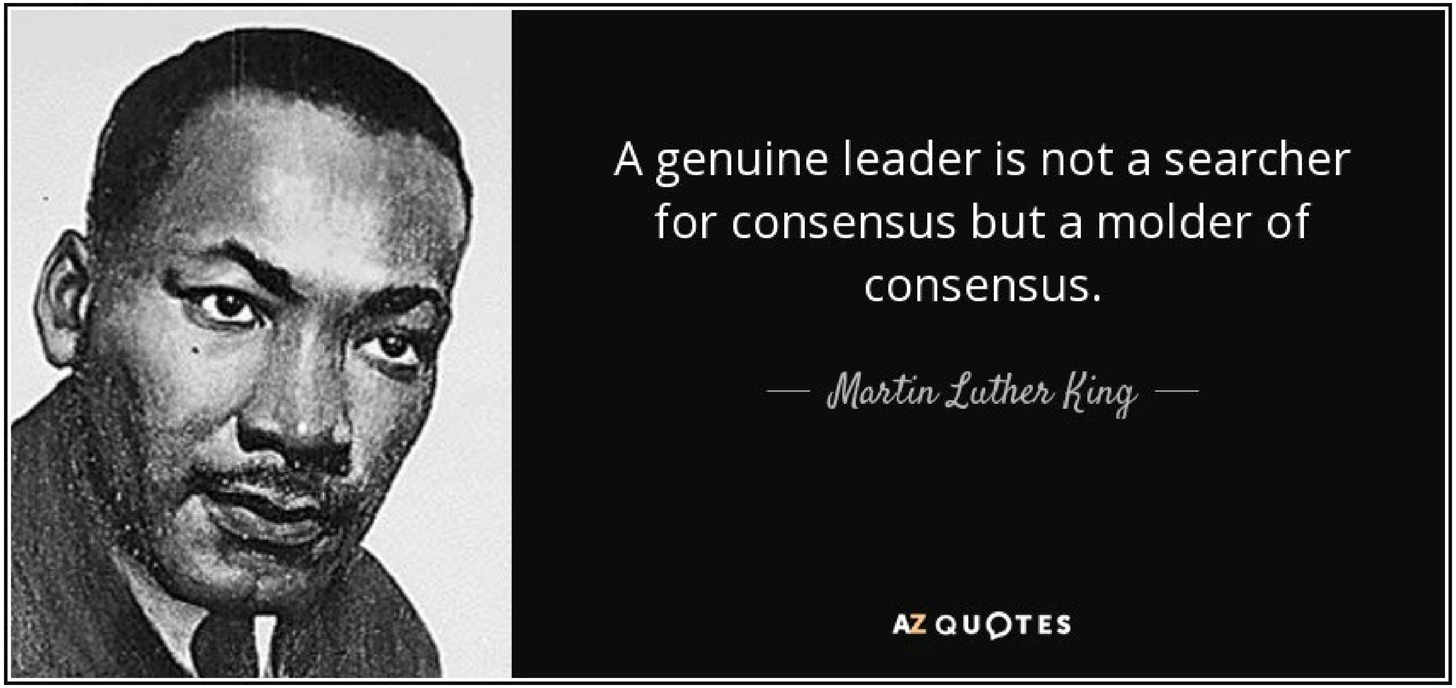
- Proskurnikov, A. , & Cao, M. Differential inequalities in multi-agent coordination and opinion dynamics modeling. Automatica, 2017, 85, 202–210.
- Proskurnikov, A. , & Cao, M. Modulus consensus in discrete-time signed net- works and properties of special recurrent inequalities. In Proceedings of IEEE CDC, 2017.

Constrained consensus and distributed algorithms to solve linear equations:

- Nedic, A., Ozdaglar, A., and Parrilo, P. Constrained consensus and optimization in multi-agent networks. IEEE Trans. Autom. Control, 55(4), 2010.
- S. Mou, J. Liu, and A. Morse, A distributed algorithm for solving a linear algebraic equation, IEEE Trans. Autom. Control, vol. 60, no. 11, pp. 2863–2878, 2015
- K. You, S. Song, and R. Tempo, A networked parallel algorithm for solving linear algebraic equations, in Proc. of IEEE Conf. Decision and Control, 2016

Just a few works on distributed optimization (huge topic):

- J. Tsitsiklis, D. Bertsekas, and M. Athans, “Distributed asynchronous deterministic and stochastic gradient optimization algorithms,” IEEE Transactions on Automatic Control, vol. 31, no. 9, 1986
- A. Nedic and A. Ozdaglar, Distributed subgradient methods for multi-agent optimization, IEEE Transactions on Automatic Control, vol. 54, no. 1, 2009
- P. Giselsson, M.D. Doan, T. Keviczky, B. De Schutter, and A. Rantzer, Accelerated gradient methods and dual decomposition in distributed model predictive control, Automatica, vol. 49, no. 3, pp. 829–833, Mar. 2013
    - N Bof, R Carli, G Notarstefano, L Schenato, D Varagnolo, Multi-Agent Newton-Raphson Optimizaton Over Lossy Networks, IEEE Transactions on Automatic Control (published online)



Истинный лидер не ищет консенсуса, а его формирует

(Мартин Лютер Кинг)

Спасибо за внимание!