DET
Department of Electronics and Telecommunications

POLITECNICO DI TORINO

Networks of Social Influence: Some Models of Opinion Dynamics

A teaser of a planned mini-course on models of social processes
(Uppsala University, Politecnico di Torino, Somewhere in Russia?)

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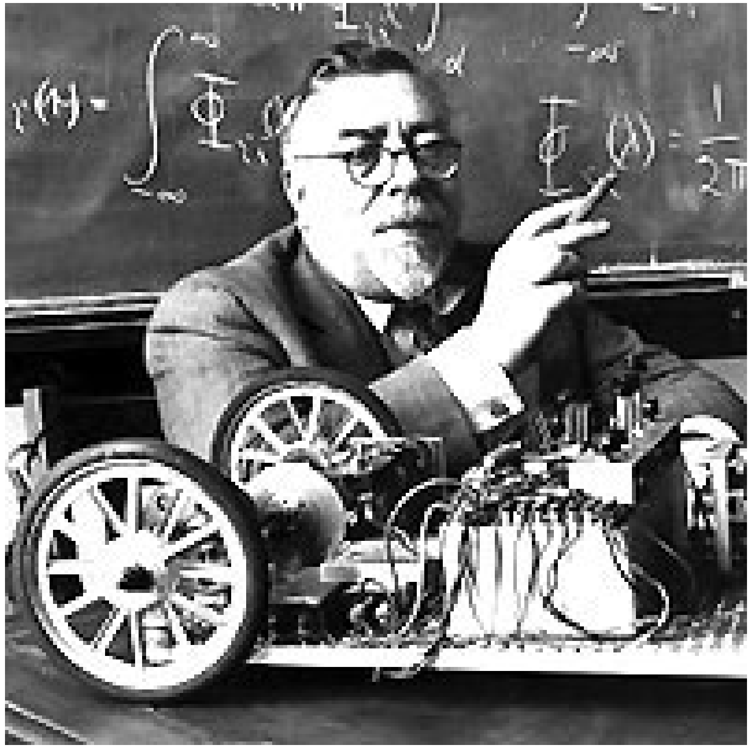
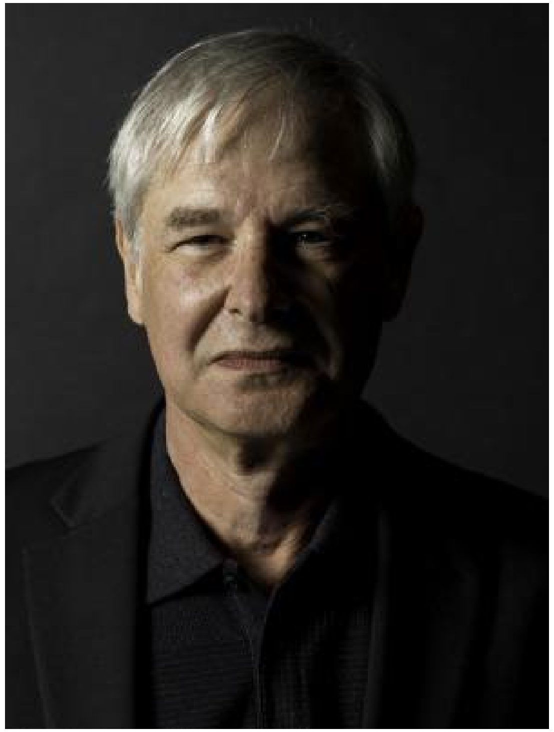
Cybernetics and mathematics in social sciences: from Wiener to our days



Society can only be understood through a study of the messages and the communication facilities which belong to it… **[N. Wiener**, The Human use of human beings: Cybernetics and society, 1st ed., 1950**]**.

Now we would say: study on how agents in a social network interact.



The coordination and **control** *of social systems is the foundational problem of sociology.* **[N.E. Friedkin**, IEEE Control Systems, 2015, 35(3)**]**.

One of the most renowned American sociologists; recipient of James S. Coleman Career Achievement Award (2020) from

- **Explosive growth of interest**

My 7 papers published in 2014-2018 are cited >800 times (WoS Core Collection)

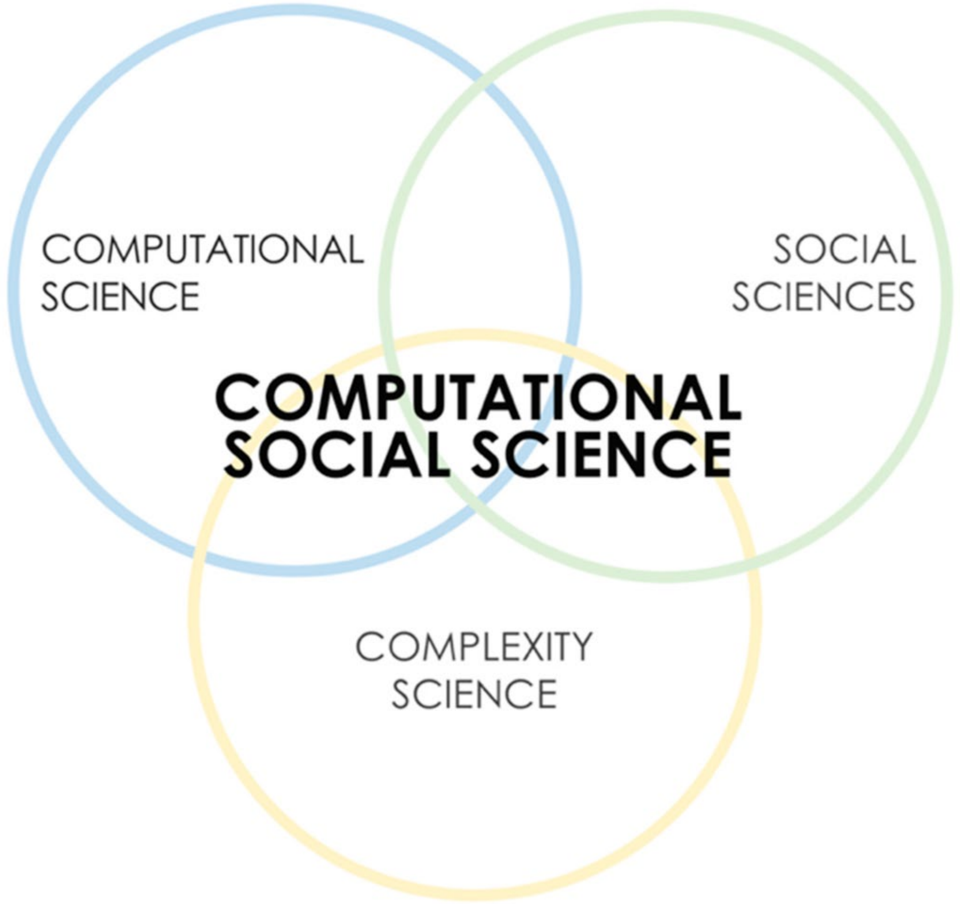
IEEExplore: **730 papers mention opinion dynamics (391 published since 2017)**

DBLP: **604 papers mention opinion dynamics (385 published since 2017)**

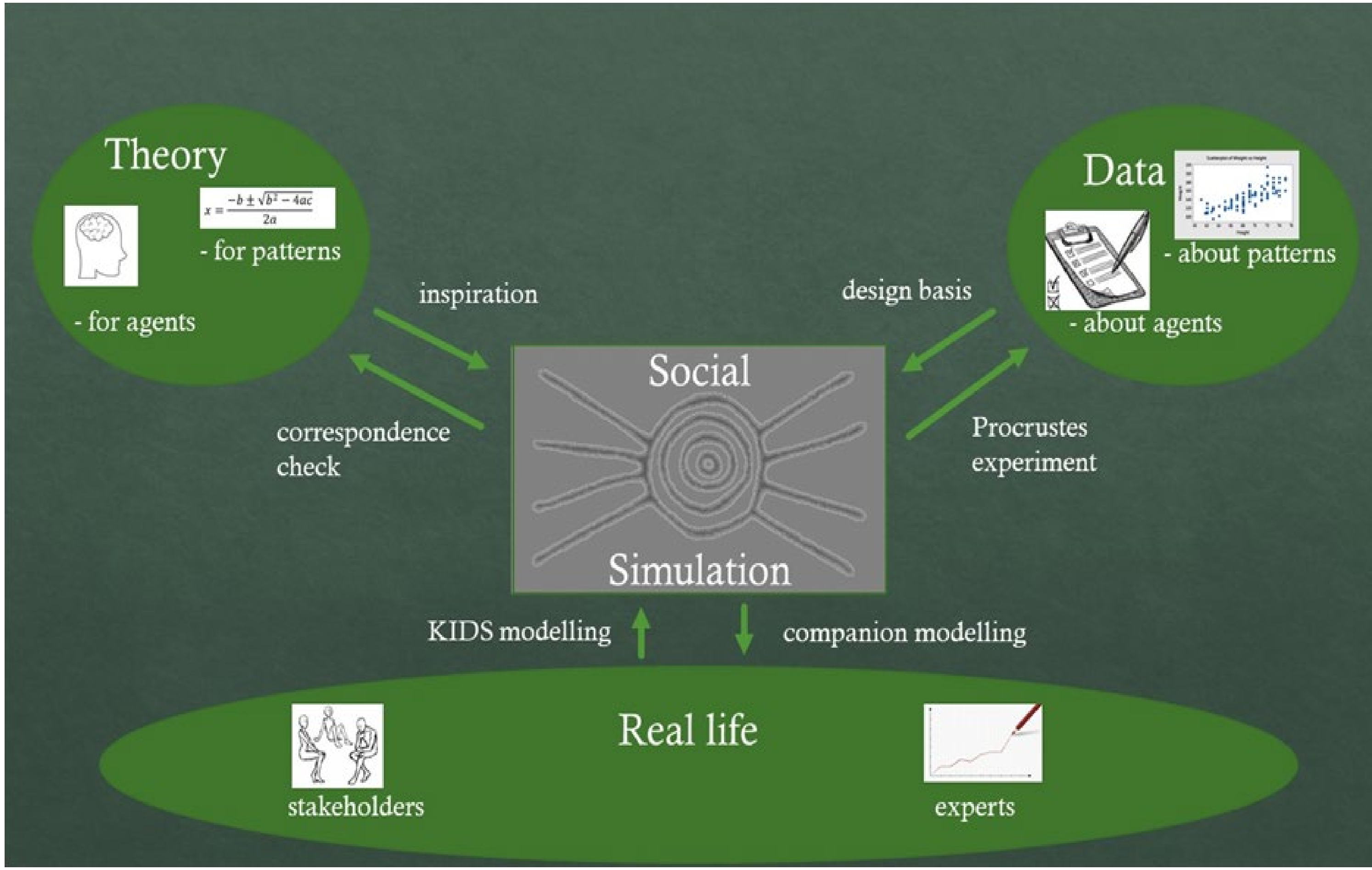
ACM DL: 282273 mentioning of opinion dynamics (85035 since 2017)

The goal of this talk is to explain what this buzz is about.

Why this summer school?



N. Lettieri, Computational Social Science, the Evolution of Policy Design and Rule Making in Smart Societies, Future Internet, **2016**, 8(2), 19



Hofstede, G.J. (2018) Social Simulation as a meeting place: report of SSC 2018 Stockholm. *Review of Artificial Societies and Social Simulation*

We witness the birth of new interdisciplinary sciences!

CUI PRODEST? What is the benefit of this merge?

Profit for socio-economic sciences, business, marketing etc.:

computational models able to explain (predict?) behavior of rational/irrational individuals and thus all human-centered and human-controller processes (crowds, markets etc.)

Profit for mathematics (control, computing, data science etc.):

Animals --> bio-mimetic algorithms (bees, ants etc.)
Human decision making --> **socio-mimetic** algorithms

Society = Large-scale and complex system exhibiting very rich behaviours. Inspires new mathematical tools (for instance, graph theory!)

Plan of the talk

1. **Social influence. Social graphs. “Opinions”.**

2. **Iterative averaging: the mature models looking too simplistic…**
   - Convergence and consensus of opinions
   - Social power
   - From consensus to disagreement. Abelson’s puzzle
   - Stubborn individuals

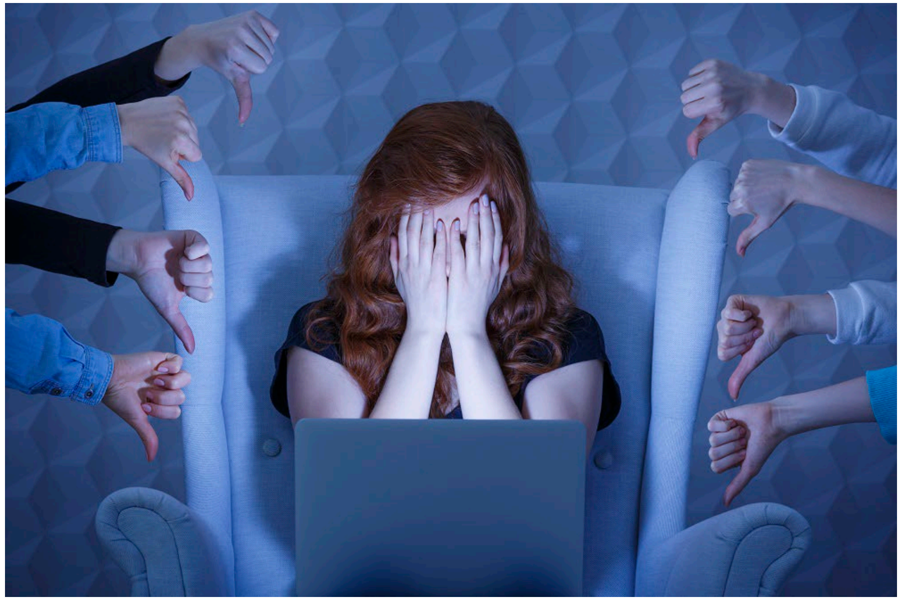
3. **Some experiments. Is it really that simple?**
   - Is convexity assumption realistic?
   - Can we somehow validate the model?

Literature.

4. **Advanced models, new directions and open problems.**

1. Social influence. Social graphs. “Opinions”.

We influence the others and are influenced by them



Social influence is any change in an individual's thoughts, feelings, or behaviours caused by other people, who may be actually present or whose presence is imagined, expected, or only implied.

[APA (American Physychological Association) Dictionary on Psychology]

Social and behavioral sciences distinguish many types of social influence (SI):
conformity, reactance, obedience, compliance, peer pressure, leadership, persuasion…

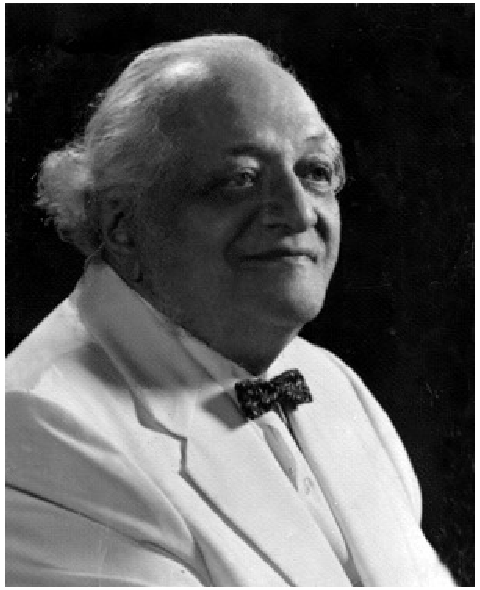
It is very difficult to find the structure of influence relations (ties): who influences whom.

Which social ties are most important?

Which individuals are most influential?

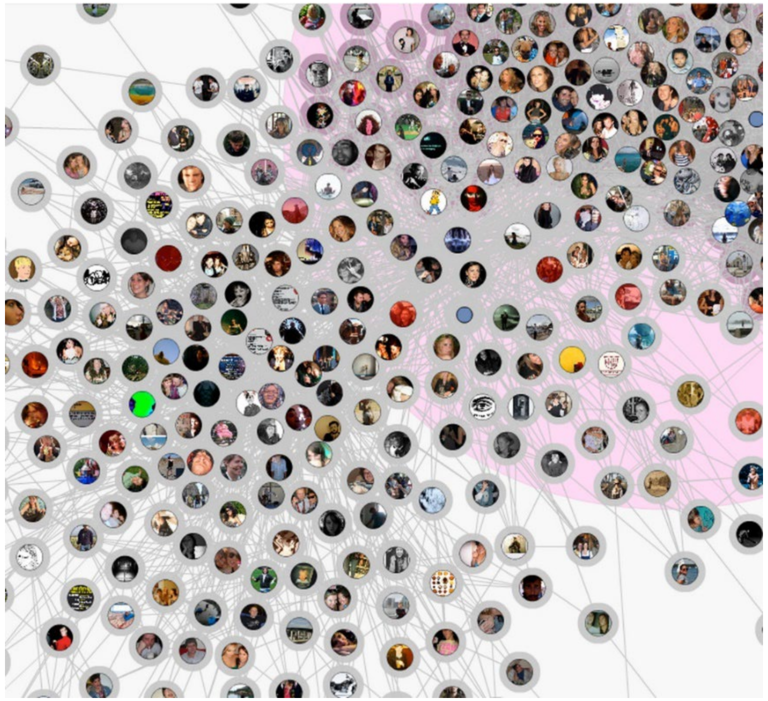
Can we describe social influence mathematically?

Graph theory: a key tool in social groups modeling, inspired by sociology!



Jacob Levy Moreno
(1899-1974)
American psychiatrist, jointly with Helen Jennings pioneered Social Network Analysis (now part of **network science**).

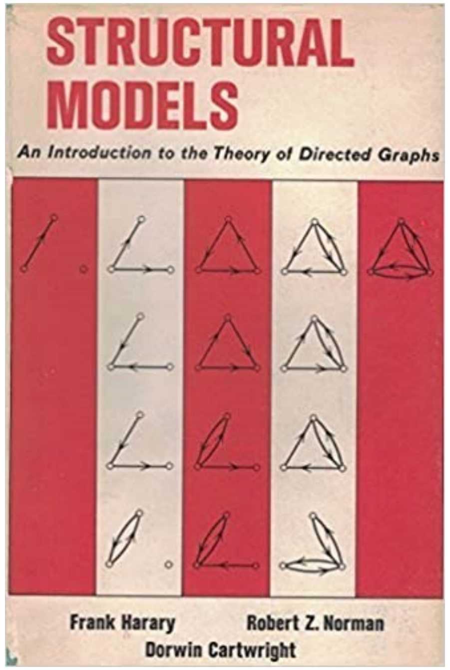
Method of sociograms (graphs of social relations).



- **Nodes** (vertices) = individuals or organizations;
- **Arcs** (edges, links) = social relations, e.g., influence between people
- Generally, graphs are **directed**

(A influences B yet is not influenced by them, or the influences between A and B are unequal;

- Arcs may also have **weights** (strength of the tie);



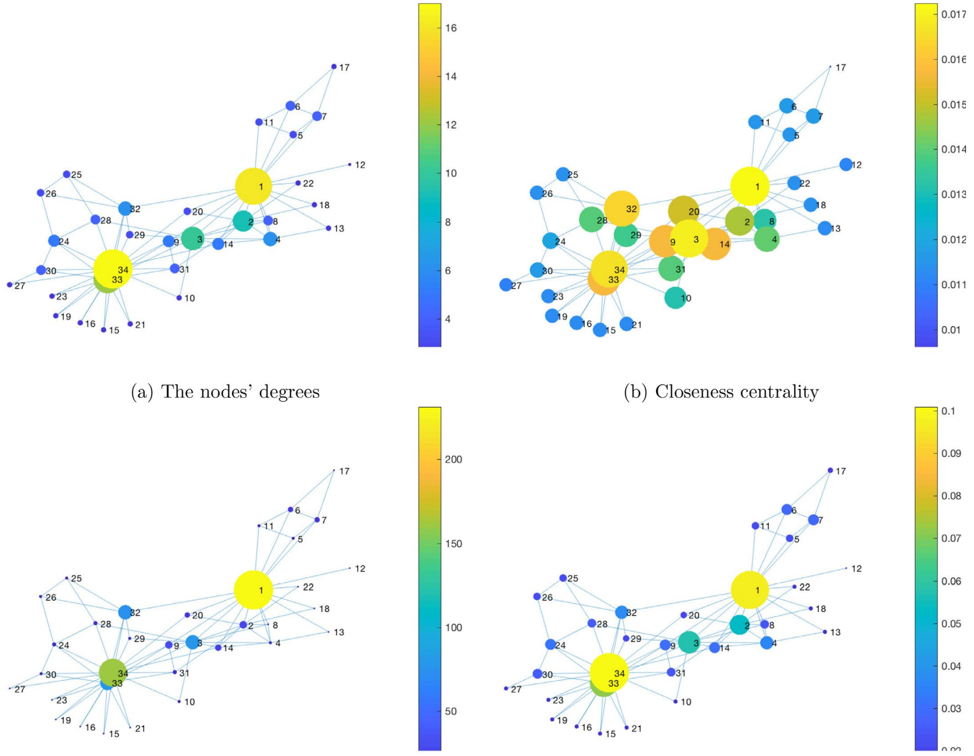
Sociology gave a strong impact to the development of **directed graph theory** (F. Harary et al.) and multi-agent systems theory.

This book (appeared in 1965!) contains a section on **consensus in an iterative averaging procedure**, or the French model of opinion formation (*hidden very well!*)

What is the social influence from the mathematical viewpoint (1)

Social influence from network science viewpoint: centrality of nodes and arcs.

- Various **centrality measures** on the nodes (in- or out- degree, closeness, betweenness, PageRank etc.).
- Measures (weights) on the edges (**edge centrality** measures). Inspired by the theory of **strong** and **weak** ties (Granovetter). Strong ties build dense communities, weak ties serve as “bridges” different communities.
- Related to resilience of a general complex network. Important concepts in networks science.

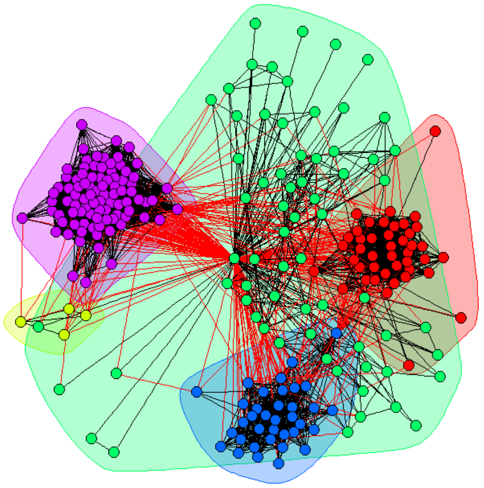


(a) The nodes’ degrees

(b) Closeness centrality

(c) Betweenness centrality

(d) PageRank centrality



Freeman (1979) *Centrality in social networks conceptual clarification*, Social networks 1(3): 215–239

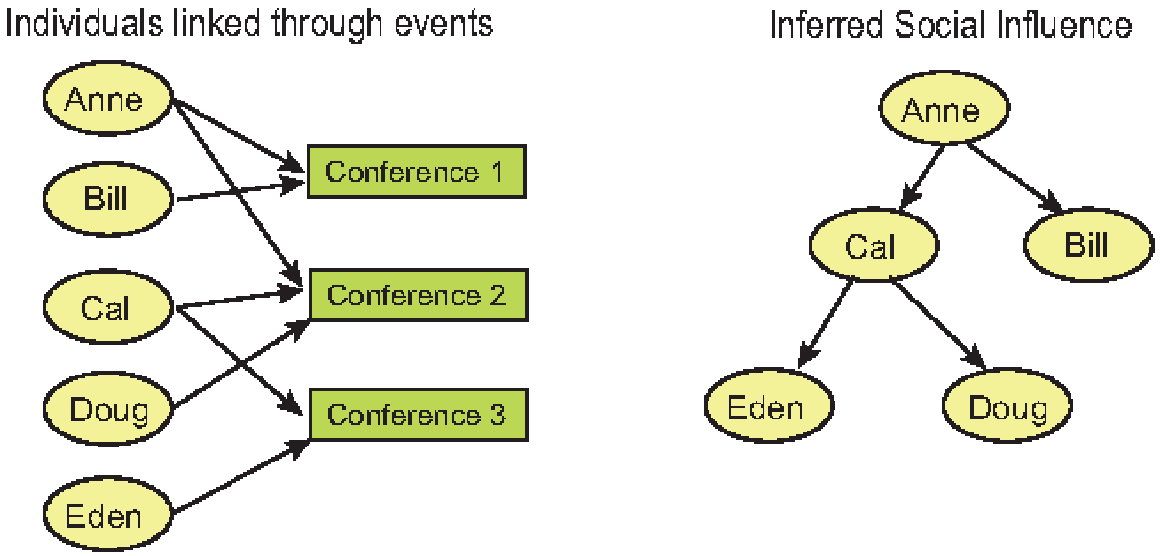
Proskurnikov et al. (2018), *Dynamics and structure of social networks from a systems and control viewpoint: A survey of Roberto Tempo’s contributions*, Online Social Networks and Media, 7, 45-59

Sun and Tang (2011). *A survey of models and algorithms for social influence analysis*. In Social Network Data Analytics, pp. 177–214.

What is the social influence from the mathematical viewpoint (2)

Social influence from data science viewpoint: the graph is inferred from data

- Individuals are associated to some data (time series etc.);
- Social influence = statistical **dependence or correlation** in data;
- Main tools are probabilistic graphical models (Gaussian graphical model, Markov random fields etc.);

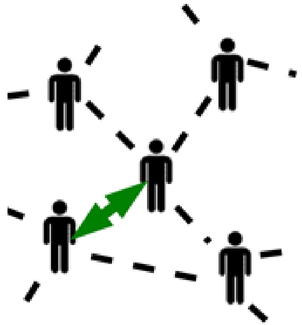


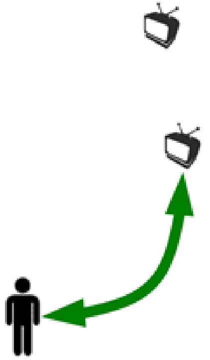
Farasat et al. (2015) *Probabilistic Graphical Models in Modern Social Network Analysis***.** Social Network Analysis and Mining, 5(1), p.62.

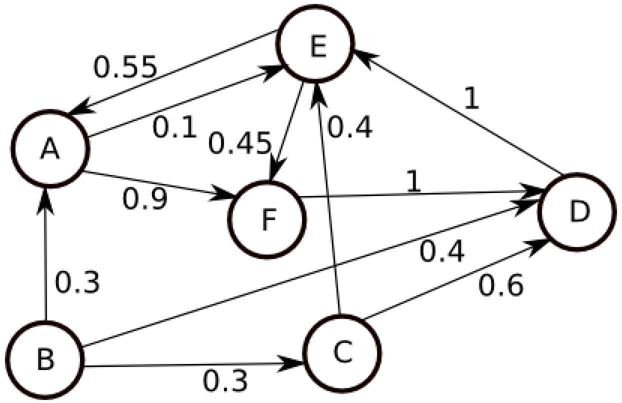
What is the social influence from the mathematical viewpoint (3)

Social influence as a dynamical system: how control comes into play!

- Social influence is a dynamical **process** unfolding over a graph of social relations.
- In this talk, we mainly consider models portraying dynamics of **"opinions"**.
- "Opinions" are some numerical characteristics of interest (we also speak about some data);
- Social actors interact, their "opinions" change due to social influence (of other individuals); external influence is also possible (media, global online social networks, etc.);
- Strength of social influence is a parameter of the dynamical system (coupling between agents) – naturally represented as a **weight** of the **directed** arc. Dynamics is not determined by the topology!
- Opinion evolution is different from processes over graphs studied in network science literature, although some processes (spread of radical ideas, innovations etc.) can be represented by epidemiological models.
- The oldest models of opinion formation have been recently **rediscovered** as "consensus algorithms".







Friedkin and Johnsen (2011) *Social Influence Network Theory*. Cambridge Univ. Press
Quattrociocchi et al. (2014) *Opinion dynamics on interacting networks: media competition and social influence*. Sci Rep **4,** 4938

Do we really study the evolution of humans' OPINIONS?

The term "opinion dynamics" is convenient but unfortunate. Interpersonal influences alter individuals' cognitive orientations to objects… **[N.E. Friedkin**, IEEE Control Systems, 2015, 35(3)**]**.

Henceforth, by an "opinion" we understand some numerical scalar/multidimensional characteristics of an individual, evolving under social influence. It can be an opinion (subjective judgement, belief) or something else.

Some examples are:

1. evaluation of some product: from -1 (strong dislike) to +1 (strong like);
2. certainty of belief that some statement is true: from 0 (no belief) to 1 (strong belief);
3. decision on the resource distribution between several entities: how to distribute budget of a research foundation between natural, social and engineering sciences?
4. level of "innovativeness" - willingness to adopt some new innovations – from laggard (0) to innovator (1);

1-2,4: scalar opinions, vary in some interval or a discrete set;

3: vector (multidimensional) opinions, vary in the unit simplex.

- Zhao et al. (2016), Competitiveness Maximization on Complex Networks, IEEE Trans. Systems, Man, Cybernetics, v.48
- Friedkin et al. (2016), Network science on belief system dynamics under logic constraints, Science, v. 354
- Friedkin et al. (2019), Mathematical Structures in Group Decision-Making on Resource Allocation Distributions, Sci. Rep., 9:1377
- E.M. Rogers (1962), Diffusions of innovations. New York: Free Press of Glencoe.

Control theory mainly deals with microscopic (agent-based) models with continuous opinions

- The equations describe how each specific individual changes their opinion;
- Opinions are represented by real numbers/vectors, which usually vary in an interval or a convex polytope (as on the previous slide);
- We consider only models based on the principle of **iterative averaging**. May look too simplistic, but the experiments shows that this principle is more than just a conjecture.
- **Final goal: capability of prediction/control of opinions, identification of "opinion leaders".**

Beyond the scope of this talk are:

- Models with discrete opinions that vary in a finite or a countable set (the vote on a presidential election, a student's grade on the exam etc.);

  Typically, examined by various tools from probability theory and/or finite automata.

- Macroscopic (statistical) models describing global group-level characteristics, e.g., the opinion probability density

  Lead to PDE or integrodifferential equations (Fokker-Planck etc.)

  Can be used to approximate microscopic models as the number of agents becomes large.

2. Iterative averaging: the mature models looking too simplistic…
(French, DeGroot, Abelson, Friedkin and Johnsen)

Some auxiliary notation

n **individuals (social actors, agents)**

opinion **of the i-th agent is a number or row vector**

$$\mathbf{x}_i = (x_{i1}, \dots, x_{im}) \in \mathbb{R}^{1\times m}, \quad m \geq 1.$$

stacking ***n*** **opinions one on top of another, one obtains the opinion matrix of the group**

$$\mathbf{X} = \begin{pmatrix} x_{11} & \dots & x_{1m} \\ \vdots & & \vdots \\ x_{n1} & & x_{nm} \end{pmatrix} = \begin{pmatrix} \mathbf{x}_1 \\ \hline \vdots \\ \hline \mathbf{x}_n \end{pmatrix} \in \mathbb{R}^{n\times m}.$$

Column vector of ones $\mathbb{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n.$

DeGroot and Abelson dynamics

$$\mathbf{A} = (a_{ij}),\quad a_{ij} \geq 0.$$

Along with the opinion matrix, the models involve a matrix of influence weights

Discrete-time averaging mechanism

$$\sum_{j=1}^{n} a_{ij} = 1\,\forall i.$$

$$\mathbf{x}_i(k+1) = \sum_{j=1}^{n} a_{ij}\mathbf{x}_j(k),$$

$$i = 1,\ldots,n, \quad k = 0,1,\ldots$$

$$\mathbf{X}(k+1) = \mathbf{A}\mathbf{X}(k).$$

DeGroot models arise as a result of Abelson's model discretization

$$\mathbf{X}((k+1)\delta) = \exp(-\delta\mathcal{L}(\mathbf{A}))\mathbf{X}(k\delta).$$

French (1956), Harary (1959, 1965), DeGroot (1974)

Continuous-time Laplacian dynamics

$$a_{ij} \geq 0, \quad a_{ii} = 0.$$

$$\dot{\mathbf{x}}_i(t) = \sum_{j \neq i} a_{ij}(\mathbf{x}_j(t) - \mathbf{x}_i(t)),$$

$$i = 1,\ldots,n, \quad t \geq 0.$$

$$\dot{\mathbf{X}}(t) = -\mathcal{L}(\mathbf{A})\mathbf{X}(t).$$

The Laplacian is defined as follows:

$$(\mathcal{L}(\mathbf{A}))_{ii} := \sum_{j} a_{ij},$$

$$(\mathcal{L}(\mathbf{A}))_{ij} := -a_{ij}\,\forall j \neq i.$$

Abelson (1964)

DeGroot and Abelson models: consensus and convergence problems

Individuals seek for *unanimity* with people whom they believe.

They wish to bring their own opinions closer to the influencers' opinions $a_{ij} \geq 0.$

We can expect that eventual consensus (unanimity) of opinions is established

$$\mathbf{x}_i \xrightarrow[t\to\infty]{k\to\infty} \mathbf{x}_i^\infty = \mathbf{x}_1^\infty = \ldots = \mathbf{x}_n^\infty \qquad \forall \mathbf{X}(0).$$

In fact, sometimes the opinions fail even to converge, exhibiting periodic oscillations

$$\mathbf{x}_1(k+1) = \mathbf{x}_2(k), \quad \mathbf{x}_2(k+1) = \mathbf{x}_1(k),$$
$$\mathbf{x}_1(0) \neq \mathbf{x}_2(0).$$

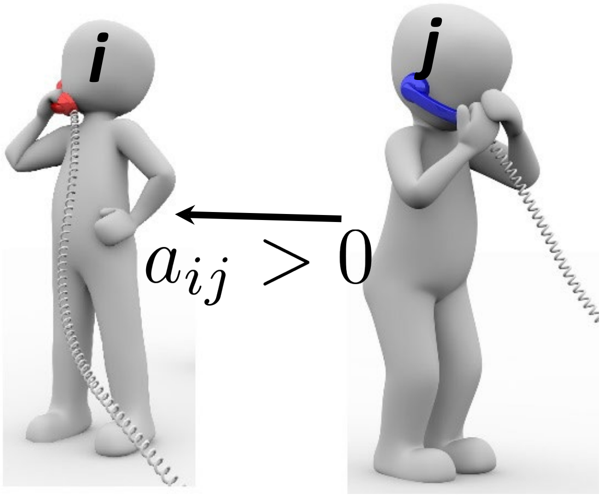
Does the group elaborate final opinions $\mathbf{x}_i^\infty$ as a result of discussion (convergence)?

Do these final opinions coincide (consensus)?

Nowadays, the answer is well known; conveniently given on graph-theoretic language.

DeGroot and Abelson dynamics: social influence graph

agent i assigns a positive influence weight to agent j



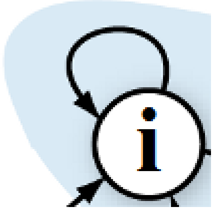
⬍

the i'th opinion is directly influenced by opinion of j at any period k/time t.

$$\mathbf{x}_i(k+1) = \sum_j a_{ij}\mathbf{x}_j(k)$$

$$\dot{\mathbf{x}}_i(t) = \sum_j a_{ij}[\mathbf{x}_j(t) - \mathbf{x}_i(t)]$$

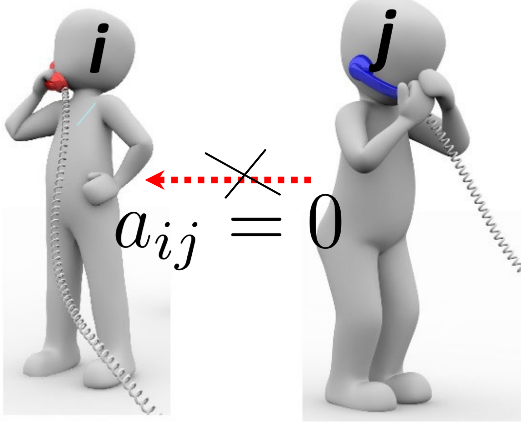
Agent i is self-confident if it assigns a positive weight to themselves (self-arc in the graph).

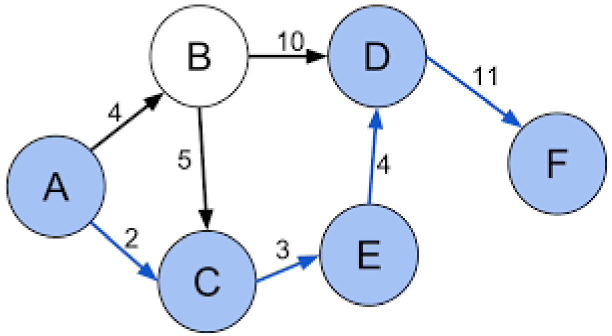


agent i assigns zero weight to agent j

⬍

opinion of i is not influenced by opinion of j directly.





However, indirect influence can exist through the opinions of other individuals: directed paths in the social influence graph.

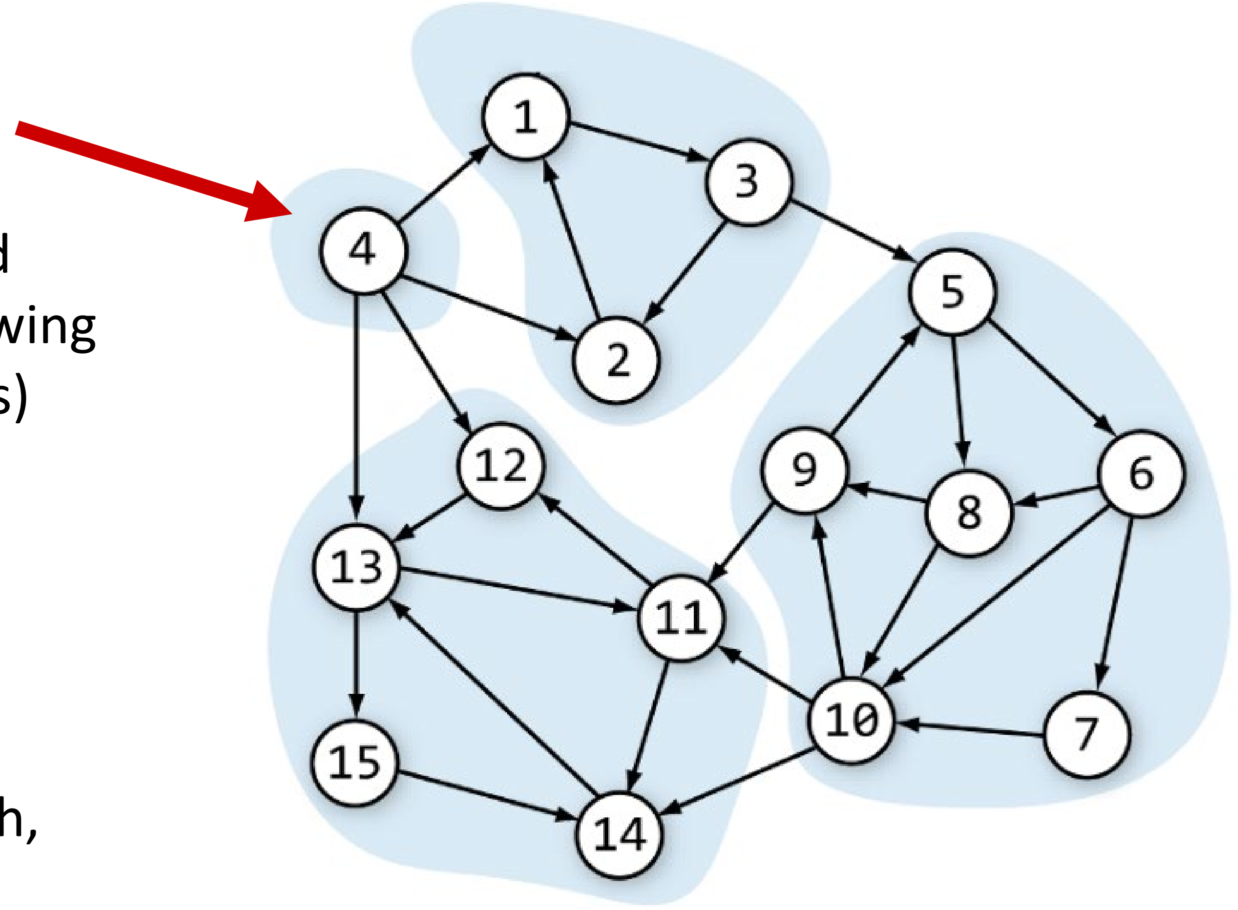
A concept of rooted graph (quasi-strongly connected, with directed spanning tree…)

- A graph is rooted if it contains a root node

- From a root, all other nodes can be reached (equivalently, there exists a **spanning tree** growing out of this node and containing all other nodes)

- The individual at such a node influences (directly on indirectly) all other individuals

- A root need not be unique, the graph may have many roots. In a strongly connected graph, all nodes are roots.



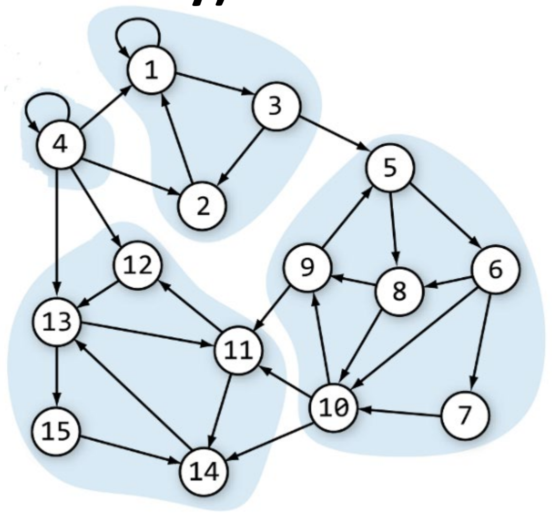
DeGroot and Abelson models: Consensus and Convergence Criteria

$$\mathbf{X}(k+1) = \mathbf{A}\mathbf{X}(k) \qquad (\mathbf{A}\mathbf{1} = \mathbf{1}).$$

Consensus is established if and only if

$$1 = \lambda_1(\mathbf{A}) > |\lambda_j(\mathbf{A})| \, \forall j = 2, \ldots, n.$$

Aperiodicity. **Each agent is either self-confident or influenced by some self-confident agent (at least indirectly).** **Can be relaxed, not discarded.**



Opinions converge $\mathbf{X}(k) \xrightarrow[k\to\infty]{} \mathbf{X}^{\infty}.$

Consensus is reached iff the graph is rooted

$$\dot{\mathbf{X}}(t) = -\mathcal{L}(\mathbf{A})\mathbf{X}(t) \qquad (\mathcal{L}(\mathbf{A})\mathbf{1} = \mathbf{0}).$$

$$\lambda_j(\mathcal{L}(\mathbf{A})) \in \{0\} \cup \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$$

Consensus $\leftrightarrow$ 0 is a simple eigenvalue.

Consensus is reached if and only if the graph of social influence is rooted: someone influences all other people.

Even if consensus is not reached, the opinions always converge $\mathbf{X}(t) \xrightarrow[t\to\infty]{} \mathbf{X}^{\infty}.$

Proskurnikov, Tempo, A Tutorial on Modeling and Analysis of Dynamic Social Networks. Part I, vol. 43

DeGroot and Abelson models: Consensus Opinion and "Social Power"

$$\mathbf{X}(k+1) = \mathbf{A}\mathbf{X}(k) \qquad (\mathbf{A}\mathbf{1} = \mathbf{1}). \quad \dot{\mathbf{X}}(t) = -\mathcal{L}(\mathbf{A})\mathbf{X}(t) \qquad (\mathcal{L}(\mathbf{A})\mathbf{1} = \mathbf{0}).$$

Assume that the model establish consensus. What is the unanimous opinion?

$$\mathbf{A}^\top \boldsymbol{\zeta} = \boldsymbol{\zeta}, \ \sum_i \zeta_i = 1.$$

$$\mathbf{A}^k \xrightarrow[k\to\infty]{} \mathbf{1}\boldsymbol{\zeta}^\top = \begin{bmatrix} \boldsymbol{\zeta}^\top \\ \vdots \\ \boldsymbol{\zeta}^\top \end{bmatrix}.$$

A **is SIA (stochastic, indecomposable, aperiodic)**

$$\mathcal{L}(\mathbf{A})^\top \boldsymbol{\zeta} = 0, \ \sum_i \zeta_i = 1,$$

$$\exp(-t\mathcal{L}(\mathbf{A})) \xrightarrow[t\to\infty]{} \mathbf{1}\boldsymbol{\zeta}^\top.$$

$$\mathbf{x}_1^\infty = \ldots = \mathbf{x}_n^\infty = \boldsymbol{\zeta}^\top \mathbf{X}(0) = \sum_{i=1}^n \zeta_i \mathbf{x}_i(0), \qquad \zeta_i \geq 0, \sum_i \zeta_i = 1.$$

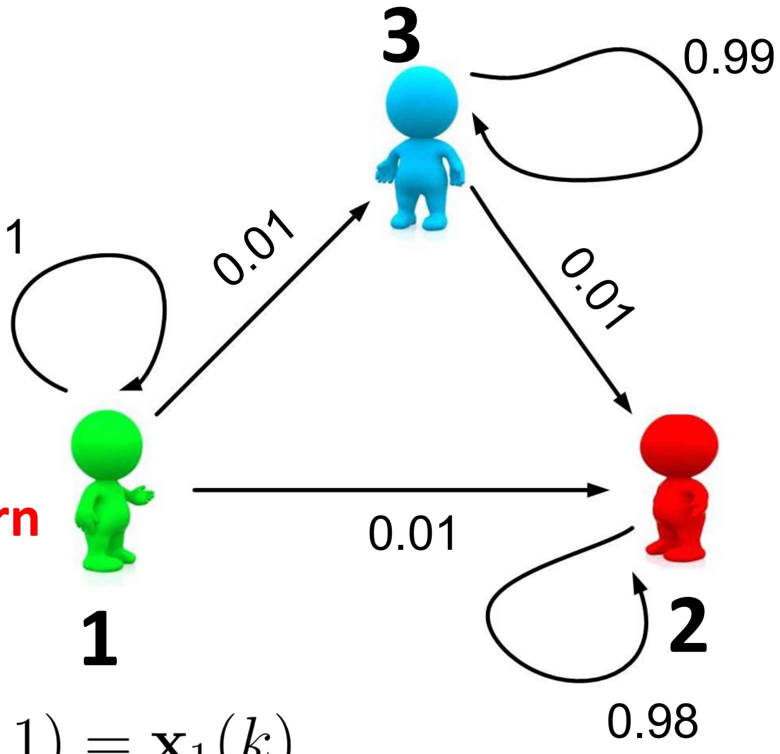
ζ_i **may be considered as a measure of social power: agent i's capability of controlling the ultimate opinion of the group. One of natural centrality measures (eigenvalue centrality)**

French (1956), Friedkin (1986)

DeGroot model: a toy example

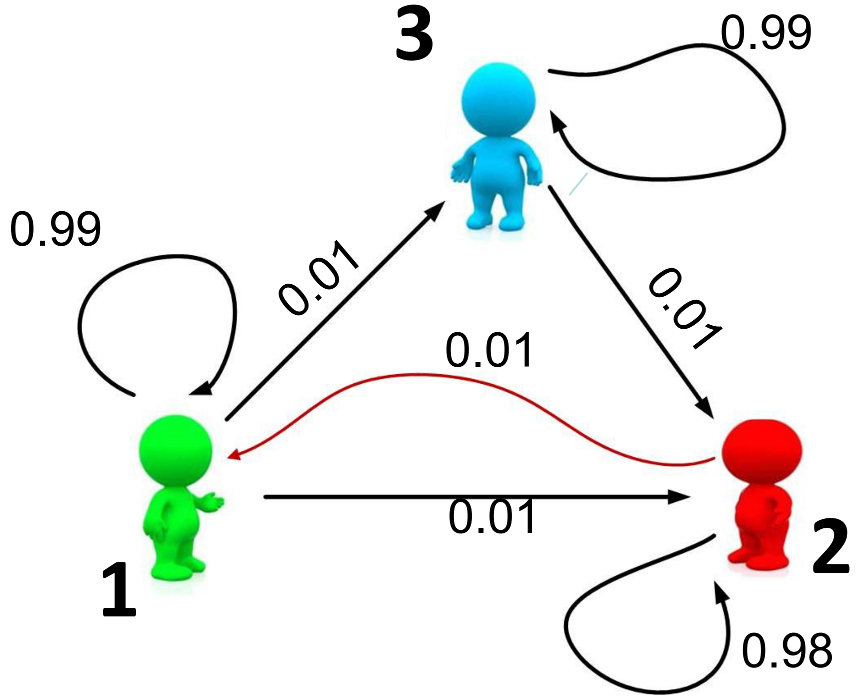
Distribution of social power has no relation to self-influence and is very sensitive to A.

Totally stubborn



$$\mathbf{x}_1(k+1) = \mathbf{x}_1(k),$$
$$\mathbf{x}_2(k+1) = 0.01\mathbf{x}_1(k) + 0.98\mathbf{x}_2(k) + 0.01\mathbf{x}_3(k)$$
$$\mathbf{x}_3(k+1) = 0.01\mathbf{x}_1(k) + 0.99\mathbf{x}_3(k)$$
$$\zeta = (1, 0, 0)^\top$$

"Totally stubborn" agent 1 is a dictator.



$$\mathbf{x}_1(k+1) = 0.99\mathbf{x}_1(k) + 0.01\mathbf{x}_2(k),$$
$$\mathbf{x}_2(k+1) = 0.01\mathbf{x}_1(k) + 0.98\mathbf{x}_2(k) + 0.01\mathbf{x}_3(k$$
$$\mathbf{x}_3(k+1) = 0.01\mathbf{x}_1(k) + 0.99\mathbf{x}_3(k)$$
$$\zeta = (0.5, 0.25, 0.25)^\top$$

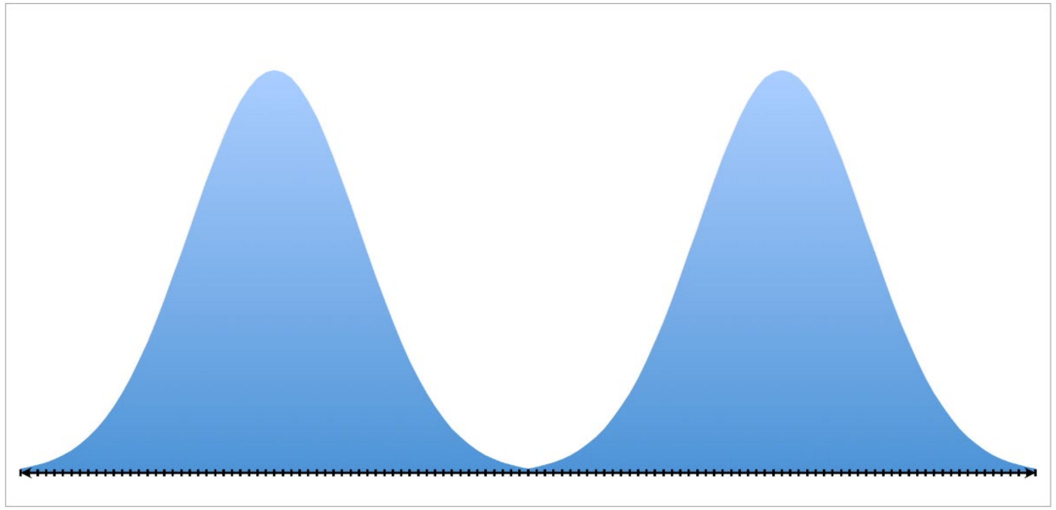
Agents 1 lost half of their power, agents 2 and 3 are equally powerful

Abelson's puzzle, or community cleavage problem:

How to explain disagreement under (and despite) social influence?

R.P. Abelson (1964), In Frederiksen and Gulliksen (Eds.), Contributions to Mathematical Psychology, NY: Holt, Rinehart and Winston.

"… we are naturally led to inquire what on earth one must assume in order to generate the bimodal outcome of community cleavage studies."



Abelson's suggestions:

- **Negative (repulsive) influence: beyond the scope of this lecture.**
- **Nonlinear influence mechanisms, naturally led to idea of bounded confidence.**

A simpler idea: let all the agents to be fully or partially stubborn (as in example).

Leads to Friedkin-Johnsen and Taylor models.

Friedkin-Johnsen and Taylor models

Characterized by three matrices:

$$\mathbf{A}, \quad \mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & 0 & & \lambda_n \end{bmatrix} \quad \mathbf{U} = \begin{pmatrix} \mathbf{u}_1 \\ \hline \vdots \\ \hline \mathbf{u}_n \end{pmatrix} \in \mathbb{R}^{n \times m}.$$

Stubbornness and persistent external influence:

$$\sum_{j=1}^{n} a_{ij} = 1,\, 0 \le \lambda_i \le 1 \;\; \forall i = 1, \ldots, n$$

$$\mathbf{x}_i(k+1) = \lambda_i \sum_{j=1}^{n} a_{ij}\mathbf{x}_j(k) + (1-\lambda_i)\mathbf{u}_i$$

$$\mathbf{u}_i = \mathbf{x}_i(0), \quad i = 1, \ldots, n.$$

λ_i **- susceptibility to social influence (0=total stubbornness, 1=French-DeGroot)**

$$\mathbf{X}(k+1) = \mathbf{\Lambda}\mathbf{A}\mathbf{X}(k) + (\mathbf{I}-\mathbf{\Lambda})\mathbf{U}.$$

Friedkin and Johnsen (1990, 1999, 2011)

$$a_{ij} \ge 0,\; \lambda_i \ge 0 \;\; \forall i, j.$$

$$\dot{\mathbf{x}}_i = \sum_j a_{ij}(\mathbf{x}_j - \mathbf{x}_i) + \lambda_i(\mathbf{u}_i - \mathbf{x}_i)$$

$$i = 1, \ldots, n.$$

$\mathbf{u}_i$ **- some external source of information (this interpretation is applicable also to the Friedkin-Johnsen model!)**

$$\dot{\mathbf{X}}(t) = -(\mathbf{\Lambda} + \mathcal{L}(\mathbf{A}))\mathbf{X}(t) + \mathbf{\Lambda}\mathbf{U}.$$

Taylor (1968)

Friedkin-Johnsen and Taylor models: stability and convergence

$$\mathbf{X}(k+1) = \mathbf{\Lambda}\mathbf{A}\mathbf{X}(k) + (\mathbf{I} - \mathbf{\Lambda})\mathbf{U}. \qquad \Bigg| \qquad \dot{\mathbf{X}}(t) = -(\mathbf{\Lambda} + \mathcal{L}(\mathbf{A}))\mathbf{X}(t) + \mathbf{\Lambda}\mathbf{U}.$$

In the generic situation, the linear system is exponentially stable (the respective matrix is Schur or Hurwitz). This holds, e.g., whenever the graph of social influence is strongly connected and at least one agent is stubborn/externally influenced, that is

$$\lambda_i < 1 \qquad \Bigg| \qquad \lambda_i > 0$$

$$\mathbf{X} \xrightarrow[t\to\infty]{k\to\infty} \mathbf{X}^\infty = \mathbf{V}\mathbf{U} \quad \forall \mathbf{X}(0).$$

$$\mathbf{V} = (\mathbf{I} - \mathbf{\Lambda}\mathbf{A})^{-1}(\mathbf{I} - \mathbf{\Lambda}). \qquad \Bigg| \qquad \mathbf{V} = (\mathbf{\Lambda} + \mathcal{L}(\mathbf{A}))^{-1}\mathbf{\Lambda}.$$

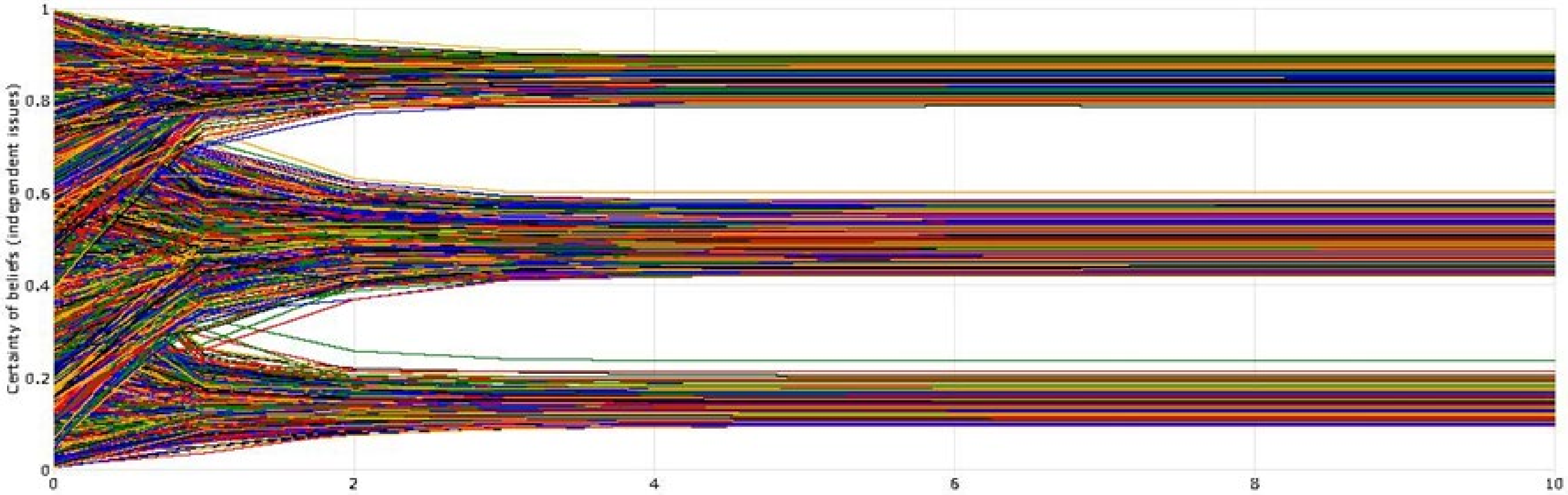
Matrix V is stochastic and characterizes the social power

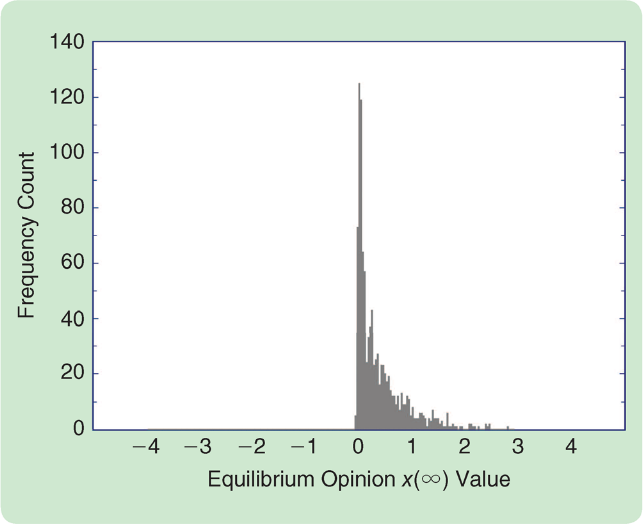
$$\mathbf{x}_i^\infty = \sum_j v_{ij}\mathbf{u}_j, \quad v_{ij} \geq 0, \sum_j v_{ij} = 1.$$

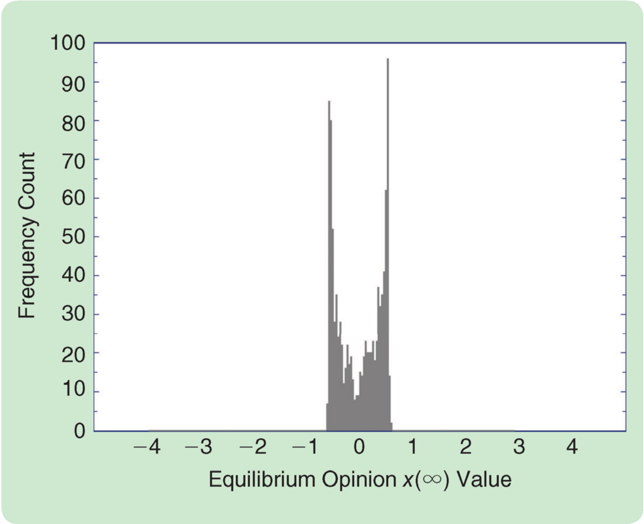
Proskurnikov, Tempo, A Tutorial on Modeling and Analysis of Dynamic Social Networks. Part I, Ann. Rev. Control, vol. 43

Stubbornness may lead to interesting configurations of final opinions

Final opinions are usually different, but (depending on parameters) can organize into clusters.







Friedkin et al., Science, v. 354, 2016; **Friedkin**, IEEE Control Systems, 2015, 35(3)

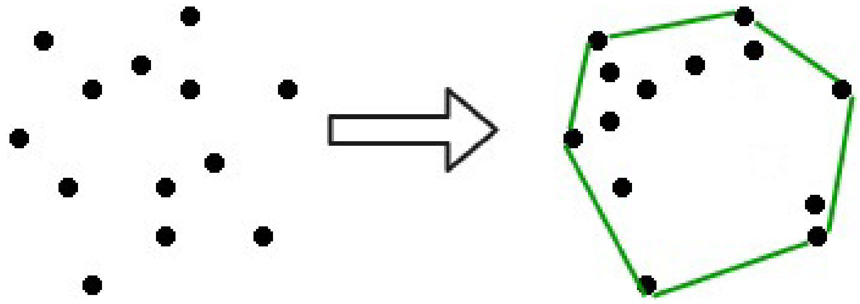
3. Some experiments. Is it that simple?

Convex (averaging) mechanism of opinion update

The important feature of all models we have considered: convex constraint preservation

If the initial opinions of the agents are contained by a convex set, their opinions will never leave this set.

Equivalently: the opinions (as points) are confined to the convex hull of the initial opinions



Implicit structure of informal decision space: the convex hull spanned by the initial opinions

Do opinion dynamics in a social group enjoy this property?

A number of experiments documented in the literature

- Sentiments about people, emotional appraisals: **Friedkin and Johnsen**, in Advances in Group Processes, vol. 20, 1999
- Discussions in book clubs: **Childress and Friedkin**, American Soc. Rev., 2012, 77:1
- Intellective tasks (solving mathematical or logical problems): **Friedkin and Bullo**, PNAS, 2017, 114:43
- Decisions on Resource distribution: **Friedkin, Proskurnikov, Mei and Bullo**, Scientific Reports, 2019, 9:1377. **Multidimensional opinions!**
- Threat appraisals (how dangerous some object is): **Friedkin, Proskurnikov and Bullo**, Social Networks, 2021, 65

- Political views of online social network VK users (determined by their subscriptions to politically-related groups (VK=Vkontakte = Russian acronym for "in contact"): **Kozitsin**, Journal of Mathematical Sociology, 2020, DOI; **Kozitsin et al.**, Mathematical Models and Computer Simulations, 12(2), 2020.

- ➢ There is a tendency to keep the opinion unchanged or move it towards the average of peers' opinions, staying thus within the hull of initial opinions.
- ➢ However, there are deviations from this rule ("skipping") even for the scalar opinion case.
- ➢ *Despite the fraction of such instances being relatively small … this phenomenon is not accidental* (Kozitsin, 2020)

Example with multidimensional opinions (UCSB, 2017)

Conditions: 80 individuals, organized into small groups of 3-5 people, each group sits at a table (face-to-face interactions). Participants give a written consent to participate in the experiment. The discussion is opened with instruction: "**Discuss the problem with the other members of your group. The conversation that you will have may or may not lead you to alter your initial answer, and you may not come to an agreement as a group**."

Your group is planning to sail from California to Hawaii. Storage space is limited, so careful planning is required. All food brought aboard the boat must be essential to maintain the health and morale of the group, and acceptable to everyone. We need foods that provide Carbohydrates, Protein, and Fat all which contribute to our daily caloric intake, which must be at least 2,000 calories. Our body needs carbohydrates, protein and fats to fuel its physical activity and metabolic needs. The Food and Nutrition Board of the Institute of Medicine (IOM) provides a range of your total caloric intake that is acceptable for each nutrient. The IOM Acceptable Macronutrient Distribution Range (AMDR) is: ***45–65%*** *Carbohydrates,* ***10–35%*** *Protein, and* ***20–35%*** *Fat. Any blend (which must sum to 100%) that falls within the IOM's AMDR will ensure adequate nutrition. What is your preferred blend?*

Multidimensional opinion: sum of three components is 100%, each component is within a given range.

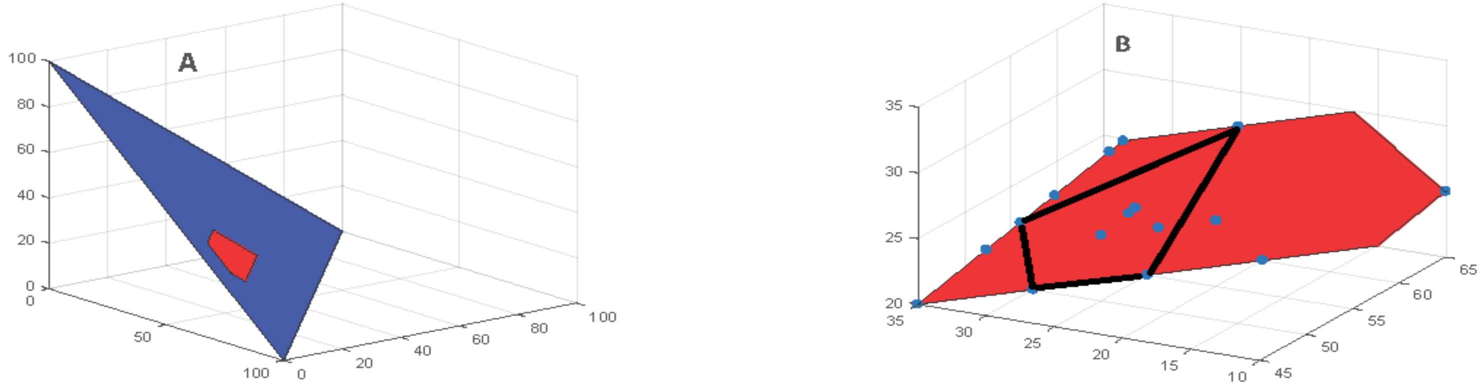
Friedkin, Proskurnikov, Mei and Bullo, *Scientific Reports*, 2019, 9:1377

The idea of experiment is borrowed from optimal diet problems in **Stigler, G. J**. *The cost of subsistence*, J. Farm. Econ., 1945

Example with multidimensional opinions (UCSB, 2017)

Conditions: 80 individuals, organized into small groups of 3-5 people, each group sits at a table (face-to-face interactions). How many groups satisfy the following property: the final opinions fall into the convex hull, spanned by the initial opinions.

Note: individuals had no computers, so they could not visualize the convex hull as we did – some automatic mechanism provides the "non-expansion" of the convex hull.



17 of 23 groups reach consensus, 72 of 80 (90%) individuals' final positions are found in the convex hulls of the respective initial opinions

Similar results on 4-dimensional opinions vectors (**impossible to visualize!!!)**

Friedkin, Proskurnikov, Mei and Bullo, *Scientific Reports*, 2019, 9:1377

Literature.

- **C. Ravazzi et al.,** *Learning hidden influences in large-scale dynamical social networks: A data-driven sparsity-based approach,* IEEE Control Systems 41 (5), 61-103, 2021
- **A. Proskurnikov and R. Tempo,** *A Tutorial on Modeling and Analysis of Dynamic Social Networks*. *Part I*, Ann. Rev. Control, vol. 43, 2017; *Part II*, Ann. Rev. Control, vol. 45.
- **N. Friedkin et al.,** *Network science on belief system dynamics under logic constraints*, Science, v. 354, 2016.
- **N. Friedkin et al.,** *Mathematical Structures in Group Decision-Making on Resource Allocation Distributions*, Scientific Reports, 9:1377, 2019
- **N. Friedkin et al.,** *Group dynamics on multidimensional object threat appraisals*, Social Networks 65, 157-167, 2021

- **N. Friedkin,** *The problem of social control and coordination of complex systems in sociology: A community cleavage problem*, IEEE Control Systems Magazine, 35(3), 2015.
- **J. French, Jr.** *A formal theory of social power.* Physchological Review, 63, 1956
- **F. Harary** *A criterion for unanimity in French's theory of social power*. In D. Cartwright (Ed.), Studies in social power Ann Arbor, MI: Univ. of Michigan Press, 1956
- **R. Abelson.** *Mathematical models of the distribution of attitudes under controversy*. In N. Frederiksen, & H. Gulliksen (Eds.), *Contributions to mathematical psychology*. New York: Holt, Rinehart & Winston, Inc., 1964
- **DeGroot, M.** *Reaching a consensus*. Journal of the American Statistical Association, 69, 1974
- **Taylor, M.** *Towards a mathematical theory of influence and attitude change*. Human Relations. 21(2), 1968
- **Dong et al.,** *Dynamics of linguistic opinion formation in bounded confidence model,* Information Fusion, v.32, 2016
- **Hegselmann, R. , & Krause, U.** *Opinion dynamics and bounded confidence models, analysis, and simulation*. Journal of Artifical Societies and Social Simulation (JASSS), 5 (3), 2002.
- **Kozitsin, I**. *Formal models of opinion formation and their application to real data: evidence from online social networks*. Journal of Mathematical Sociology, 2020 (published online)

4. Advanced models, new directions and open problems.

Other culprits of disagreement: evolution of social ties.

- **We accept similar opinions more readily than dissimilar ones.**
- **We establish strong relations with people with similar interests, behaviors and cultural traits.**

Biased assimilation, homophily, social selection.

- These effects are captured by **nonlinear** models with **opinion-dependent** influence weights.
- They can explain "partial consensus": splitting of opinions into **few** clusters (exceptional case in the F.-J. and Taylor's models).

Seminal idea of Abelson: opinion-dependent weights

$$\mathbf{x}_i(k+1) = \sum_{j=1}^{n} a_{ij}\mathbf{x}_j(k),$$
$$i = 1,\ldots,n, \quad k = 0,1,\ldots$$

- **each agent has a confidence set (open ball);**
- **all opinions falling into this set get equal weights;**

$$\mathcal{N}_i(\mathbf{X}) = \{j : |\mathbf{x}_j - \mathbf{x}_i| < R_i\}$$
$$a_{ij}(\mathbf{x}_i,\mathbf{x}_j) = \frac{1}{|\mathcal{N}_i(\mathbf{X})|}\,\forall j \in \mathcal{N}_i(\mathbf{X})$$

- **All other opinions are given zero weight**

$$a_{ij}(\mathbf{x}_i,\mathbf{x}_j) = 0\,\forall j \notin \mathcal{N}_i(\mathbf{X})$$

- **Terminates in finite time if** $R_1 = R_2 = \ldots = R$
- **Otherwise, even convergence is an open problem**

Krause (2000), Hegselmann, Krause (2002): bounded confidence

$$\dot{\mathbf{x}}_i(t) = \sum_{j\neq i} a_{ij}(\mathbf{x}_j(t) - \mathbf{x}_i(t)),$$
$$i = 1,\ldots,n, \quad t \geq 0.$$
$$a_{ij}(\mathbf{x}_i,\mathbf{x}_j) = e^{-|\mathbf{x}_i - \mathbf{x}_j|}$$

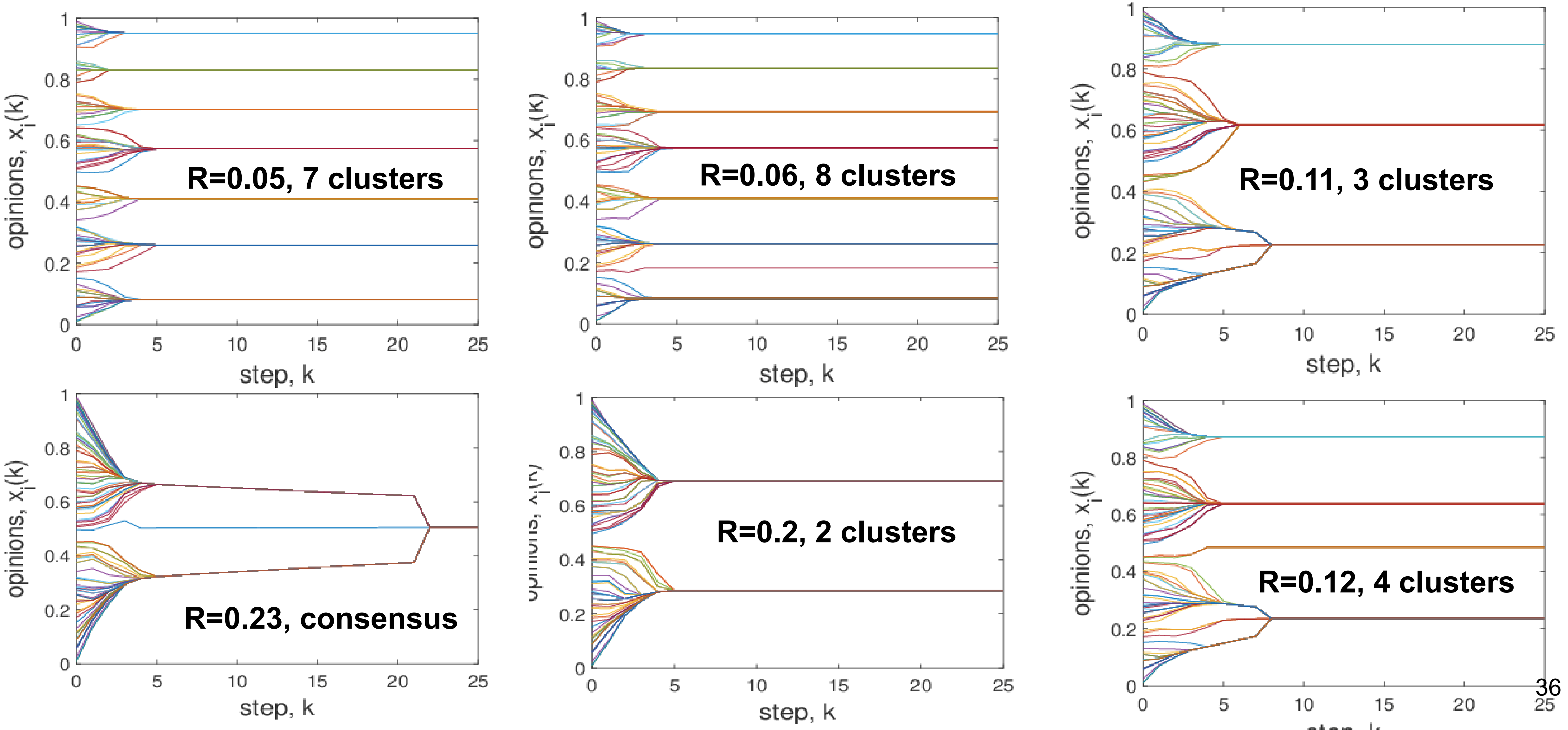
- Complicated nonlinear dynamics. **Convergence still can be proved due to symmetry of the weights**.

Abelson (1964)

Clusters for n=100 opinions on [0,1] (all *R* are same)

The dependence between final and initial opinions is very sensitive to R.

As R grows, the number of clusters can both decrease and increase!



Dynamic models are different, but their structures are similar:

- **Averaging mechanism:** desire to reach consensus**;**
- **Stubborness (**attachment to the initial or external opinion**);**
- **Homophily (**opinion-dependent graphs, nonlinear dynamics)**;**

These and other features (e.g., random graphs, disturbances, negative influence) can be mixed, producing highly non-trivial dynamics.

Explanations for non-convexity (convex hull is not always preserved):

- **Negative couplings (reactance, boomerang effects etc.);**
- **Nonlinear mechanisms of information integration (e.g. recurrent neural networks with general activation functions).**

Proskurnikov, Tempo, A Tutorial on Modeling and Analysis of Dynamic Social Networks. Part II, Ann. Rev. Control, vol. 45
Zhu et al., Neural opinion dynamics model for the prediction of user-level stance dynamics, Inform. Process. Manag., 57(2)

New fields: Dynamics of opinions on several logically related topics.

- **We allow multidimensional opinions, but their coordinates evolve independently.**

- **Such models are not very suitable to deal with opinions on interrelated topics. Some mechanisms maintaining consistency of opinions should be included e.g.**
    - If you promote veganism, you can hardly recommend meat burgers;
    - If you are libertarian, you are most probably against death penalty.

- **Although some linear models are published, we rather need some tools from formal logic and/or artificial intelligence to describe such a mechanism.**

Friedkin et al. (2016), Network science on belief system dynamics under logic constraints, Science, v. 354
Ye et al. (2020), Continuous-time opinion dynamics on multiple interdependent topics. Automatica, 115, 108884.

New fields: Numerical Information vs. Natural Language.

- **Numbers are not used in our everyday discussions. If someone asks you about a new movie, will you answer: "I estimate it as 0.07"? But mathematical models need numbers and vectors!**

- **Some numerical information can be extracted from text and speech (using e.g. sentiment analysis). The joint efforts from computational linguists, data scientists and system theorists are needed.**

- **Another approach: opinions as "linguistic variables" (strings of symbols).**
- Some models can be extended in this direction.

Dong et al. (2016), Dynamics of linguistic opinion formation in bounded confidence model, Information Fusion, v.32

New fields: Micro vs Macro

What happens as the number of agents grows?

- Equations of high dimensions are difficult to solve, especially if they are nonlinear. What can we do with dynamical models as the number of agents becomes large?

- Statistical physicists know answer: macroscopic (mean-field) approximation. Replace individual opinions by the density function. How accurate is it? Do we need n=100, n=1000, n=1 000 000 000 of agents to use macroscopic models?

- We can solve infinite-dimensional equations (PDEs, integral equations etc.) efficiently. So if the approximation is accurate, we have numerical tools to work with huge networks.

New fields: From closed to open social groups.

The community can lose its members and take new ones aboard.

How to modify the equations of opinion formation in the case where not only social ties, but even the number of individuals can change?

de Galland, Hendrickx (2019), Lower bound performances for average consensus in open multi-agent systems, IEEE CDC 2019
Hendrickx, Martin (2017), Open multi-agent systems: Gossiping with random arrivals and departures, IEEE CDC 2017